A Structurally Stable Realization for Jacobi Elliptic Functions

Honghao Ji and Robert W. Newcomb
Microsystems Laboratory, ECE Department
University of Maryland, College Park, MD 20742 USA
Email: newcomb@eng.umd.edu , jhonghao@glue.umd.edu

Abstract: By adding convergence terms, the dynamical equations for the generation of elliptic functions versus time are presented. This results in a structurally stable oscillator with limit cycles, which are Jacobi elliptic functions. From these equations a CMOS realization is developed with the nonlinearities obtained by using analog four-quadrant multipliers of the type developed by Kimura.

I. Introduction:

Recently Meyer [1] has presented the state variable equations which yield the Jacobi elliptic functions as their solution. This is an important result since it allows for the simple generation of driving signals for soliton computers [2] since there exist soliton systems where the solitons are based upon Jacobi elliptic functions [3, pp. 15 & 134][4, p. 17]. And practically this is quite important since it allows for the VLSI realization of multi-soliton systems, which can act as several simultaneous computers using the same hardware [5].

Here we present the state variable equations which generate the Jacobi elliptic functions, in Section II. Since the equations are not structurally stable they are not the most ideal for realization by electronic hardware. Consequently, in Section III, we modify these equations to be structurally stable with a single nonzero limit cycle. In Section IV we discuss the realization in terms of CMOS VLSI where the four-quadrant multiplier of Kimura [5] is used to realize the nonlinearities.

II. The Basic Equations

The basic state variable equations are [1, p. 730]

$$\frac{dx}{dt} = yz \qquad (1a)$$

$$\frac{dy}{dt} = -xz \qquad (1b)$$

$$\frac{dz}{dt} = -k^2xy \qquad (1c)$$

where $0 \le k \le 1$ is the modulus parameter and initial conditions are as followings,

$$x(0) = 0,\ y(0) = z(0) = 1 \qquad (2a,b,c)$$

With these initial conditions the solution x is the elliptic sine, sn(t,k), y is the elliptic cosine, cn(t,k), and z is the elliptic delta, dn(t,k). Figure 1 shows a PSpice run from an ideal circuit realizing equations (1) with k=1/2.

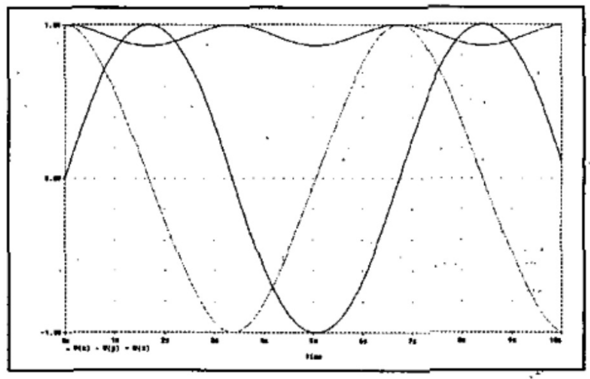

Figure 1: The three Jacobi elliptic functions

Of considerable interest is the fact that equations (1) admits two positive definite invariants of the motion

$$I = x^2 + y^2,\quad J = k^2x^2 + z^2 \qquad (3a,b)$$

Here a given set of initial conditions fixes the constants I and J on the left of (3) and then the solutions should be held to those values when evaluated on the right. Unfortunately, due to inaccuracies in the system these invariants become violated in practical realizations, this being one problem faced in constructing electronic circuits for this system.

Since we will wish to work with different amplitudes to fit vlsi multipliers we first consider an amplitude scaling by a real constant factor A. Thus let

$$X = Ax,\ Y = Ay,\ Z = Az \qquad (4a,b,c)$$

Substituting into (1) shows that the same differential equation is satisfied if we set T=At and use dT as the time differential. Thus, by scaling the size of the solutions to fit our multipliers we will still obtain the elliptic functions but scaled in both amplitude and time. The initial conditions of course will correspondingly change to that these will be

$$X(0) = 0,\ Y(0) = A = Z(0) \qquad (5a,b,c)$$

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Out of this we note that if we change the initial conditions to x (0) = 0, y (0) = z (0) = A we still obtain the Jacobi elliptic functions as the solution of (1) but their amplitudes are scaled as are their time scales. Consequently, a small perturbation in initial conditions does not lead to a return to the unperturbed limit cycle, in which the system of (1) is not structurally stable. The situation is similar to that for LC resonators for which a stabilization of the structure is obtained in the van der Pol oscillator by the insertion of damping that is negative for small amplitudes and positive for large amplitudes.

