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APPARENTLY LOSSLESS TIME-VARYING NETWORKS

by

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ABSTRACT: Time-varying multiport transformers with secondary ports terminated in passive time-invariant capacitors are shown to be not necessarily lossless. A theorem is proved giving necessary and sufficient conditions on the transformer turns-ratio for there to be finite energy excitations at the transformer primary producing arbitrary charges on the capacitors.

1. Introduction

Statement of Problem

Recently a theory of synthesis has been proposed for linear time-variable lossless networks which is based upon loading time-variable transformers in fixed unit capacitors (1). Although the synthesis methods available guarantee that this structure will be lossless, an arbitrary connection of time-variable transformers loaded in unit capacitors need not be lossless, as will be seen here. Even though the transformer and the capacitors are independently lossless their interconnection need not be since charge can become trapped on the capacitors by suitable turns-ratio variations, as will become clear. Consequently we examine in this paper the conditions for such an occurrence. In particular, a theorem is proven which gives necessary and sufficient conditions for attaining arbitrary charges at time $t = \infty$ on capacitors loading a transformer secondary, when voltage and current excitations at the primary of the transformer are described by square-integrable functions of time; the losslessness of the network is considered in the light of this theorem.

It should be pointed out that, paradoxical though it might be that an interconnection of lossless networks be not lossless, this situation is certainly known for nonlinear networks. The series connection of an ideal diode and a time-invariant capacitor, both lossless elements, is not lossless in the sense that energy put into the network will always be returned to the source impedance when the excitation is reduced to zero.

The Multiport Transformer and Its Losslessness

Belevitch (2) has given a canonical form for multiport transformers. The circuit of the transformer is shown in Fig. 1(a) and the symbol in Fig. 1(b).

The m primary and n secondary voltages are written as vectors $\underline{v}_1$, $\underline{v}_2$ where $\underline{v}_1$ is an m-vector of functions of time, $\underline{v}_2$ an n-vector of functions of time. $\underline{i}_1$ and $\underline{i}_2$ are defined in an analogous fashion. The equations describing the operation of the transformer are

$$\underline{v}_1(t) = \underline{\tilde{N}}(t)\, \underline{v}_2(t) \tag{1a}$$

$$\underline{i}_2(t) = -\underline{N}(t)\, \underline{i}_1(t) \tag{1b}$$

$\underline{N}(t)$ is an $n \times m$ matrix with i,j element equal to $n_{ij}(t)$. The superscript tilde denotes transposition.

It follows that the instantaneous power input to the transformer is $\underset{\sim}{\tilde{v}}_1 \underset{\sim}{i}_1 + \underset{\sim}{\tilde{v}}_2 \underset{\sim}{i}_2 = \underset{\sim}{\tilde{v}}_2 \underset{\sim}{N} \underset{\sim}{i}_1 + \underset{\sim}{\tilde{v}}_2 (-\underset{\sim}{N}) \underset{\sim}{i}_1 = 0$. Thus the transformer is lossless, according to the precise definition of Eq. (9) below.

Example of a Non-Lossless Network

Consider the circuit shown in Fig. 2, together with the network $\mathfrak{N}$ enclosed in the dotted line.

For convenience we shall refer to the whole circuit above, excepting the source, as the augmented network, reserving the term network, with no qualification, for that part of the augmented network enclosed by the dotted line, that is, the transformer and capacitor only.

Suppose the augmented network is excited by a voltage $e_1(t)$ with

$$e_1(t) = 0 \qquad \text{for} \quad t < 0;$$

$$= (1 + \frac{1}{2a})e^{-at} - \frac{1}{2a} e^{-3at} \qquad \text{for} \quad t \geq 0. \qquad (2a)$$

and the transformer has turns-ratio

$$n(t) = 0 \qquad \text{for} \quad t < 0;$$

$$= e^{-at} \qquad \text{for} \quad t \geq 0. \qquad (2b)$$

Some straightforward calculations will verify that with this excitation and turns-ratio, the primary and secondary currents $i_1(t)$ and $i_2(t)$ are given by

$$i_1(t) = 0 \qquad \text{for} \quad t < 0;$$

$$= e^{-at} \qquad \text{for} \quad t \geq 0. \tag{3a}$$

and

$$i_2(t) = 0 \qquad t < 0;$$

$$= e^{-2at} \qquad \text{for} \quad t \geq 0. \tag{3b}$$

Thus the charge $q(t)$ on the capacitor as $t \to \infty$ approaches a value of $\frac{1}{2a}$. For the network therefore, at $t = \infty$ there is energy storage, and over all time there is a finite non-zero energy input. At the same time, the excitation $e_1(t)$ of the augmented network is a square-integrable function of time.

In terms of the precise definition of the term "lossless" given later in Eq. (9), the network is not lossless. Intuitively, we can also see this is indeed a reasonable result; that part of the energy input to the network which ends up as stored energy is just as much an energy loss as far as the driving source is concerned as if it had been dissipated in a resistor.

2. Preliminaries and Statement of Main Result

Definitions

We shall use the following notation:

(a) $\mathcal{L}_2^+$: The space of real-valued functions of time $f(t)$ for which $\int_T^\infty f^2(t)dt < \infty$ for some finite T. $\mathcal{L}_2$ has its usual meaning, that is $f(t) \in \mathcal{L}_2$ if $\int_{-\infty}^\infty f^2(t)dt < \infty$. $\underline{\mathcal{L}}_2$ and $\underline{\mathcal{L}}_2^+$ are the corresponding vector spaces; a real-valued n-vector of functions $\underline{f}(t)$ is in the space $\underline{\mathcal{L}}_2^+$ if $\int_T^\infty \underline{\tilde{f}}(t)\, \underline{f}(t)dt < \infty$ for some finite T; T is replaced by $-\infty$ for the $\underline{\mathcal{L}}_2$ definition.

