On Reciprocity in Linear Time-Invariant Networks

BY

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Summary.—A general time-domain definition of reciprocity is given in terms of network-port variables, and this definition is applied to conclude the symmetry of network matrices, even in the case of active, nonfinite networks. A new proof is presented of the reciprocity of finite networks composed of time-invariant resistors, capacitors, inductors and transformers, and this proof is also applied to show the reciprocity of interconnections of a wide class of reciprocal networks.

1.—Introduction.

We first consider a general definition of reciprocity for linear time-invariant networks and from this deduce certain equivalent definitions which depend on additional assumptions being made, such as the existence of an impedance matrix. The initial definition is a time-domain one; the property of reciprocity of a network is independent of whether or not the network inputs are Laplace-transformable, and to discuss reciprocity (as is often done) with inputs being restricted to being Laplace-transformable is therefore better avoided if possible.

The time-domain definition is initially in terms of voltage and current variables. An equivalent definition in terms of scattering variables (incident and reflected voltage waves) is given. This allows the effect the reciprocity constraint has on the scattering matrix, when this matrix exists, to be examined. Here, as in the immittance matrix case, it is concluded that symmetry of the matrix is a consequence of reciprocity.

The third section of the paper considers primarily the case of networks composed of a finite number of positive-valued (passive) resistors, capacitors and inductors, together with multiport transformers. The scattering matrix (though not the impedance matrix) for such a network always exists (Ref. 1), and the matrix is shown to be symmetric. The method used to prove this result is based on a representation of the network as a " cascade loading " (Ref. 1), that is, the cascade connection of a network consisting entirely of opens and shorts, and a loading network that is a multiport of uncoupled resistors, capacitors, inductors, and transformers. Finally, it is shown that any interconnection of linear, passive, solvable, reciprocal time-invariant networks is again a reciprocal network.

2.—Reciprocity for Linear Time-Invariant Networks.

As explained in the introductory section, we are interested in defining reciprocity in the widest possible sense, that is, assuming linearity and time-invariance but as little else as possible. Definitions requiring, for example, that a unit impulse of current at port i gives a voltage at port j which is the same as the voltage at port i produced by a unit impulse of current at port j are unnecessarily restrictive. This particular definition requires the existence of an impedance matrix, and, as such, does not even apply to a network consisting of a simple transformer.

We offer what we believe to be the most general definition, put in the terms of earlier discussions of network definitions (Refs. 2 and 3).

We assume that an n-port network $\mathbf{N}$ permits voltage-current pairs $[\mathbf{v}, \mathbf{i}]$ (termed allowed pairs) at its terminals, where $\mathbf{v}, \mathbf{i}$ are real-valued n-vector functions of time, zero up until some finite time, and infinitely differentiable after this time. The physical reasoning behind these assumptions may be found in Refs. 2 and 3. We shall write $\mathbf{v} \epsilon \mathcal{D}_+$ to signify that $\mathbf{v}$ has these properties.

Suppose $[\mathbf{v}_1, \mathbf{i}_1]$ and $[\mathbf{v}_2, \mathbf{i}_2]$ are two arbitrary allowed pairs. Then by definition $\mathbf{N}$ is reciprocal if for all such choices

$$\tilde{\mathbf{v}}_1 * \mathbf{i}_2 = \tilde{\mathbf{v}}_2 * \mathbf{i}_1 \quad \ldots\ldots\ldots\ldots(1)$$

The symbol $*$ denotes convolution; the superscript tilde ~ denotes matrix transposition.

If $\mathbf{v}_1 = [v_{1\alpha}]$, $\mathbf{i}_2 = [i_{2\alpha}]$ etc., Eq. (1) written in full becomes

$$\sum_{\alpha=1}^{n} \int_{-\infty}^{+\infty} v_{1\alpha}(t-\tau) i_{2\alpha}(\tau)\, d\tau = \sum_{\alpha=1}^{n} \int_{-\infty}^{+\infty} v_{2\alpha}(t-\tau) i_{1\alpha}(\tau)\, d\tau \quad \ldots(1')$$

Definition 1, which appears elsewhere (Ref. 2 and Ref. 4, p. 236), has been termed Lorentz reciprocity, as it is suggested by a theorem originally stated by Lorentz for electromagnetic systems. We comment further that since the various functions involved in Eq. (1) are zero up until some finite time the convolutions are well defined, and the convolution product is commutative.