III. Structurally Stable Equations

We modify the basic equation of (1) by adding damping terms which give positive damping when the right side of (3) becomes bigger than the constants on the left and gives negative damping when smaller. Our structurally stable equations become

$$\frac{dx}{dt} = yz - \varepsilon_x (x^2 + y^2 - I_0) \tag{6a}$$

$$\frac{dy}{dt} = -xz \tag{6b}$$

$$\frac{dz}{dt} = -k^2 xy - \varepsilon_z (k^2 x^2 + z^2 - J_0) \tag{6c}$$

Here ε_x and ε_z are small positive constants which determine the size of the damping on the system; the smaller they are the closer to the Jacobi elliptic functions are the trajectories. I_0 and J_0 are the constants, which determine toward which limit cycle the trajectories tend. Figure 2 shows how the trajectories tend to the desired ones, of valleys and peaks $\mp$ ½ for x and y and to ½ and 0.433 = square_root $(J_0 - k^2 x_{max}^2)$ for z; these are for k = 1/2, $\varepsilon_x = \varepsilon_z = 0.1$, $I_0 = J_0 = ((0)^2 + (1/2)^2)$ and x(0) = 0.1, y(0) = 0.2, z(0) = 0.3.

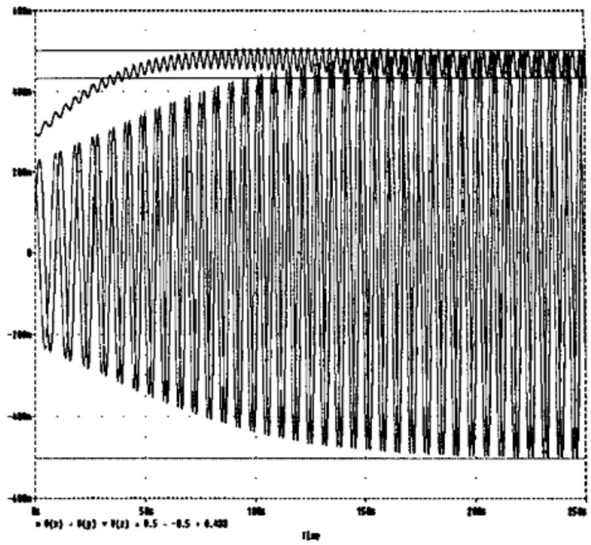

Figure 2: Illustration of convergence to limit cycle for (6)

The structural stability of the system results from the fact that there are Lyapunov type functions, a suitable one being

$$L(x, y, z) = ([k^2 + 1]x)^2 + y^2 + z^2 \tag{7a}$$

which is positive definite (and zero only if all variables are zero) with the derivative

$$\frac{dL}{dt} = -\varepsilon_X [k^2 + 1] x^2 ((x^2 + y^2) - I_0) - \varepsilon_Z z^2 ((k^2 x^2 + z^2) - J_0) \tag{7b}$$

From (7b) we see that the derivative is negative for small x, y, z and positive for large x, y, z. So, the trajectory converges to a limit cycle (note that the situation where a positive and a negative term cancel in (7b) only occurs at isolated times so has no real effect).

IV. CMOS Realization

As seen from equations (6) the nonlinearities can all be realized by four-quadrant multipliers. Consequently, for realization in CMOS VLSI we turn to the multiplier discussed by Kimura. This multiplier gives an output current in terms of the two input voltages with a multiplier g_{mult}.

$$I_{out} = g_{mult} V_x V_y \tag{8}$$

In order to set up the final circuit we need the value of this multiplier. Figure 3 shows the circuit as set up for the MOSIS 1.6u process and Fig. 4 shows the results.

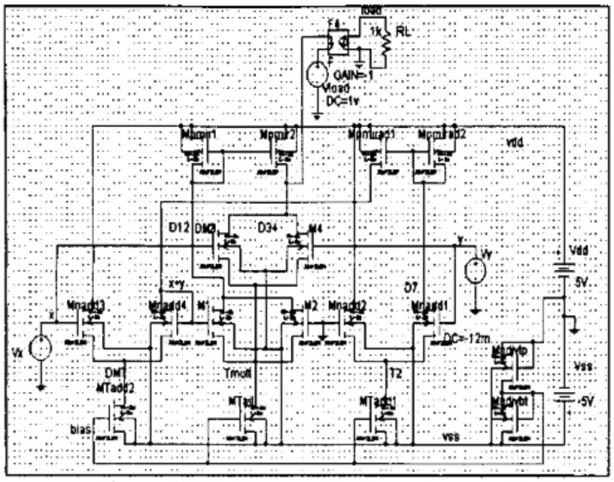

Figure 3: PSpice setup for 4-quadrant multiplier

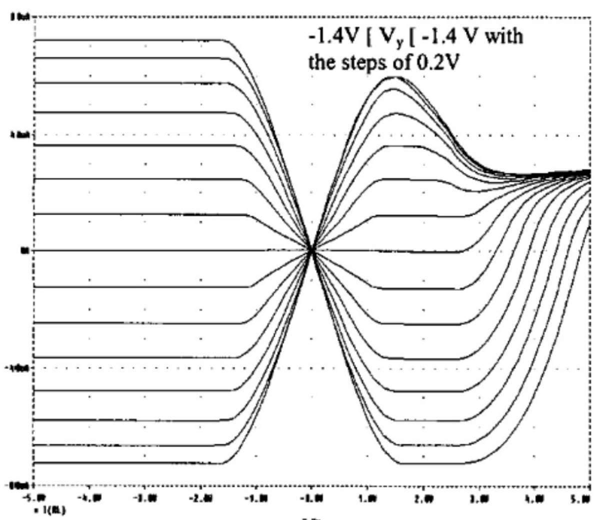

Figure 4: Results of multiplier giving g_{mult} = 52.5u (amps/volt^2)

