On the Energy in Passive Systems

At present, several notions of a passive system are available and useful in different contexts. Here we investigate two of the widely used passive definitions showing that they are equivalent for a large class of systems.

Consider a multivariable system which can be represented by a transformation mapping an input $\mathbf{x}$ into an output $\mathbf{y}$; if $\mathbf{y}$ is so obtained from an allowed input the pair $[\mathbf{y}, \mathbf{x}]$ will be called *admissible*. Both $\mathbf{x}$ and $\mathbf{y}$ are assumed to be n-vectors of complex-valued functions of time t whose entries, for our purposes, are assumed to be infinitely differentiable and zero until some finite time. If $\mathbf{x}$ and $\mathbf{y}$ are measured at the same points and taken such that their scalar product has the dimension of power, then two reasonable definitions of the *total input energy* into the system are

$$\mathcal{E}_1(t) = \int_{-\infty}^{t} [\operatorname{Re} \tilde{\mathbf{y}}(\tau)][\operatorname{Re} \mathbf{x}(\tau)] d\tau \qquad (1)$$

and

$$\mathcal{E}_2(t) = \int_{-\infty}^{t} \operatorname{Re} [\tilde{\mathbf{y}}^*(\tau)\mathbf{x}(\tau)] d\tau. \qquad (2)$$

In these expressions we have used Re $\boldsymbol{A}$, $\tilde{\boldsymbol{A}}$, and $\boldsymbol{A}^*$ to respectively denote the real part, the transpose, and the complex conjugate of a matrix $\boldsymbol{A}$. The previous restrictions placed on $\mathbf{x}$ and $\mathbf{y}$ guarantee that both $\mathcal{E}_1$ and $\mathcal{E}_2$ exist, however other types of variables can be used if only energy is considered, such as square integrable or exponentially increasing $\mathbf{x}$ and $\mathbf{y}$. The first type of energy $\mathcal{E}_1$ is common in electromagnetism ([1], p. 135) and can be found in system studies ([2], p. 417). The second $\mathcal{E}_2$ has been used by Dolph in studying the properties of passive systems ([3], p. 13), and can be found in rigorous network studies ([4], p. 110). Thus, both definitions can be found in the literature and lead to two definitions of passive systems. A system is defined to be *$\mathcal{E}_1$-passive* if $\mathcal{E}_1(t) \geq 0$ and *$\mathcal{E}_2$-passive* if $\mathcal{E}_2(t) \geq 0$, both for all finite t and all admissible output-input pairs $[\mathbf{y}, \mathbf{x}]$. As a preliminary to later discussion we point out that (1) and (2) are merely definitions which one can consider for any type of system, linear or not time-variable or not, etc.

We first show that the two definitions are completely equivalent for real systems. If we let

$$\mathbf{x} = \mathbf{x}_r + j\mathbf{x}_i, \qquad \mathbf{y} = \mathbf{y}_r + j\mathbf{y}_i \qquad (3)$$

with $\mathbf{x}_r$, $\mathbf{x}_i$, $\mathbf{y}_r$ and $\mathbf{y}_i$ real-valued n-vectors, then the system is called *real* if $[\mathbf{y}, \mathbf{x}]$ admissible implies that both $[\mathbf{y}_r, \mathbf{x}_r]$ and $[\mathbf{y}_i, \mathbf{x}_i]$ are admissible.

Theorem

If a system S is real then a necessary and sufficient condition for S to be $\mathcal{E}_1$-passive is that S be $\mathcal{E}_2$-passive.

Proof: We must show that if $\mathcal{E}_1(t) \geq 0$ for a given admissible pair $[\mathbf{y}, \mathbf{x}]$ then $\mathcal{E}_2(t) \geq 0$ and vice versa. Writing $[\mathbf{y}_1, \mathbf{x}_1] = [\mathbf{y}_2, \mathbf{x}_2]$ to denote $\mathbf{x}_1 = \mathbf{x}_2$ and $\mathbf{y}_1 = \mathbf{y}_2$, we first consider the real pair $[\mathbf{y}, \mathbf{x}] = [\mathbf{y}_r, \mathbf{x}_r]$ for which, in suggestive notation,

$$_r\mathcal{E}_1(t) = {}_r\mathcal{E}_2(t) = \int_{-\infty}^{t} \tilde{\mathbf{y}}_r(\tau)\mathbf{x}_r(\tau) d\tau \geq 0. \qquad (4a)$$

