observed on an average day, during daytime hours. The layer reflection, if that is what it was, shown in Fig. 2(a) and (b), was encountered during an early morning period on February 18th. By 0955, the time the sweep of Fig. 1 was taken, the layer had completely disappeared.

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The Time-Variable Scattering Matrix

In scattering systems [1], the scattering matrix is one of the more important descriptions which has important physical uses, as in the design of microwave structures. Until now the treatment of such systems has been almost exclusively limited to the time-invariant case. However, it seems that a time-varying treatment is in order, especially since transmission through time-variable media, as the ionosphere, presents important practical problems. Here we describe in the network theory context the time-variable scattering matrix giving some of its properties.

As a preliminary, let $\boldsymbol{D}_+$ denote the set of real-valued n-vector functions of time t which are infinitely differentiable and are zero until a finite time; $\boldsymbol{D}$ is the same as $\boldsymbol{D}_+$ except that the functions are also zero after a finite time, while $\boldsymbol{D}'$ is the set of real-valued n-vector distributions [2]. Then consider an n-port network $\boldsymbol{N}$ with $[v, i]$ as an allowed pair of port voltages and currents [3]; we assume $v \in \boldsymbol{D}_+$ and $i \in \boldsymbol{D}_+$ and write $[v, i] \in \boldsymbol{N}$, if $[v, i]$ is an allowed pair. By definition the network is called *linear* if

$$[\alpha v_1 + \beta v_2, \alpha i_1 + \beta i_2] \in N \tag{1}$$

for all $[v_1, i_1] \in \boldsymbol{N}$, $[v_2, i_2] \in \boldsymbol{N}$, and all real constants α and β. For linear networks it is convenient to consider $\boldsymbol{N}$ on the scattering basis by introducing incident and reflected voltages v^i and v^r, respectively, which in the normalized case are defined through

$$2v^i = v + i \tag{2a}$$

$$2v^r = v - i. \tag{2b}$$

Now consider the augmented network $\boldsymbol{N}_a$ defined by $[v+i, i]$ allowed for $\boldsymbol{N}_a$ if $[v, i]$ is allowed for $\boldsymbol{N}$ and illustrated in Fig. 1, where the $n \times n$ identity matrix I_n denotes n-unit resistors, each in series with one of the ports. Physically it is only reasonable to expect a unique current i to flow in Fig. 1 when a given $e = 2v^i$ is applied. We, therefore, define $\boldsymbol{N}$ to be *solvable* [4] if for every $e \in \boldsymbol{D}_+$

$$e = v + i \tag{3}$$

has a unique solution $[v, i] \in \boldsymbol{N}$. If $\boldsymbol{N}$ is solvable there is a unique mapping $Y_a[\ ]$ of $e \in \boldsymbol{D}_+$ into $i \in \boldsymbol{D}_+$

$$i = Y_a[e]. \tag{4}$$

Physically it is only further reasonable to expect this mapping to be continuous in some sense. Consequently, we define a solvable $\boldsymbol{N}$ to be *completely solvable* if

$$\lim_{j\to\infty} Y_a[e_j] = Y_a\left[\lim_{j\to\infty} e_j\right] \tag{5}$$

whenever a sequence $\{e_j\}$, $e_j \in \boldsymbol{D}$, is convergent; the convergence on the right of (5) is taken to be in $\boldsymbol{D}$ [5], while that on the right is taken in $\boldsymbol{D}'$.

If we define

$$K \cdot x = \int_{-\infty}^{\infty} K(t, \tau) x(\tau) d\tau \tag{6}$$

for any $n \times n$ matrix K of distributions in two variables t and τ (K is called a distributional kernel) and any $x \in \boldsymbol{D}$, we see by a theorem of L. Schwartz [6], [7] that every linear, completely solvable $\boldsymbol{N}$ can be described by a distributional kernel $y_a(t, \tau)$ through

$$i = y_a \cdot e. \tag{7}$$

If we now define the *time-variable scattering matrix* $s(t, \tau)$ through

$$v^r = s \cdot v^i \tag{8}$$

then (8), (2), (3), and (7) show that

$$s(t, \tau) = \delta(t - \tau) I_n - 2y_a(t, \tau) \tag{9}$$

where δ is the unit impulse and I_n the $n \times n$ identity matrix. Since $Y_a[\ ]$ is a linear continuous mapping of $\boldsymbol{D}_+$ into $\boldsymbol{D}_+$, (8) can be used [8] for all $v^i \in \boldsymbol{D}_+$. An almost obvious argument shows that a network completely described by (2) and (8) for all $v^i \in \boldsymbol{D}_+$ must be linear and completely solvable; we conclude the following result.