In preparation for dealing with networks possessing a scattering matrix, we introduce at this stage scattering variables $\mathbf{v}^i$, $\mathbf{v}^r$ (incident and reflected voltage vectors) for the network $\mathbf{N}$ (Ref. 5, p. 142). These quantities are defined through

$$2\mathbf{v}^i = \mathbf{v} + \mathbf{i} \quad \ldots\ldots\ldots\ldots(2a)$$

$$2\mathbf{v}^r = \mathbf{v} - \mathbf{i} \quad \ldots\ldots\ldots\ldots(2b)$$

Note that by defining scattering variables we are not predicating the existence of a scattering matrix. Corresponding to allowed pairs $[\mathbf{v}_1, \mathbf{i}_1]$ and $[\mathbf{v}_2, \mathbf{i}_2]$ we have pairs $\{\mathbf{v}_1^i, \mathbf{v}_1^r\}$ and $\{\mathbf{v}_2^i, \mathbf{v}_2^r\}$, where the different brackets are to distinguish scattering variables from the ordinary variables.

Then we claim that an alternate statement of reciprocity is: $\mathbf{N}$ is reciprocal if for arbitrary allowed $\{\mathbf{v}_1^i, \mathbf{v}_1^r\}$ and $\{\mathbf{v}_2^i, \mathbf{v}_2^r\}$

$$\tilde{\mathbf{v}}_1^i * \mathbf{v}_2^r = \tilde{\mathbf{v}}_2^i * \mathbf{v}_1^r \quad \ldots\ldots\ldots\ldots(3)$$

Eqs. (1) and (3) are strictly equivalent, so that Eq. (3) could, equally well, be taken as the reciprocity definition. To see for example that Eq. (3) implies Eq. (1), substitute for the variables $\mathbf{v}_1^i$ etc. in Eq. (3) by using Eq. (2). On cancelling out like terms on each side of the resulting equation, Eq. (1) follows. In an equally simple fashion Eq. (3) follows from Eq. (1).

Let us now specialize $\mathbf{N}$ so that it is linear, time-invariant and completely solvable, that is, $\mathbf{N}$ possesses a (time-domain) scattering matrix (Ref. 6). The class of networks possessing scattering matrices is a more general one than the class possessing impedance matrices (Ref. 7, p. 122) for example, any finite linear time-invariant $\mathbf{N}$ composed of standard passive-network elements which possesses an impedance matrix also possesses a scattering matrix,

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not vice versa (Ref. 1). Accordingly we choose first to make ; specialization, rather than assume the existence of an immittance trix. The time-domain scattering matrix is used as this is a re fundamental matrix than the more common frequency-domain trix, which exists if and only if the time-domain matrix is Laplace-ısformable.

Since any physical network **N** (active or passive) is non-anti-ıtive, we shall assume that the scattering matrix $\mathbf{s} \equiv \mathbf{s}(t)$ is ɔ for $t < 0$, and maps incident voltages into reflected voltages ɔugh

$$\mathbf{v}^r = \mathbf{s} * \mathbf{v}^i \quad \ldots\ldots\ldots\ldots(4)$$

ɜre $\mathbf{v}^i \epsilon \mathcal{D}_+$ but is otherwise arbitrary. If all quantities are ɔlace-transformable, then Eq. (4) implies the more familiar ıation $V^r(p) = S(p)\, V^i(p)$.

In general **s** will not be a function; however, it is shown in . 6 that **s** will be a distribution in the variable t (Ref. 8), typically ɔlving a delta function and a unit step function. Because **s** is ıtically zero for negative values of the argument **s** is said to be he space $\mathcal{D}'_+$ (Ref. 9, p. 28) of distributions with support boun- on the left. The variables $\mathbf{v}^r$ and $\mathbf{v}^i$ being in the space $\mathcal{D}_+$ also in the space $\mathcal{D}'_+$.

