the temperature.

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REFERENCE TERMINATIONS FOR THE SCATTERING MATRIX*

The relationship between the scattering matrixes of a given network (or microwave junction) with different reference terminations is developed. For this the scattering matrix with respect to coupled terminations is introduced, and a 2-port example is considered.

In engineering analysis and design, the scattering matrix often occurs as a natural description to use. This is particularly true in microwave circuitry, where the scattering matrix allows a useful means of describing the operation of a junction operating between waveguides,[1] and in network theory, where the scattering matrix allows general theories of synthesis and equivalence.[2] For such considerations, it is convenient to take the port reference impedances for the scattering matrix as those of transmission lines (possibly of zero length[3]) in which the junction or network is embedded. Since a given junction is often used between different transmission lines, it is of interest to know the relationship between the scattering matrixes calculated under two different reference conditions. Because the transmission lines are often electromagnetically coupled, we here consider the relationships for coupled reference impedances.[4] We give a network treatment, since the results are directly applicable in network theory, and one of the purposes of using scattering matrixes in microwave theory is to use circuit considerations, avoiding the solution of complicated boundary-valued problems.

We assume a Laplace-transform description exists and treat the n-port case. Let E_R be an $n$vector of source voltages feeding an n-port N, connected in series with a resistive n-port of impedance matrix $Z_0 = R$, as shown in Fig. 1a. We assume that the $n \times n$ constant matrix R is symmetric and positive definite, from which we can write

$$R = \tilde{T}T \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

where T is an $n \times n$ constant nonsingular matrix.[5] Incident and reflected voltages with reference to R are then defined by[6]

$$2V_R{}^i = V + RI = E_R \quad . \; . \; . \; . \quad (2a)$$

$$2V_R{}^r = V - RI \quad . \; . \; . \; . \; . \; . \quad (2b)$$

where V and I are the n-vector voltage and current for N. Physically, $V_R{}^i$ and $V_R{}^r$ can be measured[3] as incident and reflected voltages on zero-length transmission lines of characteristic impedance matrix $Z_0 = R$, placed between R and N in Fig. 1a.

The (voltage) scattering matrix S_R with respect to (the reference) R is in turn defined by

$$V_R{}^r = S_R V_R{}^i \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

If R is chosen as the $n \times n$ identity matrix, $R = 1_n$, we write $S_R = S$ and simply call S the (normalised) scattering matrix for N. Given Fig. 1a, we can always insert a transformer 2n-port of turns-ratio matrix T, defined by[7]

$$V = \tilde{T}V^n \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4a)$$

$$I^n = TI \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4b)$$

so that N still sees R, as shown in Fig. 1b. This insertion allows us to consider a new (normalised) network N^n as N loading the transformer, and allows us to define S^n as the scattering matrix for N^n; we call S^n the scattering matrix of N normalised to R. For N^n,

$$V^{in} = V^n + I^n \quad . \; . \; . \; . \; . \; . \; . \quad (5a)$$

$$V^{rn} = V^n - I^n \quad . \; . \; . \; . \; . \; . \; . \quad (5b)$$

Inserting eqns. 4 for V^n and I^n, premultiplying by $\tilde{T}$ and using eqn. 1 gives

$$\tilde{T}V^{in} = V + RI = V_R{}^i \quad . \; . \; . \quad (6a)$$

$$\tilde{T}V^{rn} = V - RI = V_R{}^r \quad . \; . \; . \quad (6b)$$

which shows that $S^n = \tilde{T}^{-1}S_R T$, where naturally $V^{rn} = S^n V^{in}$. Fig. 1 shows why one would consider the somewhat strange change of variables of eqn. 6, which generalises the result[8] for diagonal $R = \tilde{T}T$.