Similarly for the real pair $[\mathbf{y}, \mathbf{x}] = [\mathbf{y}_i, \mathbf{x}_i]$

$$_i\mathcal{E}_1(t) = {}_i\mathcal{E}_2(t) = \int_{-\infty}^{t} \tilde{\mathbf{y}}_i(\tau)\mathbf{x}_i(\tau) d\tau \geq 0. \qquad (4b)$$

Now consider $[\mathbf{y}, \mathbf{x}] = [\mathbf{y}_r + j\mathbf{y}_i, \ \mathbf{x}_r + j\mathbf{x}_i]$, then from (1) and (4a) one has

$$\mathcal{E}_1(t) = {}_r\mathcal{E}_2(t) \geq 0 \qquad (5)$$

and from (2) and (4a), (4b)

$$\mathcal{E}_2(t) = {}_r\mathcal{E}_1(t) + {}_i\mathcal{E}_1(t) \geq 0. \qquad (6)$$

Since $[\mathbf{y}_r + j\mathbf{y}_i, \ \mathbf{x}_r + j\mathbf{x}_i]$ admissible implies $[\mathbf{y}_r, \mathbf{x}_r]$ and $[\mathbf{y}_i, \mathbf{x}_i]$ admissible, (6) shows that $\mathcal{E}_1$-passivity implies $\mathcal{E}_2$-passivity, while (5) shows the converse, these results holding for any admissible $[\mathbf{y}, \mathbf{x}]$. Q.E.D.

As a consequence of the theorem, results derived using one definition of passivity are valid for the other in the case of real systems. In particular let us consider the linear case where we allow exponentially increasing variables

$$\mathbf{x}(t) = X \exp(pt),$$
$$\mathbf{y}(t) = Y \exp(pt), \qquad p = \sigma + j\omega \qquad (7a)$$

for $\sigma > 0$. If we further assume the $n \times n$ system-function, or transfer-function, matrix $\boldsymbol{T}(p)$ is defined and exists in $\sigma > 0$, then

$$Y = \boldsymbol{T}(p)X. \qquad (7b)$$

If the system is $\mathcal{E}_1$- or $\mathcal{E}_2$-passive, then by various means one can show that $\boldsymbol{T}(p)$ is holomorphic in $\sigma > 0$ {for instance, one can assume singularities and show that (1) or (2) are violated using (7), or one can introduce unity feedback working with a new passive system of admissible pairs $[\mathbf{y}, \mathbf{x}+\mathbf{y}]$ ([4], pp. 111 and 122)}. Further if the system is real then $\boldsymbol{T}(\sigma)$ is real in $\sigma > 0$, which, by Schwarz's Reflection Principle ([5], p. 186) implies $\boldsymbol{T}^*(p) = \boldsymbol{T}(p^*)$ in $\sigma > 0$.

Assuming then the two conditions

$$1) \ \boldsymbol{T}(p) \text{ holomorphic} \quad \text{in } \sigma > 0 \qquad (8a)$$

$$2) \ \boldsymbol{T}^*(p) = \boldsymbol{T}(p^*) \quad \text{in } \sigma > 0 \qquad (8b)$$

consider an $\mathcal{E}_2$-passive system. Letting $\boldsymbol{T}_H$ denote the Hermitian part, $2\boldsymbol{T}_H = \boldsymbol{T} + \tilde{\boldsymbol{T}}^*$, and noting that Re $\tilde{X}^*\boldsymbol{T}X = \tilde{X}^*\boldsymbol{T}_H X$, substitution of (7) into (2) immediately shows that

$$3) \ \tilde{X}^*\boldsymbol{T}_H(p)X \geq 0 \quad \text{in } \sigma > 0 \qquad (8c)$$

for all complex constant n vectors X. Any matrix $\boldsymbol{T}$ satisfying conditions 1), 2) and 3) is called *positive-real*. Since (8c) under (8a), (8b) implies $\mathcal{E}_2(t) \geq 0$ ([6], p. 273), every real system with $\boldsymbol{T}(p)$ defined in $\sigma > 0$ is $\mathcal{E}_2$-passive if and only if $\boldsymbol{T}$ is positive-real.