Substituting from Eq. (4) into Eq. (3) we have

$$\tilde{\mathbf{v}}_1{}^i * (\mathbf{s} * \mathbf{v}_2{}^i) = \tilde{\mathbf{v}}_2{}^i * (\mathbf{s} * \mathbf{v}_1{}^i) \quad \ldots\ldots\ldots(5a)$$

ıeorem of L. Schwartz (Ref. 9, p. 28) shows that the convolution ducts in this equation are associative and commutative since various factors are all in the space $\mathcal{D}'_+$. Accordingly,

$$\tilde{\mathbf{v}}_1{}^i * \mathbf{s} * \mathbf{v}_2{}^i = \tilde{\mathbf{v}}_2{}^i * \mathbf{s} * \mathbf{v}_1{}^i$$

$$= \tilde{\mathbf{v}}_1{}^i * \tilde{\mathbf{s}} * \mathbf{v}_2{}^i \quad \ldots\ldots\ldots\ldots(5b)$$

We may choose $\mathbf{v}_1{}^i$ and $\mathbf{v}_2{}^i$ arbitrarily; take in particular $\mathbf{v}_1{}^i$ ıave all entries zero except the k-th, and $\mathbf{v}_2{}^i$ to have all entries except the l-th. The non-zero entries of $\mathbf{v}_1{}^i$ and $\mathbf{v}_2{}^i$ are trary $\mathcal{D}_+$ functions. Then

$$v^i_{1k} * s_{kl} * v^i_{2l} = v^i_{1k} * s_{lk} * v^i_{2l} \quad \ldots\ldots\ldots(6)$$

ch implies that

$$s_{kl} = s_{lk} \quad \ldots\ldots\ldots\ldots(7)$$

erve also that Eq. (7) implies Eq. (5*a*). This yields the follow- theorem.

orem 1:

A linear time-invariant completely solvable network **N** is procal if and only if its (time-domain) scattering matrix is metric.

In the case where the Laplace transform $\mathcal{L}[\mathbf{s}(t)] = S(p)$ exists ıave

ɔllary:

A linear time-invariant solvable network **N** possessing a tering matrix $S(p)$ is reciprocal if and only if $S(p)$ is symmetric.

We comment that the preceding proof does not assume the tence $\mathcal{L}[\mathbf{s}]$, $\mathcal{L}[\mathbf{v}^i]$, $\mathcal{L}[\mathbf{v}^r]$, and is, accordingly, more general ı any proof considering all quantities to be defined in the (com-) frequency domain. It should also be noted that the result is pendent of the passivity (or lack of it) of the network **N**. Neither **N** have to be finite.

What now of the linear time-invariant network which possesses mpedance matrix $\mathbf{z}(t)$, with possibly $Z(p)$ existing as well? ıments similar to those used to establish *Theorem* 1, but start- from Eq. (1) rather than Eq. (3) yield:

ırem 2:

A linear time-invariant network **N** possessing an impedance ix $\mathbf{z}(t)$ (and perhaps $Z(p)$) is reciprocal if and only if $\mathbf{z}(t)$ (and) is symmetric.

A similar theorem holds of course for admittance matrices.

Note that if $S(p)$ and $Z(p)$ both exist, then the symmetry of one follows immediately from the symmetry of the other through the equations (Ref. 10, p. 242)

$$S(p) = [Z(p) + 1]^{-1}\,[Z(p) - 1] \quad \ldots\ldots\ldots\ldots(8a)$$

$$Z(p) = [1 - S(p)]^{-1}\,[1 + S(p)] \quad \ldots\ldots\ldots\ldots(8b)$$

3.—Reciprocity for Finite Passive Linear Time-Invariant Networks and Interconnections of Reciprocal Networks.