(b) $\|\underline{g}\|$: $\underline{g}$ is a vector, and $\|\underline{g}\|^2 = g_1^2 + g_2^2 + \ldots + g_k^2$ where $g_1, g_2, \ldots, g_k$ are the components of $\underline{g}$, assumed k-dimensional. Obviously if $\underline{g}$ is a time-variable vector, then $\|\underline{g}\|$ is a time-variable function.

(c) $\|\underline{A}\|$: $\underline{A}$ is a matrix, and $\|\underline{A}\| = \sup \|\underline{A}\, \underline{x}\|$ for all $\underline{x}$ of $\|\underline{x}\| = 1$.

The following facts should be clear:

$$\|\underline{g}\| \in \mathcal{L}_2\ (\mathcal{L}_2^+) \iff \text{each component of } \underline{g} \in \mathcal{L}_2\ (\mathcal{L}_2^+) \qquad (4)$$

$$\|\underline{A}\| \in \mathcal{L}_2\ (\mathcal{L}_2^+) \iff \text{each element of } \underline{A} \in \mathcal{L}_2\ (\mathcal{L}_2^+) \qquad (5)$$

In order to obtain a precise meaning for losslessness we need some properties of networks. We adopt the point of view given in (3), where linear, finite, passive, solvable time-varying networks are defined. For our purposes, the following brief remarks should serve as sufficient background.

(a) A network $\mathfrak{N}$ with n ports permits port voltages and currents $\underline{v} = [v_j(t)]$, $\underline{i} = [i_j(t)]$ at its ports, where $\underline{v}$ and $\underline{i}$ are n-vectors with elements in $\mathfrak{D}_+$, the space of infinitely differentiable real-valued functions whose values are zero until some finite time. For brevity, we shall write $\underline{v}, \underline{i} \in \underline{\mathfrak{D}}_+$. The couple $[\underline{v},\underline{i}]$ is termed an allowed pair if the network constraints permit this voltage couple.

(b) For an allowed pair $[\underline{v},\underline{i}]$, the energy input to the network up till time t is given by

$$\mathcal{E}(t) = \int_{-\infty}^{t} \tilde{\underline{v}}(\lambda)\, \underline{i}(\lambda) d\lambda \qquad (6)$$

The lower limit of integration can be changed to t_o, where $\underline{v} = \underline{0}$, $\underline{i} = \underline{0}$ for $t < t_o$. Such a t_o exists by the remarks in (a). The network $\mathfrak{N}$ is passive if for all allowed $[\underline{v},\underline{i}]$ and finite t,

$$\mathcal{E}(t) \geq 0. \qquad (7)$$

(c) The network $\mathfrak{N}$ is solvable if for every $\underline{e} \in \underline{\mathfrak{D}}_+$, there exists a unique allowed $[\underline{v},\underline{i}]$ such that

$$\underline{e} = \underline{v} + \underline{i}. \qquad (8)$$

Physically this means that if unit resistors are connected in series with each port of $\mathcal{N}$ and a voltage excitation $\underline{e}$ is applied to this "augmented" network, as shown in Fig. 3, then a uniquely determined current $\underline{i}$ flows, and the port voltage of $\mathcal{N}$ is $\underline{v}$.

(d) The network $\mathcal{N}$ is called lossless if $\mathcal{N}$ is passive and solvable and for every $\underline{e} = \underline{v} + \underline{i}$ with $e_j \in \mathfrak{D}_+ \cap \mathcal{L}_2$,

$$\mathcal{E}(\infty) = 0 \tag{9}$$

The voltage-current pair $[\underline{v}, \underline{i}]$ is computed from (8) for the given $\underline{e}$, and then $\mathcal{E}(\infty)$ found as the limit of $\mathcal{E}(t)$ given in (6). (Note that if $\mathcal{E}(\infty) \neq 0$ with some $e_j \notin \mathcal{L}_2$, we cannot conclude that $\mathcal{N}$ is nonlossless.)

In terms of the augmented network description, the energy stored by $\mathcal{N}$ is returned to and dissipated in the unit resistors of the augmented network connected to each port of $\mathcal{N}$.

We draw attention to a result first established by Youla et al. (4), that for a passive network, $\underline{e} \in \underline{\mathfrak{D}}_+ \cap \underline{\mathcal{L}}_2 \Leftrightarrow \underline{v}, \underline{i} \in \underline{\mathfrak{D}}_+ \cap \underline{\mathcal{L}}_2$. Equation (8) proves the arrow pointing to the left. To prove the arrow pointing to the right, we have

$$\infty > \int_{-\infty}^{+\infty} \underset{\sim}{\tilde{e}}\, \underset{\sim}{e}\, dt = \int_{-\infty}^{+\infty} \underset{\sim}{\tilde{v}}\, \underset{\sim}{v}\, dt + \int_{-\infty}^{+\infty} \underset{\sim}{\tilde{i}}\, \underset{\sim}{i}\, dt + 2 \int_{-\infty}^{+\infty} \underset{\sim}{\tilde{v}}\, \underset{\sim}{i}\, dt \quad \text{using (8)}$$

$$> \int_{-\infty}^{+\infty} \underset{\sim}{\tilde{v}}\, \underset{\sim}{v}\, dt + \int_{-\infty}^{+\infty} \underset{\sim}{\tilde{i}}\, \underset{\sim}{i}\, dt \quad \text{using (6).}$$

This result obviously holds when $\underline{\mathfrak{D}}_+ \cap \underline{\mathfrak{L}}_2$ is replaced by $\underline{\mathfrak{L}}_2^+$, assuming $\underline{v}$ and $\underline{i}$ are well defined.

Statement of Conditions Under Which Main Theorem Holds

We consider a transformer with turns-ratio matrix $\underline{N}(t)$, m primary ports and n secondary ports, the secondary ports being loaded with unit capacitors as shown in Fig. 4.

