Synthesis of ARMA Filters by Real Lossless Digital Lattices

Louiza Sellami and Robert W. Newcomb

***Abstract*—A new method is presented to obtain degree-one or real degree-two transfer scattering matrices of two-port lossless lattice filters through the use of complex Richard's function extractions for the minimum degree cascade synthesis of real, stable, single-input, single-output ARMA (n, m) filters from the transfer function or the input reflection coefficient. The method relies on a four-step Richard's function extraction where two steps are used for reducing the degree of the transfer function and two for obtaining real lattices. We treat the cases where the zeros of transmission are inside and outside the unit circle but not on the unit circle.**

I. Introduction

The SYNTHESIS of lattice digital filters has been investigated by several authors [1]–[3]. In their paper [1], Deprettere and Dewilde deal with the realization of a cascade of orthogonal multiport digital filters from the overall transfer scattering matrix of the system. In the second paper [2], Vaidyanathan and Mitra introduce a discrete version of Richard's function and use it to extract degree-one real sections each of which realizes a real zero of transmission. Following up on that, the authors of the third paper [3] extend the ideas presented in [2] to complex zeros. But because the zeros of transmission are complex, the degree-one sections extracted are also complex and the cascade of two degree-one complex sections does not necessarily yield a degree-two real section.

Here we propose a new technique to synthesize a real, stable, single-input, single-output, digital ARMA (n, m) filter in the form of a cascade of degree-one or degree-two real lossless two-port lattice filters, from the reflection coefficient or the transfer function, and using complex or real zeros of transmission. The technique relies on a four-step complex Richard's function extraction to calculate the transfer scattering matrices that characterize the lattices. Two of the steps are used to reduce the degree of the transfer function or the reflection coefficient of the filter through the extraction of the zeros of transmission and the other two steps are used to obtain a real realization. The synthesis is minimum, the number of lattice sections being dictated by the degree of the transfer function. Compared to [2], [3], the proposed technique offers the advantage of realizing real degree-two sections from complex degree-one lattices. A typical application of this technique is the modeling of the mechanics of the cochlea [4], [5] for which a transfer function, assumed of minimum phase, has been estimated, without a priori knowledge of either the numerator or the denominator degrees, through a new ARMA system identification technique developed by Youla, Shim, and Pillai [6].

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L. Sellami is with the Electrical Engineering Department, US Naval Academy, Annapolis, MD 21401 USA and Microsystems Laboratory, Electrical Engineering Department, University of Maryland, College Park, MD 20742 USA.

R. W. Newcomb is with the Microsystems Laboratory, University of Maryland, College Park, MD 20742 USA.

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II. Preliminaries

Our technique relies on the concepts of two-port synthesis, through the use of scattering and transfer scattering matrices, developed in [7], and at this point we give definitions of the concepts to be used later. Each lattice section will be described by its 2×2 scattering and transfer scattering matrices, denoted by S and θ, respectively. We recall that the scattering matrix, $S(z)$, relates the reflected signals v^r to the incident signals v^i (see Fig. 1) as follows:

$$\begin{bmatrix} v_1^r \\ v_2^r \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} v_1^i \\ v_2^i \end{bmatrix}. \tag{1}$$

The transfer scattering matrix, $\theta(z)$, relates the signals at the left port to the signals at the right port, according to the following equation

$$\begin{bmatrix} v_1^i \\ v_1^r \end{bmatrix} = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \begin{bmatrix} v_2^r \\ v_2^i \end{bmatrix}. \tag{2}$$

Because the lattices we will consider are lossless, we recall that for lossless structures $S(z)$ is para-unitary and $\theta(z)$ is J-para-unitary, that is

$$S(z)S_*(z) = I_2, \quad \theta_*(z)J\theta(z) = J \tag{3}$$

where $S_*(z) = S^{*T}(1/z^*)$, $\theta_*(z) = \theta^{*T}(1/z^*)$, I_2 is the identity matrix, and J is given below. The superscripts * and ^{T} denote complex conjugation and matrix transposition, respectively.

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4}$$

Also, since the ARMA filter is assumed finite and, hence, described by a rational transfer function with real coefficients, $S(z)$ and $\theta(z)$ are both rational matrices with real coefficients. Furthermore, since the lattice structure of interest is passive, $S(z)$ is bounded real in $|z| > 1$.

The proposed synthesis method utilizes the overall input reflection coefficient to extract one zero of transmission for

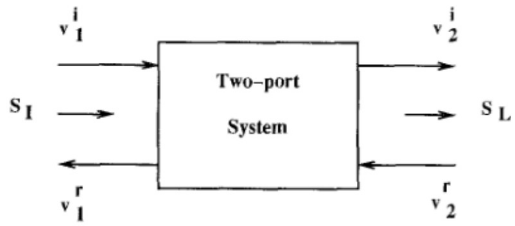

Fig. 1. Signal flow-graph like representation of a two-port system.