Theorem: An n-port $\boldsymbol{N}$ is linear and completely solvable if and only if a time-variable scattering matrix $s(t, \tau)$ exists as a distributional kernel mapping $\boldsymbol{D}_+$ into $\boldsymbol{D}_+$ through (8).

The importance of the theorem is that it points out the necessary and sufficient conditions on $\boldsymbol{N}$ for s to exist. Because of the generality of these conditions, s appears as one of the most useful descriptions in network theory. This usefulness is certainly borne out in the time-invariant case where an almost complete theory of synthesis and equivalence of finite passive $\boldsymbol{N}$ proceeds from the scattering matrix [9].

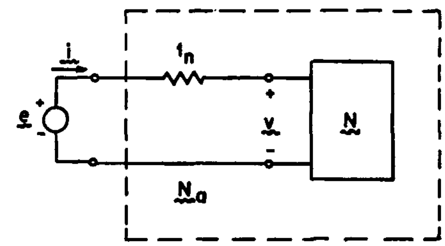

Fig. 1. The augmented network N_a.

As an example, the time-variable inductor of inductance $l(t)$, $v = d[li]/dt$, has y_a found from the first-order equation

$$e = \frac{dx}{dt} + \frac{x}{l}, x = li, l \neq 0,$$

giving from (9)

$$s(t\ \tau) = \delta(t - \tau) - \frac{2}{l(t)} \cdot \exp\left\{-\int_\tau^t \frac{d\lambda}{l(\lambda)}\right\} u(t - \tau)\ [\text{inductor}] \tag{10}$$

where u is the unit step function. Neither the nullator nor the norator [9], nor the -1-ohm resistor, have a scattering matrix since they are not solvable.

In the time-varying case, various properties of s can be determined, several of which we mention. Because (3) is valid for solvable $\boldsymbol{N}$, we note that $v = i = 0$ for $t < t_0$ whenever $e = 0$ for $t < t_0$ (by the uniqueness of $[v, i] \in \boldsymbol{N}$) if s exists. This implies that

$$s(t, \tau) = 0_n \quad \text{for } t < \tau \tag{11}$$

where 0_n is the $n \times n$ zero matrix. If case $\boldsymbol{N}$ is passive, then somewhat complicated necessary and sufficient conditions [11] can be given on s. However, when $\boldsymbol{N}$ is lossless [12], [13] these conditions essentially reduce to

$$\int_{-\infty}^{\infty} \tilde{s}(\lambda, t) s(\lambda, \tau) d\lambda = \delta(t - \tau) I_n. \tag{12}$$

Here the superscript tilde ~ denotes the matrix transpose. By definition, $\boldsymbol{N}$ is lossless if its input energy is non-negative at any instant of time [that is, passive], and if this energy is zero at infinite time when any $e \in \boldsymbol{D}$ is applied to N_a of Fig. 1. Equation (12) is recognizable as a generalization of the para-unitary frequency domain result [14]. Unless $l(t)$ is a positive, constant $s(t, \tau)$ in (10) does not satisfy the lossless constraint [15] of (12). The time-variable $(l+m)$-port transformer [16] of $m \times l$ turns-ratio matrix $T(t)$ has [17]

$$s(t, \tau) = \delta(t - \tau)\begin{bmatrix} (I_l + \tilde{T}T)^{-1}(\tilde{T}T - 1_l) & 2(1_l + \tilde{T}T)^{-1}\tilde{T} \\ 2T(I_l + \tilde{T}T)^{-1} & (I_m + T\tilde{T})^{-1}(I_m - T\tilde{T}) \end{bmatrix} \tag{13}$$

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which does satisfy (12).

In summary, we have presented a theorem which states the conditions under which a time-variable scattering matrix $\mathbf{s}(t, \tau)$ exists, showing that it satisfies an antecedance (causality) restriction (11). Because of its importance, the lossless condition (12) has also been stated. The statements have been worded in terms of network theory, but because of the importance of general scattering systems, the results should be of use in other disciplines.