In this section we consider networks composed of a finite number of positive resistors, capacitors, inductors, and multiport transformers, that is, the most general kind of linear, passive, finite time-invariant network composed of reciprocal elements. The reciprocity of such a network was established by McMillan (Ref. 4), and more recently for example by de Buda (Ref. 11). We present here a simple proof relying on the result established elsewhere (Ref. 1) that the scattering matrix of such a network exists in the complex frequency domain (p-domain), and further apply this result to the interconnections of passive reciprocal networks.

It is clear, and shown in Ref. 1, that any finite linear time-invariant n-port network **N** composed of positive resistors, capacitors and inductors, together with multiport transformers, can be represented as the cascade connection of a network $\mathbf{N}_\Sigma$ composed entirely of opens and shorts, terminated in a network $\mathbf{N}_l$ consisting of all the resistors, capacitors, etc., of **N** unconnected to each other. This is shown in Fig. 1.

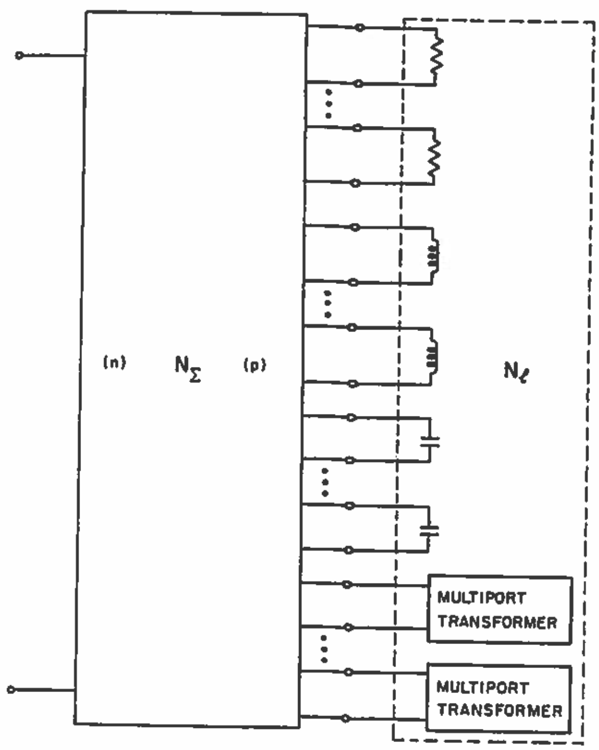

Fig. 1.—Cascade Loading Representation of N.

A multiport transformer may be described by the equations

$$\mathbf{v}_1 = \tilde{T}\mathbf{v}_2 \quad \ldots\ldots\ldots\ldots(9a)$$

$$\mathbf{i}_2 = -T\mathbf{i}_1 \quad \ldots\ldots\ldots\ldots(9b)$$

where T is a constant $m \times n$ matrix, the transformer having n primary and m secondary ports. The vectors $\mathbf{v}_1$, $\mathbf{v}_2$ are the primary and secondary voltages, and $\mathbf{i}_1$, $\mathbf{i}_2$ the primary and secondary currents. The transformer has the scattering matrix

$$S(p) = \begin{bmatrix} (1_n + \tilde{T}T)^{-1}(\tilde{T}T - 1_n) & 2(1_n + \tilde{T}T)^{-1}\tilde{T} \\ 2T(1_n + \tilde{T}T)^{-1} & (1_m + T\tilde{T})^{-1}(1_m - T\tilde{T}) \end{bmatrix} \quad ..(10)$$

hich is easily verified to be symmetric. As a consequence, $\mathbf{N}_\Sigma$ as a symmetric scattering matrix Σ and is therefore reciprocal, nce, as is shown in Ref. 1, $\mathbf{N}_\Sigma$ is subject to the same constraining lations as a multiport transformer. The network $\mathbf{N}_l$ also has a mmetric scattering matrix S_l, this being the (matrix) direct sum f a number of symmetric matrices (corresponding to each of the etwork elements of $\mathbf{N}$, each of which is reciprocal).

We partition Σ as

$$\Sigma = \left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array}\right] = \widetilde{\Sigma}$$

here Σ_{11} is $n \times n$, corresponding to the input ports of $\mathbf{N}$, Σ_{22} $p \times p$, where $\mathbf{N}_l$ has p ports.