The following conditions will be placed on $\underline{N}(t)$; a discussion of them will be found following the statement of the Main Theorem.

(a) $\underline{N}(t)$ is a matrix of bounded infinitely differentiable elements.

(b) No row of $\underline{N}(t)$ is identically zero for all time; i.e., for each row there is a time T for which the row is not identically zero.

(c) Denote the rows of $\underline{N}(t)$ by the row vectors $\underline{\tilde{r}}_1(t)$, $\underline{\tilde{r}}_2, \ldots, \underline{\tilde{r}}_n(t)$. There is no relation valid for all t and some set of constants α_i, not all zero, of the form

$$\alpha_1\underline{\tilde{r}}_1(t) + \alpha_2\underline{\tilde{r}}_2(t) + \ldots + \alpha_n\underline{\tilde{r}}_n(t) = \underline{0}$$

The network formed by the transformer and capacitors is seen to be linear, solvable and passive.

Statement of Main Theorem

The following statements are equivalent:

I. $\|\underline{N}(t)\| \in \mathcal{L}_2^+$

II. Arbitrary charges can be induced on all the capacitors at time $t = \infty$ with a suitably chosen excitation $\underline{e}$ of the "augmented" network, $\underline{e} \in \underline{\mathcal{L}}_2 \cap \underline{\mathcal{D}}_+$.

We point out that II implies the network is not lossless. It should be noted that the excitation $\underline{e}$ of the "augmented" network determines by solvability an excitation $\underline{v}_1$ of the (unaugmented) network, with $\underline{v}_1 \in \underline{\mathcal{L}}_2 \cap \underline{\mathcal{D}}_+$, and with the associated $\underline{i}_1 \in \underline{\mathcal{L}}_2 \cap \underline{\mathcal{D}}_+$.

Discussion of Conditions and Main Theorem Statement

The main theorem essentially states that under certain conditions, an m-dimensional space of functions $\underline{e}(t)$ can be mapped onto an n-dimensional euclidian space (the charges at $t = \infty$ forming the vectors in this space). For the case $m \geq n$, there are more input ports than capacitors, (or at least an equal number of each), and the result seems intuitively reasonable. For the case $m < n$ however, the input ports are fewer in number than the capacitors. Nevertheless there is no paradox in this: the space of inputs, being a function space, can be regarded as being infinite dimensional.

In order that the various port variables be infinitely differentiable, we see that the differentiability condition on $\underline{N}(t)$ is clearly a reasonable one. The condition on the rows not being identically zero is also sensible; if one row were identically zero there would be no point in terminating the secondary port associated with that row, as there would never be any current flowing at this secondary port. The third condition, on the linear independence of the rows, is one that might readily be violated in practice; it is always violated for example in the case of a time-invariant transformer if $n > m$. In discussions following the proof of the main theorem, we shall point out how this condition may be removed, at the same time modifying the statement of the theorem to cope with this removal. For clarity, however, such discussion will be postponed.

3. Proof of the Main Result

Statement I Implies Statement II

The conditions imposed on $\underline{N}(t)$ mean that there exists a finite interval in which

(a) none of the rows of $\underline{N}$ is identically zero; and

(b) there is no linear relation valid over the whole interval among the rows $\underline{\tilde{r}}_i(t)$ of $\underline{N}(t)$.

Let α, β be the endpoints of such an interval, with $\alpha < \beta$.

By the theorem proved in the appendix, there exist n infinitely differentiable m-vector functions $\underline{\phi}_j(t)$ such that

$$\int_\alpha^\beta \tilde{\underline{r}}_1(t)\, \underline{\phi}_j(t)dt = \delta_{ij} \qquad (10)$$

where δ_{ij} is the Kronecker δ.

For each j we may define an input current $\underline{i}_1^j(t)$ at the transformer primary by

$$\begin{aligned} \underline{i}_1^j(t) &= \underline{\phi}_j(t) \qquad && \alpha \leq t \leq \beta \\ &= \underline{0} && \text{elsewhere} \end{aligned}$$

Let $\underline{q}(t) = [q_i(t)]$ denote the charge on the capacitors at time t. Let $\underline{Q}$ denote the charge at time $t = \infty$. The polarity of $\underline{q}(t)$ is taken as the same as the polarity of $\underline{v}_2$. Let $\underline{q}^j(t)$, $\underline{Q}^j$ refer to the case where the input current is $\underline{i}_1^j(t)$.

We then have

$$\begin{aligned} \underline{Q}^j &= \int_{-\infty}^{+\infty} - \underline{i}_2^j(t)dt = \int_{-\infty}^{+\infty} \underline{N}(t)\, \underline{i}_1^j(t)dt = \int_\alpha^\beta \underline{N}(t)\, \underline{\phi}_j(t)dt \\ &= [\delta_{ij}] \qquad \text{using (10)} \qquad (11) \end{aligned}$$

$[\delta_{ij}]$ is of course the vector with 1 for the j-th entry, zero other entries.

It is clear that by a superposition of such $\underline{i}_1^j(t)$ we can obtain any desired $\underline{Q}$. To establish this part of the theorem it only remains to see that the $\underline{v}_1$ and $\underline{i}_1$ used to establish $\underline{Q}$ are $\underline{\mathcal{L}}_2^+$ functions. That $\underline{i}_1$ has this property follows from the fact that $\underline{i}_1$ is non-zero on a finite interval only.