At this point we have three different scattering matrixes on hand: S, S_R and S^n, all of which completely describe N. Two of these we have interrelated by $S_R\tilde{T} = \tilde{T}S^n$, while the remaining relationships can be obtained as special cases of the general result on change of reference to now be developed. For this we consider how the scattering matrixes S_R and S_P with respect to two references compare for the same n-port N. For this, let P have the same properties as R, in which case

$$2V_P{}^i = V + PI \quad . \; . \; . \; . \; . \; . \quad (7a)$$

$$2V_P{}^r = V - PI = 2S_P V_P{}^i \quad . \; . \quad (7b)$$

Solving eqns. 7 and 2 for V and equating and then repeating for I yields

$$V = (1_n + S_P)\, V_P{}^i$$
$$= (1_n + S_R)\, V_R{}^i \quad . \; . \; . \; . \; . \quad (8a)$$

$$I = P^{-1}(1_n - S_P)\, V_P{}^i$$
$$= R^{-1}(1_n - S_R)\, V_R{}^i \quad . \; . \; . \quad (8b)$$

Inserting the left sides of eqns. 8a and b into eqn. 2a yields $V_R{}^i$ in terms of $V_P{}^i$, which can be inserted in eqn. 2b using eqn. 3, which yields, as $V_P{}^i$ is arbitrary,

$$S_R[(1_n + S_P) + RP^{-1}(1_n - S_P)]$$
$$= (1_n + S_P) - RP^{-1}(1_n - S_P) \quad . \quad (8c)$$

Solving for S_R and S_P finally gives

$$S_R = [(1_n - RP^{-1}) + (1_n + RP^{-1})\, S_P][(1_n + RP^{-1}) + (1_n - RP^{-1})\, S_P]^{-1} \quad . \; . \quad (9a)$$

$$S_P = P[(R + P) + S_R(R - P)]^{-1}\, [(R - P) + S_R(P + R)]P^{-1} \quad . \; . \; . \; . \; . \; . \; . \quad (9b)$$

In these equations, R and P can also be interchanged, since there is essentially no difference between the two. Of special interest is the case where $P = 1_n$, for which we get the useful relation for unnormalising:

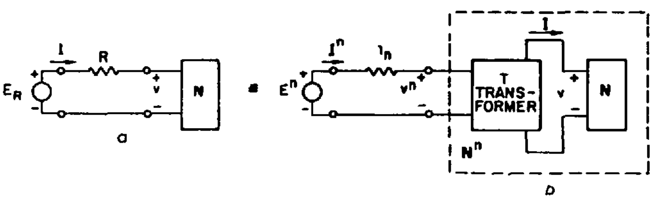

Fig. 1 *Reference impedances and normalisation*

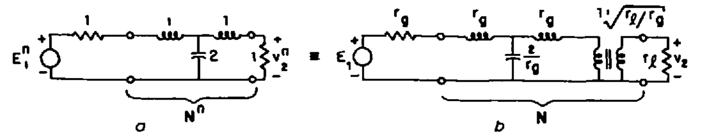

Fig. 2 *Interpretation for a maximally flat filter*

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$$\tilde{T}S^nT^{-1} = S_R = [(1_n - R) + (1_n + R) S][(1_n + R) + (1_n - R) S]^{-1} \quad . . \quad (10)$$

By simple matrix manipulations, one can obtain the equivalent form:

$$S_R = [1_n - \rho]^{-1}[S - \rho][1_n - \rho S]^{-1}[1_n - \rho] \quad . . . (11a)$$

where

$$\rho = [R + 1_n]^{-1}[R - 1_n] = [R - 1_n][R + 1_n]^{-1} \quad . . (11b)$$

is the scattering matrix of the reference resistance matrix. Eqns. 11 are similar to a result of Bayard for the uncoupled case,[9] a treatment of which we are unaware of being available in English. However, if Z denotes the impedance matrix of N, Bayard defines the scattering matrix with reference to R (in his case diagonal) by

$$S_R{}^B = [Z + R]^{-1}[Z - R]$$

while the above theory, which we believe to be more physically motivated, yields

$$S_R = R[Z + R]^{-1}[Z - R]\,R^{-1} = [Z - R][Z + R]^{-1} \quad . . (12)$$

If one uses a current-scattering matrix[3] (i.e. replace all Vs by Is and vice versa, with or without subscripts, in eqns. 2 and 3), one still obtains a slightly different theory than Bayard.