Similarly, assuming conditions 1) and 2), consider an $\mathcal{E}_1$-passive system. After some manipulation ([2], pp. 421–423), ([7], p. 146) (7) substituted into (1) yield

$$4) \ Q_-(p, X) \geq 0 \quad \text{in } \sigma > 0 \qquad (8d)$$

for all complex constant n-vectors X, where

$$Q_-(p, X) = \begin{cases} \tilde{X}^*\boldsymbol{T}_H(p)X & \text{if } \omega = 0 \\ \tilde{X}^*\boldsymbol{T}_H(p)X - \dfrac{\sigma}{|p|} \, |\tilde{X}\boldsymbol{T}(p)X| & \text{if } \omega \neq 0. \end{cases} \qquad (9)$$

We can now state that under the assumption of conditions 1) and 2), *conditions 3) and 4) are completely equivalent*, that is, $\tilde{X}^*\boldsymbol{T}_H(p)X \geq 0$ in $\sigma > 0$ implies (and is implied by) $Q_-(p, X) \geq 0$ in $\sigma > 0$ for the reason which follows. If $\boldsymbol{T}$ exists satisfying conditions 1), 2), and 3) then it corresponds to a real linear $\mathcal{E}_2$-passive system. But, by the theorem, such a system is $\mathcal{E}_1$-passive, hence, condition 4) must hold. Conversely, if condition 4) holds one immediately has $\tilde{X}^*\boldsymbol{T}(p)X \geq 0$ in $\sigma > 0$ and condition 3) holds showing that the system must be $\mathcal{E}_2$-passive and, thus, also (by the last sentence) $\mathcal{E}_1$-passive.

One can go further and show that conditions 1) and 4) imply 2). Although space considerations preclude presenting the calculations, this equivalence follows from the fact that $Q \geq 0$ in $\sigma > 0$ implies by the continuity of holomorphic matrices that

$$4') \ \operatorname{Re} \tilde{X}^*\boldsymbol{T}(p)X \geq \frac{\sigma}{|p|} \, |X\boldsymbol{T}(p)X| \geq 0 \quad \text{in } \sigma > 0 \qquad (10)$$

for even real p in $\sigma > 0$. Choosing X properly for (10) shows that conditions 1) and 4′) (or 4), imply and are implied by positive-reality. Equation (10) is analogous to the angle constraint for positive-real functions ([8], p. 114).

Although $\mathcal{E}_1$ and $\mathcal{E}_2$ yield equivalent results for real systems it should be pointed out that $\mathcal{E}_1$ is not necessarily equal to $\mathcal{E}_2$ for the same admissible pair. Further, the two concepts of passivity differ for systems which are not real. For example $y = jx$ has

$$\mathcal{E}_1(t) = -\int_{-\infty}^{t} x_r(\tau)x_i(\tau) d\tau$$

and $\mathcal{E}_2 \equiv 0$; when complex excitations are allowed, this system (which is equivalent to a complex resistor) is $\mathcal{E}_2$-passive but not $\mathcal{E}_1$-passive. Such an example is of interest in network synthesis where complex resistors have been used in passive synthesis [9], ([10], p. 35), and must indirectly be considered in complex normalizations ([11], p. 19). Since complex resistors are usually cancelled at the end of a synthesis, the results of this correspondence show that the final network obtained will be passive. The results are also of some use in nonlinear system analysis where real systems of higher order can be used to model nonreal systems; for example $y = (x_1 + jx_2)^2$ can be modelled for some purposes by the two-dimensional system with $y_1 = x_1^2 - x_2^2$, $y_2 = 2x_1x_2$.

In summary, we have shown that two definitions in common use yield similar results for real systems. Consequently, the use of one over the other is a matter of personal preference. Using these ideas the equivalence between the angle constraint, (8a) and (10), and the normal definition, (8a)–(8c) of positive-real matrices has been shown. Except for minor details, an alternate proof of the equivalence between (8c) and (8d), using eigenvector calculations, can be found in Youla ([11], pp. 42 and 95–96).

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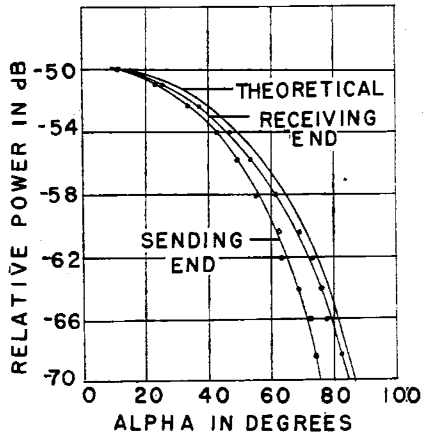

Fig. 1. A plot of the relative power in decibels vs. the angle alpha in degrees.