The scattering matrix S of the cascade loading interconnection ay be evaluated using methods outlined in Ref. 12 as

$$S = \Sigma_{11} + \Sigma_{12} S_l (1_p - \Sigma_{22} S_l)^{-1} \Sigma_{21} \quad \ldots\ldots\ldots (11a)$$

r equivalently

$$S = \Sigma_{11} + \Sigma_{12} (1_p - S_l \Sigma_{22})^{-1} S_l \Sigma_{21} \quad \ldots\ldots\ldots (11b)$$

Difficulties arise due to the question of the existence of the nverse of $(1_p - \Sigma_{22} S_l)$ or $(1_p - S_l \Sigma_{22})$ and are dealt with in the ppendix. Here we shall simply assume the inverses exist. aking the transpose of Eq. (11*a*):

$$\widetilde{S} = \widetilde{\Sigma}_{11} + \widetilde{\Sigma}_{21} (1_p - \widetilde{S}_l \widetilde{\Sigma}_{22})^{-1} \widetilde{S}_l \widetilde{\Sigma}_{12}$$

Jsing the symmetry of Σ and S_l,

$$\widetilde{S} = \Sigma_{11} + \Sigma_{12} (1_p - S_l \Sigma_{22})^{-1} S_l \Sigma_{21}$$

Jsing Eq. (11*b*) it follows that

$$S = \widetilde{S} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

By the corollary to *Theorem* 1, this implies that the network $\mathbf{N}$ s reciprocal. Hence we have the following statement.

Theorem 3:

A network consisting of a finite number of passive time-invariant resistors, capacitors, inductors and multiport transformers, s reciprocal, and in fact possesses a symmetric scattering matrix.

This result is capable of almost immediate generalization. Suppose $\mathbf{N}_1$ and $\mathbf{N}_2$ are any two linear, passive, time-invariant reciprocal networks possessing scattering matrices $S_1(p)$ and $S_2(p)$ and hat $\mathbf{N}_1$ and $\mathbf{N}_2$ are connected together in some arbitrary fashion. As before, we represent the interconnecting network as $\mathbf{N}_\Sigma$ with scattering matrix Σ, and the load network as $\mathbf{N}_l$, consisting of $\mathbf{N}_1$ and $\mathbf{N}_2$ uncoupled networks (see Fig. 2). The network $\mathbf{N}_l$ is of course inear, passive, time-invariant and reciprocal, and possesses a scattering matrix equal to the direct sum of $S_1(p)$ and $S_2(p)$. The arguments used to establish *Theorem* 3 depend merely on the passivity and symmetry of the matrices involved rather than, for example, the fact that these matrices are derived from finite networks, and accordingly they carry through to show:

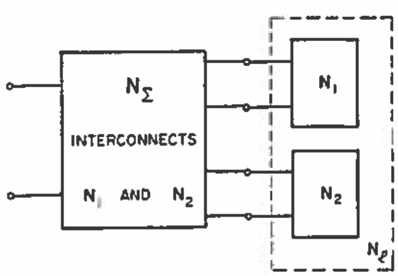

Fig. 2.—General Interconnection of Two Reciprocal Networks.

Theorem 4:

Let $\mathbf{N}_1$ and $\mathbf{N}_2$ be two linear, passive, time-invariant reciprocal networks possessing scattering matrices $S_1(p)$ and $S_2(p)$. Then any arbitrary interconnection of $\mathbf{N}_1$ and $\mathbf{N}_2$ is also reciprocal, and in fact possesses a symmetric scattering matrix $S(p)$.

Conclusions.

Reciprocity is a meaningful concept for linear time-invariant networks; existence of impedance matrices or Laplace-transformable inputs are not necessary prerequisites for a discussion of reciprocity. A time-domain definition of reciprocity which encompasses the more common but less general definitions usually given is considered, and an equivalent statement in terms of the often more useful scattering variables is given. The symmetry of network matrices when they exist is implied by reciprocity, and this property is independent of properties such as network passivity. A new proof is given of the reciprocity of finite passive resistor, capacitor, inductor, transformer networks which is an application of earlier work (Ref. 1). This proof is independent of the existence of chain or impedance matrices.