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Characteristics of Burst Noise

Germanium junctions and many other devices sometimes generate electrical-noise current as shown in Fig. 1. This correspondence reports on the occurrence and measured characteristics of this *burst-noise* [1] phenomenon.

Burst-noise current with the step waveform shown in Fig. 1(a) has been observed in selected reverse-biased germanium and silicon junctions with bias voltage V_b in the range $0.7 < V_b < V_{bd}$ volts, where V_{bd} is the breakdown voltage. Typical magnitude of the current steps (pulse amplitude) of 10^{-8}

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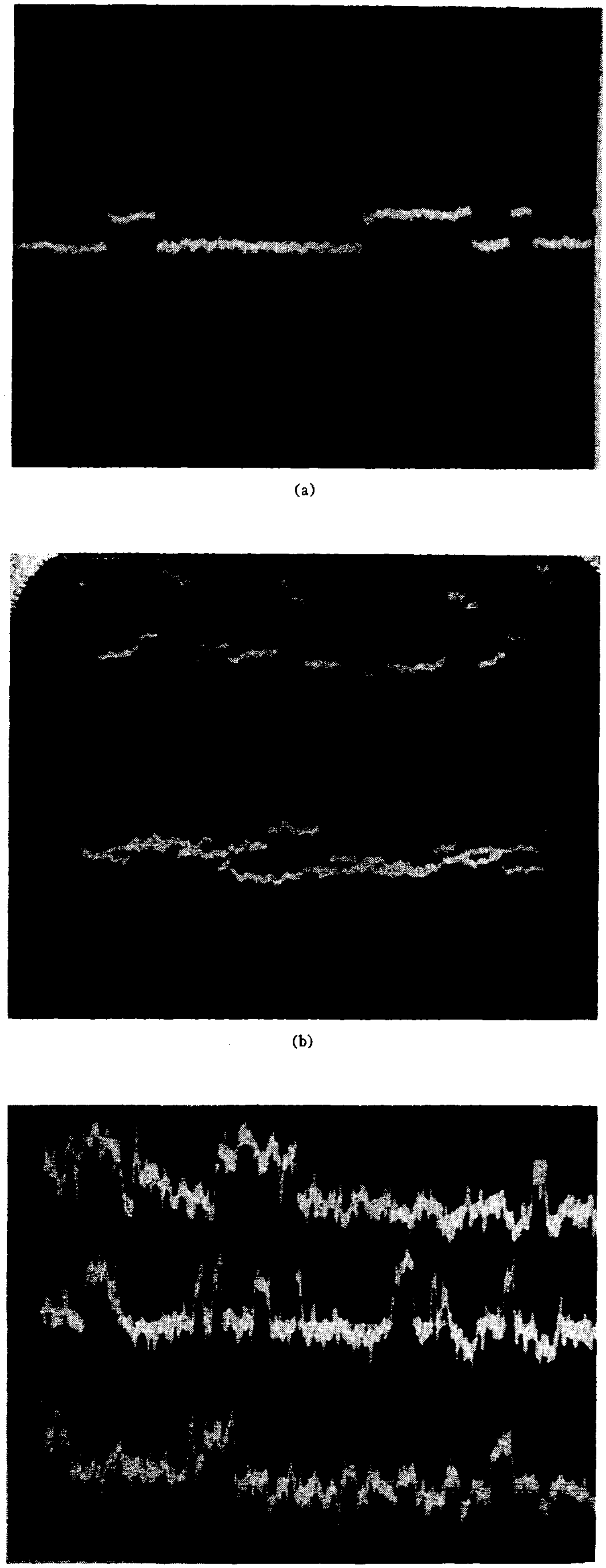

Fig. 1. Burst-noise current waveforms. (a) Ge junction with $V_b = 24$ volts, 0.05-μA/div., 2-ms/div. (b) GaSb tunnel diode with $V_b = 0.2$ volt, 10^{-5}-A/div., 5-ms/div. (c) 200-K, 2-watt carbon-composition resistor with $V_b = 9.0$ volts, 0.27-μA/div., 2-ms/div.