From Fig. 4 we determine that $g_{mult} = 52.5x10$^-6 over the range $-0.5 <= Vx, Vy <= 0.5$ in which case we will want A = 0.5 for the scale factor so that our signals are valid over the range $-0.5 <= Vx, Vy <= 0.5$. From Fig. 4 we also see that correct operation outside of this range is severely limited. We also need to realize a conversion of the current into a voltage for another multiplication in the convergence factor terms, something that can be accomplished by connecting a multiplier as a linear resistor of resistance $1/g_{mult}$. Inserting capacitors of value C to realize the derivatives we obtain the final equations to be implemented.

$$\frac{dV_x}{dt} = \frac{A}{C} g_{mult} V_y V_z - \frac{A}{C} \varepsilon_x g_{mult} V_x \frac{1}{g_{mult}} (g_{mult} A(V_x^2 + V_y^2) - I_0)$$

$$\frac{dV_y}{dt} = -\frac{A}{C} g_{mult} V_x V_z \qquad (9a, b, c)$$

$$\frac{dV_z}{dt} = -\frac{A}{C} g_{mult} k^2 V_x V_y - \frac{A}{C} \varepsilon_z g_{mult} V_z \frac{1}{g_{mult}} (g_{mult} (k^2 V_x^2 + V_z^2) - J_0)$$

The circuit to realize in VLSI results directly from a straightforward implementation of these equations by feeding three capacitors by the multipliers forming all the product terms in equations (9) as well as current sources for I_0 and J_0.

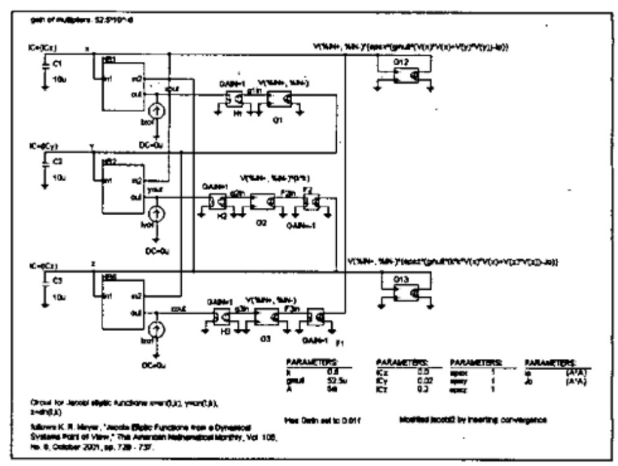

Figure 5: PSpice setup for the discussed oscillator

Fig. 5 shows a partial realization of the discussed oscillator using several ideal elements. The elliptic functions are generated as the voltages across the three capacitors C1, C2 and C3. Fig. 6 shows the simulation result of the circuit. The initial conditions are the same as those for Fig.2. From Fig. 6 we can see that with larger value of ε_X and ε_Z, the time before convergence is much shorter comparing to that shown in Fig. 2.

Due to the involvement of g_{mult}, the elliptic curves toward to different trajectories from those in Fig.2, where peak and valleys for Vx and Vy are +/- 0.688 and 0.688, 0.596 for Vz.

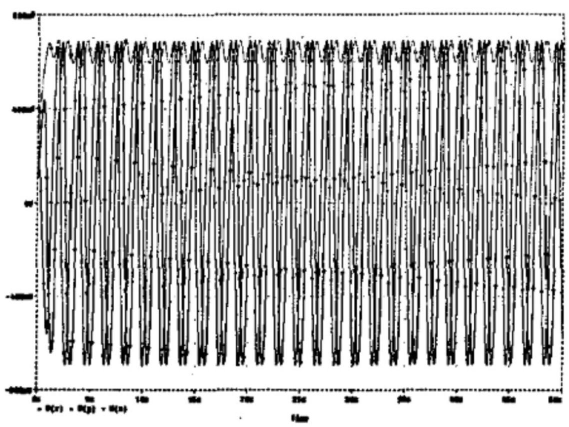

Figure 6. SPice simulation result of the circuit shown in Fig.5

V. Conclusions

VLSI circuits which yield the Jacobi elliptic functions as their limit cycles. These circuits incorporate the parameter k which for a fixed nonnegative k. Since k is bounded by unity it is readily realized by the gain of a current mirror though by the use of voltage controlled mirror gains we hope to incorporate means of easily adjusting k. Circuits for equations (9) are readily constructed in VLSI form and simulations show that they will perform well if fabricated under MOSIS 1.6u technology.

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