As a useful example of the application of eqn. 10 to the uncoupled-reference case, consider a third-order Butterworth filter designed on the normalised basis for

$$2\frac{V_2{}^n}{E_1{}^n} = s_{21}{}^n = \frac{1}{B_3} = \frac{1}{p^3 + 2p^2 + 2p + 1} \quad . . . (13)$$

From the arguments above we know S_R, with

$$\left.\begin{aligned} R &= \begin{bmatrix} r_g & 0 \\ 0 & r_l \end{bmatrix} = \tilde{T}T \\ T &= \begin{bmatrix} \sqrt{r_g} & 0 \\ 0 & \sqrt{r_l} \end{bmatrix} \end{aligned}\right\} \quad . (16a)$$

for N of Fig. 1b is given by

$$S_R = \tilde{T}S^n\tilde{T}^{-1} = \begin{bmatrix} s_{11}{}^n & \sqrt{\frac{r_g}{r_l}}s_{12}{}^n \\ \sqrt{\frac{r_l}{r_g}}s_{21}{}^n & s_{22}{}^n \end{bmatrix} = \frac{1}{B_3(p)}\begin{bmatrix} p^3 & \sqrt{\frac{r_g}{r_l}} \\ \sqrt{\frac{r_l}{r_g}} & p^3 \end{bmatrix} \quad . . (16b)$$

which serves as a check on eqn. 15. Now let us assume that N^n itself replaces N in Fig. 1b; i.e. the normalised coupling network is used between r_g and r_l. Setting $P = 1_n$ and $S_P = S^n$ in eqn. 9a shows that the scattering matrix $S_R{}^n$ of N^n with respect to R is given by

$$S_R{}^n = [(1_2 - R) + (1_2 + R) S^n][(1_2 + R) + (1_2 - R) S^n]^{-1} \quad . . . (16c)$$

which is also eqn. 10 with S replaced by S^n. Although this yields a rather complicated expression, the voltage gain of Fig. 2b with N replaced by N^n can be calculated from eqn. 16c as

$$2\frac{V_2}{E_1}\bigg|_{N\to N^n} = s_{R21}{}^n = \frac{4s_{21}{}^n}{(1 + r_g)(1 + r_l) + (1 + r_g)(1 - r_l)\,s_{22}{}^n + (1 + r_l)(1 - r_g)\,s_{11}{}^n + (1 - r_g)(1 - r_l)\,s_{11}{}^n s_{22}{}^n + (1 - r_g)(1 - r_l)\,s_{12}{}^n s_{21}{}^n} \quad . . . (16d)$$

where B_3 is the third-order Butterworth polynomial.[10] A circuit realisation[11] is given in Fig. 2a, where we have

$$S^n = \begin{bmatrix} s_{11}{}^n & s_{12}{}^n \\ s_{21}{}^n & s_{22}{}^n \end{bmatrix} = \frac{1}{B_3(p)}\begin{bmatrix} p^3 & 1 \\ 1 & p^3 \end{bmatrix} \quad . . (14)$$

If we insert transformers back to back at the input, of turns ratios $1 : \sqrt{r_g}$ and $\sqrt{r_g} : 1$, and at the output, of turns ratios $1 : \sqrt{r_l}$ and $\sqrt{r_l} : 1$, the configuration is unchanged; by using these transformers to scale the impedances, we arrive at Fig. 1b, with

$$2\frac{V_2}{E_1} = \frac{\sqrt{(r_l/r_g)}}{B_3(p)} \quad . . . . . . (15)$$

Note that eqn. 16d reduces to eqn. 13 when $r_g = r_l = 1$, and the denominator in eqn. 16d is in general a measure of the mismatch resulting from using non-unity terminations for N^n.

To summarise, we have shown how the scattering matrix with respect to a given reference changes with the reference. For generality, we have assumed coupled references, but to illustrate the main points we have considered the general 2-port situation, with uncoupled references specifying finally to a maximally flat design example. One can further extend the definitions to fixed complex references,[12, 13] of most use for single-frequency considerations, or most likely to frequency-dependent reference impedances.[14]

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TOWNSEND'S FIRST IONISATION COEFFICIENT IN COMPRESSED METHANE

Townsend's first ionisation coefficient α has been determined in methane by three independent methods for a range of pressures between atmospheric and 14 atm absolute, thus extending the range of previous measurements to lower values of E/p and much higher pressures. It is confirmed that α/p is dependent only on the value of E/p.

Despite the increasing interest in the use of high-pressure gases for high-voltage insulation, there is still a dearth of information concerning the basic ionisation processes for gases at pressures above atmospheric.

Most of the previous investigations on gases at high pressures have been concerned with the effect of electrode material