# Correspondence

## On Relations between Series- and Shunt-Augmented Networks

In an earlier note [1] the concept was presented of the scattering matrix $s(t, \tau)$ of a linear time-varying completely solvable network $N$, and the relation of $s$ to the admittance matrix $y_a(t, \tau)$ of the augmented network $N_a$ of Fig. 1 was examined. Here we introduce a second type of augmented network, $N_a'$, the shunt-augmented network, and consider some properties of the impedance matrix of this network. The shunt-augmented network is formed by connecting unit resistors in parallel with each port, as shown in Fig. 2. Oono and Yasuura [2] defined the shunt-augmented network in the time-invariant case; the relations derived here include and extend their results.

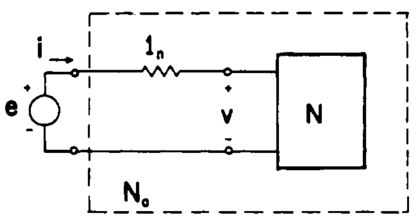


Fig. 1. The augmented network $N_a$ where $I_n$ represents $n$ series unit resistors.

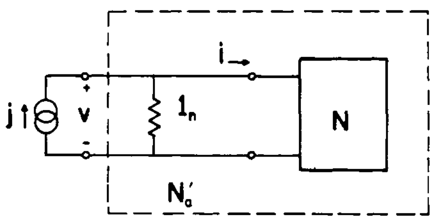


Fig. 2. The augmented network $N_a$.

We comment that although $N_a$ and $N_a'$ are different networks the voltages $v$ and currents $i$ for $N$ are identical in Figs. 1 and 2 when $j=e$ since the sources of internal impedance $I_n$ feeding $N$ are equivalent by Norton's and Thévenin's Theorems.

We shall suppose given a time-varying linear $n$-port network $N$ [1], [3]. The allowed voltages $v$ and currents $i$ for $N$ are all $D_+$ functions, i.e., infinitely differentiable $n$-vector functions zero until some finite time. The augmented admittance matrix $y_a(t, \tau)$, where it exists, is a linear continuous mapping of excitations $e \epsilon D_+$ into currents $i \epsilon D_+$ given by

$$i = y_a \cdot e \qquad (1a)$$

or in fuller notation,

$$i(t) = \int_{-\infty}^{+\infty} y_a(t, \tau) e(\tau) d\tau \qquad (1b)$$

From Fig. 1 it is clear that $v$ is determined through

$$e = v + i \qquad (2)$$

by

$$v = (\delta I_n - y_a) \cdot e \qquad (3)$$

where $\delta$ is the unit impulse and $I_n$ the $n \times n$ identity matrix.

In the event that $y_a$ exists, the network is termed completely solvable [1]. In this case there exists also a scattering matrix $s(t, \tau)$ which is a linear continuous mapping of incident voltages $v^i \epsilon D_+$ defined by

$$2v^i = v + i \qquad (4a)$$

into reflected voltages $v^r$ defined by

$$2v^r = v - i \qquad (4b)$$

through the equation

$$v^r = s \cdot v^i. \qquad (5)$$

Considering the shunt-augmented network of Fig. 2, we ask the question, under what conditions does there exist a linear continuous mapping $z_a'(t, \tau)$ of current excitations $j$ of $N_a'$ into port voltages $v$? The question is answered by the following theorem.

*Theorem:* The existence of one of $y_a$, $s$ and $z_a'$ guarantees the existence of the other two, and the following relations hold:

$$s = \delta I_n - 2y_a \qquad (6a)$$

$$s = 2z_a' - \delta I_n \qquad (6b)$$

$$\delta I_n = z_a + y_a \qquad (6c)$$

*Proof:* The existence of $y_a$ guarantees the existence of $\delta I_n - 2y_a$, which, since $y_a$ maps $e = v + i$ into $i$, maps $e = v + i$ into $e - 2i = v - i$; equivalently from (4a), (4b), $\delta I_n - 2y_a$ maps $v^i$ into $v^r$. Thus $s$ exists. Since $s$ maps $v + i$ into $v - i$, $\delta I_n + s$ maps $v + i$ into $2v$, and $\frac{1}{2}(\delta I_n + s)$ maps $v + i$ into $v$. Examination of Fig. 2 shows that $j = v + i$, and thus we conclude that $z_a'$ exists, equaling $\frac{1}{2}(\delta I_n + s)$. The proof that the existence of $z_a'$ implies the existence of $y_a$ proceeds in the same fashion. Equation (6c) following easily by subtraction. Q.E.D.

Very simple manipulations yield the following additional equations from (6a), (6b), and (6c).

$$y_a = \tfrac{1}{2}(\delta I_n - s) \qquad (6d)$$

$$z_a' = \tfrac{1}{2}(\delta I_n + s) \qquad (6e)$$

$$s = z_a' - y_a \qquad (6f)$$

As (6d), (6e) can be obtained from one another by replacing $s$ by $-s$, we conclude that $y_a$ and $z_a'$ are dual immittances.

The preceding results can be used to provide illustrations of well-known circuit theory results. For example, in the time-invariant case (6c) in the frequency domain becomes

$$I_n = Z_a'(p) + Y_a(p) \qquad (7)$$

and thus any pole of $Y_a$ is also a pole of $Z_a'$; $Y_a$ and $Z_a'$ have equal and opposite residue. Consequently, in the case of a passive network $N$, $Z_a'$, $Y_a$ and thus $S$ can have no $j\omega$-axis poles; otherwise one of $Z_a'$, $Y_a$ will violate the positive reality constraint by having a $j\omega$-axis pole of negative residue.

For finite time-varying networks it can be shown [4] that a passive immittance must be of the form

$$A_1(t)\delta'(t-\tau) + A_0(t)\delta(t-\tau) + \tilde{\Phi}(t)\psi(\tau)u(t-\tau)$$

with $A_1(t)$ positive semidefinite. Using (6c) we can conclude that for $z_a'$ and $y_a$ the $\delta'(t-\tau)$ term must be absent, and consequently the scattering matrix of a finite passive network is of the form

$$s(t, \tau) = A_0(t)\delta(t - \tau) + \tilde{\Phi}(t)\psi(\tau)u(t - \tau) \qquad (8)$$

In summary, we have introduced the shunt-augmented impedance matrix for a linear time-varying network. We have shown that the existence of this matrix guarantees and is guaranteed by the existence of the augmented network admittance matrix, or the scattering matrix. Further these three matrices are related in a simple fashion, as shown in (6).


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### References


[1] Spaulding, D. A., and R. W. Newcomb, The time-variable scattering matrix, *Proc. IEEE (Correspondence)*, vol 53, Jun 1965, p 651.
[2] Oono, Y., and K. Yasuura, Synthesis of finite passive 2n-terminal networks with prescribed scattering matrices, *Memoirs of the Faculty of Engineering*, Kyushu University, vol 14, pp 125–177, 1954.
[3] Newcomb, R. W., On the definition of a network, *Proc. IEEE (Correspondence)*, vol 53, May 1965, p 547.
[4] Spaulding, D. A., Passive time-varying networks, Ph.D. dissertation, Stanford University, Calif., Dec 1964, p 52.


## Generalized SNR and Performance Index of Filters for Waveform Estimation

### Introduction

For a random signal corrupted by additive statistically independent noise, it is common practice to use the SNR as a gain-invariant measure of the disturbance caused by noise. After filtering, however, the signal will be distorted and noisy; hence, the common definition of SNR is not directly applicable. The reciprocal of the normalized mean-square error, which has been used [1]–[3] is not satisfactory. We present here an adequate definition of a generalized

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