For $t \geq \beta$, we have

$$\underline{v}_1(t) = \underline{\tilde{N}}(t)\ \underline{v}_2(t) = \underline{\tilde{N}}(t)\ \underline{q}(t) = \underline{\tilde{N}}(t)\ \underline{Q} \tag{12}$$

and

$$\underline{v}_1(t) \in \underline{\mathcal{L}}_2^+ \qquad \text{since} \qquad \|\underline{\tilde{N}}(t)\| \in \mathcal{L}_2^+.$$

<u>Statement II Implies Statement I</u>

Our goal here is to establish that $\|\underline{N}(t)\| \in \mathcal{L}_2^+$ when there exists an excitation $\underline{e}(t)$ of the augmented network which will produce any desired set of charges $\underline{Q}$ on the capacitors at time $t = \infty$, [i.e., the instantaneous charge $\underline{q}(t) \to \underline{Q}$, where $\underline{Q}$ is arbitrary, as $t \to \infty$].

Observe that we cannot claim that $\underline{Q}$ can be attained by using a current $\underline{i}_1(t)$, derived from $\underline{e}(t)$, which is zero except on a finite interval, for this assumption is not justified by the statement of the theorem. If it were possible to assume this, we have an immediate proof, for we should have $\underline{q}(t) = \underline{Q}$ for $t \geq \beta$ where β is some constant, and then the primary voltage is given for $t \geq \beta$ by

$$\underline{v}_1(t) = \underline{\tilde{N}}(t)\underline{Q} \tag{12}$$

By the hypothesis of the theorem, $\underline{e}(t) \in \mathfrak{L}_2^+$, and thus, by an earlier remark, $\underline{v}(t) \in \mathfrak{L}_2^+$. Equation (12) then implies $\|\underline{N}(t)\| \in \mathfrak{L}_2^+$.

We shall therefore suppose that in achieving a desired $\underline{Q}$, $\underline{q}(t)$ approaches $\underline{Q}$, but does not necessarily attain $\underline{Q}$ in a finite time.

In order to establish the result, we shall use a subsidiary lemma.

Lemma. Under the conditions of the theorem, given a $\underline{Q}$ with $\|\underline{Q}\| = 1$, $\epsilon > 0$, there exists a $T_1 = T_1(\epsilon)$ and excitation $\underline{e}(t)$ such that $\|\underline{q}(t) - \underline{Q}\| < \epsilon$ for $t > T_1$.

Proof of Lemma. By hypothesis, there exist augmented network excitations $\underline{e}^1(t), \underline{e}^2(t), \dots, \underline{e}^n(t)$ producing charges $\underline{Q}^1, \underline{Q}^2, \dots, \underline{Q}^n$ where $\underline{Q}^i$ is a vector with 1 in the i^{th} row, zero elsewhere; further, given $\epsilon > 0$ there exists $T^i = T^i(\epsilon)$ with $\|\underline{q}^i(t) - \underline{Q}^i\| < \frac{\epsilon}{n}$ for $t \geq T^i$ $(i = 1,2,\dots,n)$. Then we may take $T_1 = \max\limits_{1 \leq i \leq n} T^i$. To produce a $\underline{Q} = \sum\limits_i \alpha_i \underline{Q}^i$, the excitation $\underline{e}(t) = \sum\limits_i \alpha_i \underline{e}^i(t)$ suffices; then we have for $t \geq T_1$,

$$\|\underline{q}(t) - \underline{Q}\| = \left\| \sum_{i=1}^{n} \alpha_i (\underline{q}^i(t) - \underline{Q}^i) \right\|$$

$$\leq \sum_{i=1}^{n} \|\alpha_i\| \, \|\underline{q}^i(t) - \underline{Q}^i\| < \epsilon$$

as $\|\alpha_i\| \leq 1$ for each i if $\|\underline{Q}\| = 1$.

Consider now the function of t and $\underline{Q}$ given by $\|\underline{\tilde{N}}(t)\underline{Q}\|$, where we restrict $\underline{Q}$ by requiring $\|\underline{Q}\| = 1$. By the definition of norm, we know that there exists for each t a particular $\underline{Q}$, call it $\underline{Q}^t$, such that

$$\|\underline{\tilde{N}}(t)\ \underline{Q}^t\| \geq \|\underline{\tilde{N}}(t)\| - \frac{1}{(1 + t^2)^{1/2}} \tag{13}$$

We stress at this point that $\underline{Q}^t$ for any t still denotes a set of charges on the capacitors at time infinity. $\underline{Q}^t$ is a family of such sets, parametrised by t.

By the hypothesis of the theorem, we know that there are augmented network excitations $\underline{e}^t$ (parametrised by t), producing the desired $\underline{Q}^t$. If we have

$$\underline{Q}^t = \begin{bmatrix} \alpha_1^t \\ \alpha_2^t \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n^t \end{bmatrix} \qquad \text{with} \quad \sum_{i=1}^{n} (\alpha_i^t)^2 = 1 \tag{14}$$

an appropriate excitation is

$$\underline{e}^t = \sum_{i=1}^{n} \alpha_i^t\ \underline{e}^i \tag{15}$$

where $\underline{e}^i$ has been defined as the excitation required to produce $\underline{Q}^i$, that is, $\underline{Q}$ with 1 in the i-th place, zeros elsewhere.

Associated with each augmented network excitation $\underline{e}^i$ is a voltage $\underline{v}^i$ at the primary ports of the unaugmented network, and we have in obvious notation,

$$\underline{v}^t = \sum_{i=1}^{n} \alpha_i^t \, \underline{v}^i \tag{16}$$

As pointed out earlier, each $\underline{v}^i$ is an $\underline{\mathcal{L}}_2^+$ function.

Taking τ as our running variable, for the case of excitation to produce $\underline{Q}^t$, we shall have

$$\underline{v}^t(\tau) = \underline{\tilde{N}}(\tau) \, \underline{q}^t(\tau) \tag{17}$$

where, stressing the fact that t is a parameter, $\underline{q}^t(\tau)$ is the charge at time τ when an excitation is used that will produce a charge at infinite time of $\underline{Q}^t$ needed to satisfy (13).

Now take any ϵ in $0 < \epsilon < 1$ and restrict consideration to values of the parameter t and the running variable $\tau > T_1$, T_1 being as defined in the lemma.