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APPENDIX.

Suppose $(1_p - \Sigma_{22} S_l)^{-1}$ does not exist. Now, as is pointed out in Ref. 1 $\begin{bmatrix} \Sigma_{11} & \Sigma_{12} S_l \\ \Sigma_{21} & \Sigma_{22} S_l \end{bmatrix}$ and $\Sigma_{22} S_l$ are passive scattering matrices, and accordingly (Ref. 7, p. 121), there exists a constant orthogonal matrix T such that

$$\begin{bmatrix} 1 & 0 \\ 0 & \widetilde{T} \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} S_l \\ \Sigma_{21} & \Sigma_{22} S_l \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \hat{\Sigma}_{12} & 0 \\ \hat{\Sigma}_{21} & \hat{\Sigma}_{22} & 0 \\ 0 & 0 & 1_r \end{bmatrix}$$

with $\tilde{T}\,\Sigma_{21} = \begin{bmatrix} \hat{\Sigma}_{21} \\ 0 \end{bmatrix}$, $\Sigma_{12}\,S_l\,T = [\hat{\Sigma}_{12} \;\; 0]$ and $\det(1_{p-r} - \hat{\Sigma}_{22}) \neq 0$,

that is $(1_{p-r} - \Sigma_{22})^{-1}$ exists. Range space arguments following those given in Ref. 1 then show that in place of

$$S = \Sigma_{11} + \Sigma_{12}\,S_l(1_p - \Sigma_{22}\,S_l)^{-1}\,\Sigma_{21} \quad \ldots\ldots\ldots\ldots(11a)$$

or equivalently

$$S = \Sigma_{11} + \Sigma_{12}\,S_l\,T\{\tilde{T}(1_p - \Sigma_{22}\,S_l)T\}^{-1}\,\tilde{T}\,\Sigma_{21}$$

$$= \Sigma_{11} + [\hat{\Sigma}_{12} \;\; 0] \begin{bmatrix} (1_{p-r} - \hat{\Sigma}_{22}) & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \hat{\Sigma}_{21} \\ 0 \end{bmatrix}$$

we have

$$S = \Sigma_{11} + \hat{\Sigma}_{12}(1_{p-r} - \hat{\Sigma}_{22})^{-1}\,\hat{\Sigma}_{21} \quad \ldots\ldots\ldots\ldots(13a)$$

In place of

$$S = \Sigma_{11} + \Sigma_{12}(1_p - S_l\,\Sigma_{22})^{-1}\,S_l\,\Sigma_{21} \quad \ldots\ldots\ldots\ldots(11b)$$

or equivalently

$$S = \Sigma_{11} + \tilde{\Sigma}_{21}\widetilde{(1_p - \Sigma_{22}\,S_l)}^{-1}\,\tilde{S}_l\,\tilde{\Sigma}_{12}$$

$$= \Sigma_{11} + \tilde{\Sigma}_{21}\,T\widetilde{\{\tilde{T}(1_p - \Sigma_{22}\,S_l)T\}}^{-1}\tilde{T}\,\tilde{S}_l\,\tilde{\Sigma}_{12}$$

$$= \Sigma_{11} + [\tilde{\hat{\Sigma}}_{21} \;\; 0] \begin{bmatrix} \widetilde{(1_{p-r} - \hat{\Sigma}_{22})} & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\hat{\Sigma}}_{12} \\ 0 \end{bmatrix}$$

we have

$$S = \Sigma_{11} + \tilde{\hat{\Sigma}}_{21}\widetilde{(1_{p-r} - \hat{\Sigma}_{22})}^{-1}\,\tilde{\hat{\Sigma}}_{12} \quad \ldots\ldots\ldots\ldots(13b)$$

From Eqs. (13*a*) and (13*b*) we conclude that

$$S = \tilde{S} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

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