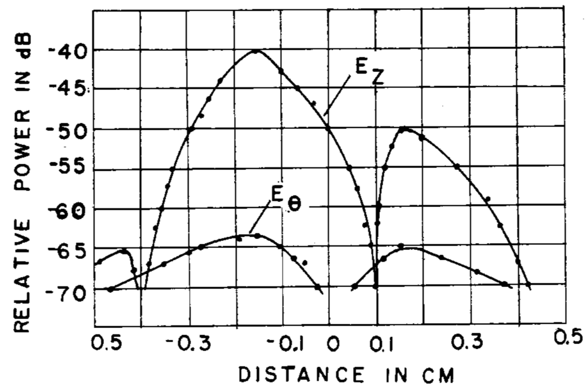

Fig. 2. A plot of the relative power in decibels for E_z and E_θ vs. the distance x in centimeters. For the measurement of E_z, the wide edge of the guide was perpendicular to the line, and the guide was moved in the x direction. The z direction is along the line, and the x direction is perpendicular to z in a horizontal plane. For the measurement of E_θ, the wide edge of the guide was parallel to the wires.

Mode of Millimeter Wave Two-Wire Surface Wave Transmission Line Fields

The purpose of this correspondence is to report the detailed experimental investigation of the mode configuration of millimeter wave two-wire surface wave transmission line. The possibility of millimeter wave surface wave transmission was predicted by Goubau.[1] Umehara performed a theoretical analysis of the surface wave two-wire line and produced some numerical examples in the X-band.[2] In millimeter wavelengths, Ishii, *et al.*,[3] demonstrated experimentally that the surface wave two-wire line has reasonable energy concentration along the wires and they predicted a hybrid mode, which agrees with Umehara. The attenuation of the two-wire line was found to be comparable to that of waveguides at millimeter wavelengths.[4]

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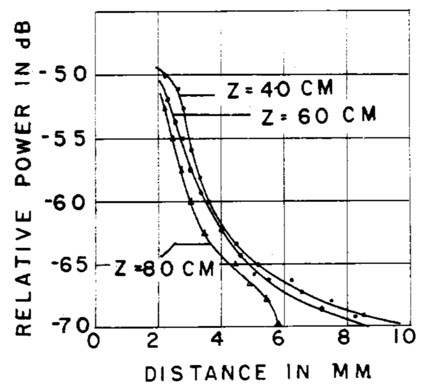

Fig. 3. A plot of E_z vs. x, for various distances (z) from the launcher. These measurements were made in a horizontal plane, the center of the wide dimension of the guide being in the horizontal plane defined by the two wires.

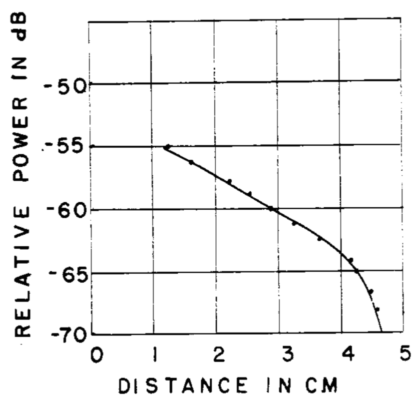

Fig. 4. A plot of the relative power in decibels for E_r vs. x in centimeters. This measurement was made by placing the wide side of the guide vertically, and the axis of the guide along the wires. This measurement required a right angle bend, and the flange on the ninety degree section limited how close the pickup could be placed near the transmission line.

The experiments were performed in the 70–71 Gc frequency range. The unique pickup device for the electric field was the open end of an RG-98/U waveguide.

The existence of a strong electric field in the z direction is demonstrated in Fig. 1. The maximum power level (alpha equals zero degrees) corresponds to narrow dimension of the waveguide parallel to the line. It can be seen on the curve measurements that were made at the receiving and sending ends of the line. The distances from the horns were 15 cm and 5 cm, respectively. The curve labeled theoretical was obtained by taking 20 log cos alpha.

Electric fields in the z and θ directions are compared in Fig. 2. Both measurements were made 25 cm from the sending end and approximately 1 mm above the wire. The