Then we have

$$\|\underline{q}^t(\tau) - \underline{Q}^t\| < \epsilon$$

We can set τ equal to any value greater than T_1, and as $t > T_1$, we claim, setting $\tau = t$,

$$\|\underline{q}^t(t) - \underline{Q}^t\| < \epsilon \tag{18}$$

Considering (17) with $\tau = t$, we have

$$\|\underline{v}^t(t)\| = \|\tilde{\underline{N}}(t)\ \underline{q}^t(t)\|$$

$$= \|\tilde{\underline{N}}(t)\left\{\underline{Q}^t + (\underline{q}^t(t) - \underline{Q}^t)\right\}\|$$

$$\geq \|\tilde{\underline{N}}(t)\ \underline{Q}^t\| - \|\tilde{\underline{N}}(t)\left\{\underline{q}^t(t) - \underline{Q}^t\right\}\|$$

$$\geq \|\tilde{\underline{N}}(t)\| - \frac{1}{(1 + t^2)^{1/2}} - \|\tilde{\underline{N}}(t)\|\ \epsilon \quad \text{using (13), (18)}$$

Further

$$\|\underline{v}^t(t)\| = \|\sum_{i=1}^{n} \alpha_i^t\ \underline{v}^i(t)\|$$

$$\leq \sum_{i=1}^{n} \|\alpha_i^t\|\ \ \|\underline{v}^i(t)\|$$

$$\leq \sum_{i=1}^{n} \|\underline{v}^i(t)\| \qquad \text{by (14).}$$

Hence

$$\|\tilde{\underline{N}}(t)\|(1-\epsilon) \leq \sum_{i=1}^{n} \|\underline{v}^i(t)\| + \frac{1}{(1 + t^2)^{1/2}} \tag{19}$$

Although we have hitherto insisted that t has the status of a parameter, it may equally well be considered as a variable, with Eq. (19) holding for all $t > T_1$.

Each $\underline{v}^i$ is an $\mathfrak{L}_2^+$ function, and $1/(1+t^2)^{1/2}$ is an $\mathfrak{L}_2^+$ function. Equation (19) then implies that

$$\int_{T_1}^{\infty} \|\underline{\tilde{N}}(t)\|^2 dt < \infty$$

that is, $\|\underline{\tilde{N}}(t)\|$ is an $\mathfrak{L}_2^+$ function.

4. Comments and Conclusions

1. It may be thought strange that the condition imposed on $\underline{N}(t)$ in the preceding theorem is that $\|\underline{N}(t)\| \in \mathfrak{L}_2^+$ rather than $\|\underline{N}(t)\| \in \mathfrak{L}_2$.

However, the reason for this is fairly simple. We have observed that to achieve arbitrary charges on the capacitors we apply a certain excitation commencing at some time β $(\beta \neq -\infty)$. The subsequent behavior of the network is dependent on the values of $\underline{N}(t)$ for $t \geq \beta$, but not for $t < \beta$. Thus no restriction need be placed on the behavior of $\underline{N}(t)$ for $t \to -\infty$. Another way of looking at this fact is to notice that all excitations are in $\underline{\mathfrak{D}}_+$, and thus zero until some finite time. Before this time, the network variations are immaterial for determining network responses.

2. It is instructive to consider an example where we do not have $\|\underline{N}(t)\| \in \mathfrak{L}_2^+$. Here it is not possible for arbitrary charges to be attained but it is still possible for the network to not be lossless. Figure 5(a) shows the circuit we shall consider and Fig. 5(b) shows the variation of the transformer turns ratio with time, and the excitation current.

It is easy to see that in the first two seconds, q_1 will rise to a stationary value of 1, and in the period between the second and third seconds, q_2 will change to a steady value of -1. Both charges will then prevail for all time, and thus the network is not lossless. $\|\underline{N}(t)\|$ being constant after $t = 3$ seconds is not an $\mathfrak{L}_2^+$ function. Finally, it is easy to verify that v is non-zero only on a finite interval, and thus $e = v + i$ is square-integrable.

We cannot however achieve arbitrary charges on the capacitors with a square integrable e. The best we can do is to have $\underline{Q}$ in the null space of $\underline{N}(\infty)$. By changing the exciting current i so that $i = a$ for $0 \leq t \leq 2$, $i = b$ for $2 \leq t \leq 3$, and $i = 0$ elsewhere, appropriate selection of a and b will yield any desired $\underline{Q}$. In general, however, we will find that v will have a constant non-zero value for $t \geq 3$ and thus not be an $\mathfrak{L}_2$ function, as required for the main theorem.

To be strictly correct, we should require $n_1(t)$, $n_2(t)$ and $i(t)$ to be infinitely differentiable. By suitably "rounding" these functions, it is clear that they could be taken thus, and the general conclusions would still hold.

3. It will be recalled that earlier a condition was placed on $\underline{N}(t)$ to the effect that its rows could not be linearly dependent over all time. We are now in a position to remove that restriction. Taking as earlier the rows as $\tilde{\underline{r}}_i(t)$ $(i = 1,2,\ldots,n)$, suppose there exist constants α_i, not all zero, such that for all t

$$\alpha_1\tilde{\underline{r}}_1(t) + \alpha_2\tilde{\underline{r}}_2(t) + \ldots + \alpha_n\tilde{\underline{r}}_n(t) = \underline{0} \tag{20}$$

Then since

$$\underline{q}(t) = \int_{-\infty}^{t} \underline{N}(\tau)\,\underline{i}_1(\tau)d\tau = \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix} \tag{21}$$

where $\underline{i}_1(t)$ is the instantaneous primary current, it follows that the entries $q_i(t)$ of $\underline{q}(t)$ must satisfy

$$\alpha_1 q_1(t) + \alpha_2 q_2(t) + \ldots + \alpha_n q_n(t) = 0 \tag{22}$$

irrespective of the excitation.

Consequently,

$$\alpha_1 Q_1 + \alpha_2 Q_2 + \ldots + \alpha_n Q_n = 0 \tag{23}$$

where $\underline{Q} = [Q_i]$ is the charge at $t = \infty$ on the capacitors.

It is then true that if $\|\underline{N}(t)\| \in \mathcal{L}_2^+$ and an equation of the form (20) holds, then any $\underline{Q}$ can be achieved using an appropriate excitation $\underline{e} \in \underline{\mathcal{D}}_+ \cap \underline{\mathcal{L}}_2$ of the augmented network, subject to the components of $\underline{Q}$ satisfying (23).

It is also easy to see that if there exists an $\underline{e} \in \underline{\mathcal{D}}_+ \cap \underline{\mathcal{L}}_2$ producing $\underline{Q}$, arbitrary except for satisfying a relation such as (23), then $\|\underline{N}(t)\| \in \mathcal{L}_2^+$ if also $\underline{N}(t)$ satisfies (20) for all time. If $\underline{N}(t)$ does not fulfill this additional requirement, we cannot conclude that $\|\underline{N}(t)\| \in \mathcal{L}_2^+$. Our second comment illustrates the case where $\underline{Q}$ is arbitrary except for $\underline{Q}_1 + \underline{Q}_2 = \underline{0}$; however, $\|\underline{N}(t)\| \notin \mathcal{L}_2^+$.

4. In the light of the second comment, it might well be asked what guarantees that we have a lossless network. While a complete answer is necessarily involved, as the second comment makes clear, it is true that for a transformer with one primary and one secondary port, and turns ratio matrix $\underline{N}(t) = n(t)$, i.e., a scalar, then the capacitively terminated transformer is lossless if and only if $\int_T^\infty n^2(t)\, dt$ is divergent for some T. The "only if" part has already been demonstrated, by the Main Theorem. To

see the "if" part, suppose the network is not lossless; then there exists a voltage excitation $v_1(t)$ of the unaugmented network giving rise to an instantaneous charge $q(t)$, with $q(t) \to Q \neq 0$ as $t \to \infty$, and

$$v_1(t) = n(t)\, q(t) \qquad (24)$$

Since $q(t) \to Q$ continuously, for $t \geq$ some T, we have

$$\left|n(t)\,\frac{Q}{2}\right| < |v_1(t)| < |n(t)\,2Q| \qquad (25)$$

from which it follows that v_1 and n are together both in or both not in the space $\mathcal{L}_2^+$. This contradicts the assumption that v_1 is an $\mathcal{L}_2$ function when n is not an $\mathcal{L}_2^+$ function.

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Appendix

Let $\underline{u}_1(t), \underline{u}_2(t), \ldots, \underline{u}_n(t)$ be a set of m-vectors whose elements are infinitely differentiable functions of t on a bounded closed interval I.

The vectors are linearly independent over the interval, i.e., the relation for constants α_i and all t in I,

$$\alpha_1 \underline{u}_1(t) + \alpha_2 \underline{u}_2(t) + \ldots + \alpha_n \underline{u}_n(t) = \underline{0}$$

implies $\alpha_i = 0$ for each $i = 1,2,\ldots,n$.

Then there exist m-vectors $\underline{\phi}_i(t)$, $i = 1,2,\ldots n$, such that

$$\int_I \tilde{\underline{u}}_i(t)\, \underline{\phi}_j(t)dt = \delta_{ij} \qquad \text{each } i,j = 1,2,\ldots n.$$

where δ_{ij} is the Kronecker delta.

$\underline{\phi}_j(t)$ can be taken as infinitely differentiable, and zero at the end points of I and outside I.

Proof

It is clearly sufficient to establish the result for one j. Take $j = n$. Let us define for vector functions $\underline{a}(t)$, $\underline{b}(t)$,

$$< \underline{a}, \underline{b} > = \int_I \tilde{\underline{a}}(t)\, \underline{b}(t)\, dt$$

and

$$\|\underline{a}\|^2 = < \underline{a}, \underline{a} >$$

Then use the Schmidt orthogonalization procedure (5) to form successively the functions $\underline{\zeta}_1(t), \underline{\zeta}_2(t), \ldots, \underline{\zeta}_n(t)$ where

$$\underline{\zeta}_1(t) = \frac{\underline{u}_1(t)}{\|\underline{u}_1(t)\|}$$

$$\underline{\zeta}_2(t) = \frac{\underline{u}_2 - \langle \underline{u}_2, \underline{\zeta}_1 \rangle \underline{\zeta}_1}{\|\underline{u}_2 - \langle \underline{u}_2, \underline{\zeta}_1 \rangle \underline{\zeta}_1\|}$$

$$\underline{\zeta}_3(t) = \frac{\underline{u}_3 - \langle \underline{u}_3, \underline{\zeta}_2 \rangle \underline{\zeta}_2 - \langle \underline{u}_3, \underline{\zeta}_1 \rangle \underline{\zeta}_1}{\|\underline{u}_3 - \langle \underline{u}_3, \underline{\zeta}_2 \rangle \underline{\zeta}_2 - \langle \underline{u}_3, \underline{\zeta}_1 \rangle \underline{\zeta}_1\|}$$

$$\vdots$$

$$\underline{\zeta}_n(t) = \frac{\underline{u}_n - \langle \underline{u}_n, \underline{\zeta}_{n-1} \rangle \underline{\zeta}_{n-1} - \cdots - \langle \underline{u}_n, \underline{\zeta}_1 \rangle \underline{\zeta}_1}{\|\underline{u}_n - \langle \underline{u}_n, \underline{\zeta}_{n-1} \rangle \underline{\zeta}_{n-1} - \cdots - \langle \underline{u}_n, \underline{\zeta}_1 \rangle \underline{\zeta}_1\|}$$

Then, as may be easily shown, the $\underline{\zeta}_i$ form an orthonormel set. The linear independence of the $\underline{u}_i$ is a necessary and sufficient condition for the existence of the $\underline{\zeta}_i$. Each $\underline{u}_i$ is a linear combination of $\underline{\zeta}_j$ with $j < i$, and it follows that $\langle \underline{u}_i, \underline{\zeta}_n \rangle = 0$ for $i \neq n$, while $\langle \underline{\zeta}_n, \underline{u}_n \rangle \neq 0$.

$\underline{\phi}_n$ can then be formed from $\underline{\zeta}_n$ by dividing by a suitable constant, while the infinite differentiability of the $\underline{u}_i$ means $\underline{\phi}_n$ is infinitely differentiable, and clearly can be "adjusted" to be zero at the end points and outside of I.

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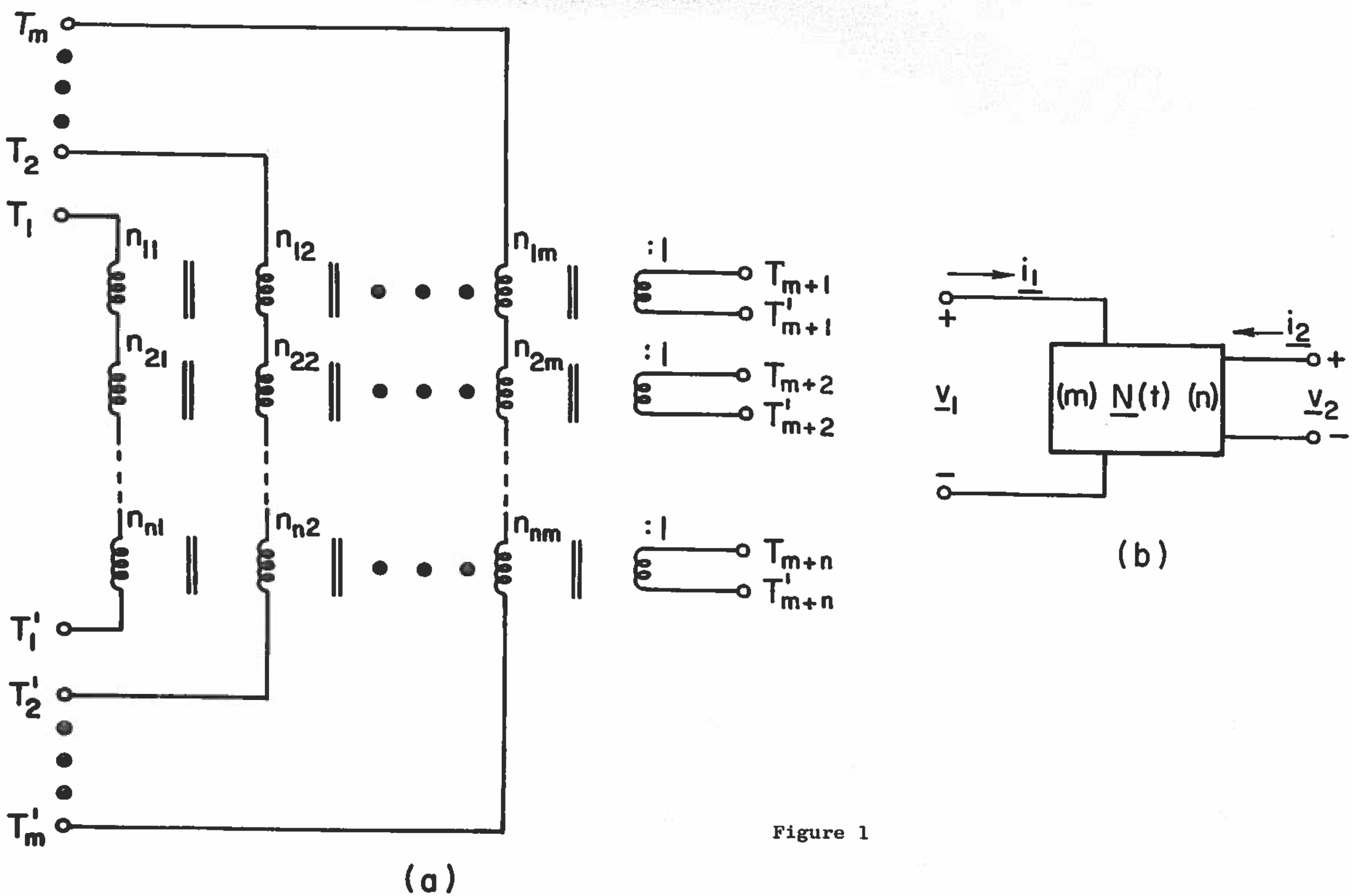
FIGURE CAPTIONS



T_m
T_2
T_1
T'_1
T'_2
T'_m
n_{11}
n_{21}
n_{n1}
n_{12}
n_{22}
n_{n2}
n_{1m}
n_{2m}
n_{nm}
:1
:1
:1
T_{m+1}
T'_{m+1}
T_{m+2}
T'_{m+2}
T_{m+n}
T'_{m+n}
(a)
$\underline{i}_1$
+
$\underline{v}_1$
−
(m) $\underline{N}(t)$ (n)
$\underline{i}_2$
+
$\underline{v}_2$
−
(b)

Figure 1

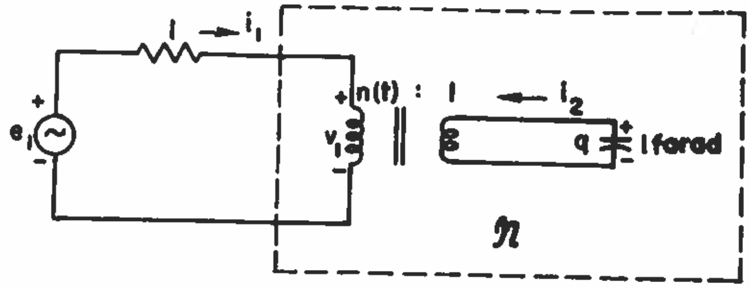


Figure 2

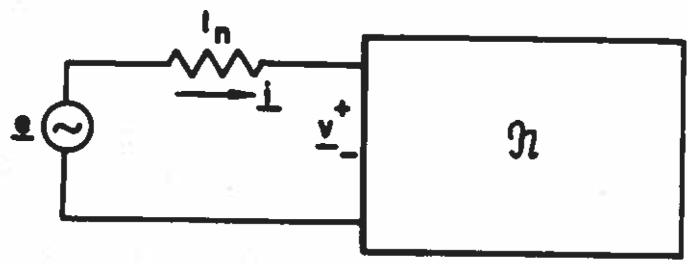


Figure 3

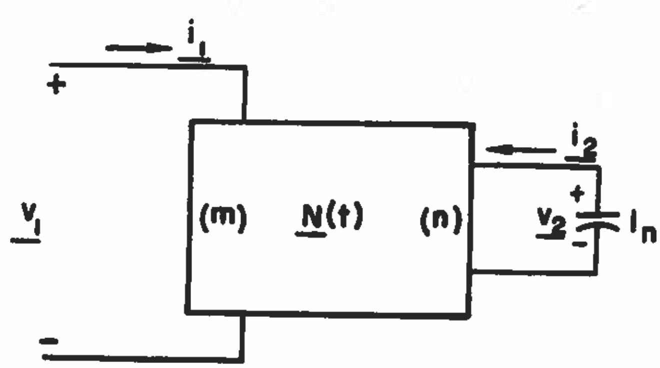


Figure 4

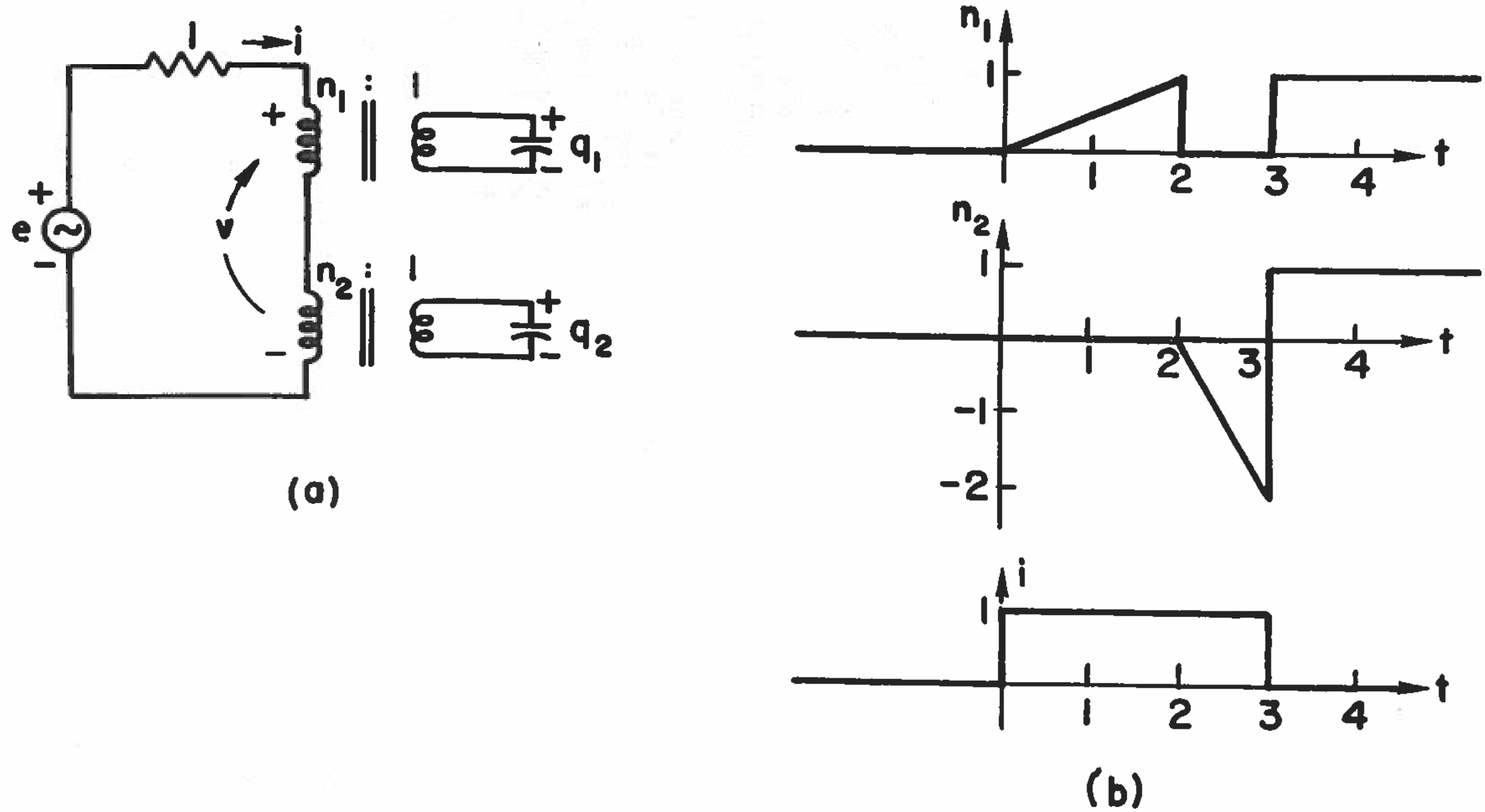


Figure 5