each section, form the transfer scattering matrix for that lattice, and calculate the output reflection coefficient whose degree will be one less than that of the input reflection coefficient. Let $S_I(z)$ be the input reflection coefficient when the two-port of Fig. 1 is loaded by $S_L(z)$, the load reflection coefficient. Then, given that S_I and S_L are such that $v_1^r = S_I v_1^i$ and $v_2^i = S_L v_2^r$, we have from (1) and (2)

$$S_I = (\theta_{21} + \theta_{22}S_L)(\theta_{11} + \theta_{12}S_L)^{-1} \tag{5}$$

$$S_L = (\theta_{22} - S_I\theta_{12})^{-1}(S_I\theta_{11} - \theta_{21}). \tag{6}$$

The zeros of transmission of the system are the zeros of $S_{21}(z)$ which, if $S_I(z)$ is given, are calculated from the para-unitary property of lossless $S(z)$, according to (3), as follows:

$$1 - S_{21}S_{21*} = S_I S_{I*}. \tag{7}$$

In order to find S_{21} from S_I, we express S_{21} and S_I as follows:

$$S_I(z) = \frac{N_I(z)}{D_I(z)}, \quad \text{and} \quad S_{21}(z) = \frac{N_{21}(z)}{D_{21}(z)} \tag{8}$$

and rewrite (7) as

$$\frac{D_{21}(z)D_{21*}(z) - N_{21}(z)N_{21*}(z)}{D_{21}(z)D_{21*}(z)} = \frac{N_I(z)N_{I*}(z)}{D_I(z)D_{I*}(z)}. \tag{9}$$

If $S_{21}(z)$ is chosen to have the same denominator as $S_I(z)$, then its numerator, $N_{21}(z)$, is a solution of

$$D_I(z)D_{I*}(z) - N_I(z)N_{I*}(z) = N_{21}(z)N_{21*}(z) \tag{10}$$

which is not unique since it is obtained by factoring $D_I(z)D_{I*}(z) - N_I(z)N_{I*}(z)$ and arbitrarily assigning half of its zeros to $N_{21}(z)$ and the other half (the images) to $N_{21*}(z)$. Also, since $D_I(z)D_{I*}(z) - N_I(z)N_{I*}(z)$ is a real even polynomial, its complex zeros occur in quadruplets of the form $(a, a^*, 1/a, 1/a^*)$ and its real zeros occur in pairs of the form $(a, 1/a)$. If instead of $S_I(z)$ a voltage transfer function $H(z)$ is given, then $S_I(z)$ can be determined via $S_{21}(z)$, through the para-unitary property of $S(z)$ as follows:

$$S_{21}(z) = \frac{H(z)}{M}, \quad M \geq \max H(z) \quad \text{for} \quad |z| = 1 \tag{11}$$

$$1 - S_{21}(z)S_{21*}(z) = S_I(z)S_{I*}(z). \tag{12}$$

In either case (given $S_I(z)$ or $H(z)$), the choice of S_{21} from (10) and the choice of S_I from (11) and (12) is made to guarantee that these functions are bounded real.

III. A Standard Degree-One Lattice Section

The key to our result is the "standard lattice". Each possibly complex standard lattice section has degree one and is realized as a delay lattice of nonstandard type, followed by a constant lattice. The nonstandard lattice is composed of a cascade of a cross-arm constant gain lattice and a simple delay lattice whose structure differs depending on whether the zero of transmission being realized is located inside or outside the unit circle; see Figs. 2 and 3. This dependence is related to the degree reduction procedure. The advantage of the chosen representation is that the cascade of two complex standard lattices of degree one, at a and a^*, will always result in a real lattice of degree two and of the same structure, as we will show in the next section, whereas the same result cannot be obtained by simply cascading two lattices of nonstandard type again at a and $a*$. The standard lattice is described by a scattering transfer matrix of the form [1]

$$\theta(z) = I_2 + \frac{(z-1)}{(1-a^*)(z-a)} x x_* J \tag{13}$$

where a is the zero of transmission being realized with (i.e., $|a| \neq 1$), and x is a constant complex vector such that

$$x_* J x = |a^2| - 1. \tag{14}$$

The above condition (14) results from the J-para-unitary property of $\theta(z)$. The components of the vector x will be calculated so that the degree of $S_L(z)$ is one less than that of $S_I(z)$ at each section extraction.

In the following two cases, we justify our previous claim about the standard lattice filter being a cascade of a delay lattice, $\theta_N(z)$, and a constant lattice, $\theta_N^{-1}(1)$, where $\theta_N(z)$ denotes the nonstandard lattice of case 1 or case 2. We express the above claim mathematically in terms of the transfer scattering matrices of each of the two lattices (delay and constant).

$$\theta(z) = \theta_N(z)\theta_N^{-1}(1) \tag{15}$$

where $\theta_N(z)$ and its inverse $\theta_N^{-1}(z)$ are both para-unitary rational transfer scattering matrices with a structure that depends on the zero of transmission being realized [3]. Now, $\theta(z) = I_2$ to match (13).

The details are as follows for the two cases of transmission zeros outside and inside the unit circle, respectively.

Case 1: $|a| > 1$ The transfer scattering matrix $\theta_N(1)$ of interest is given by

$$\theta_N(z) = \frac{1}{\sqrt{1-kk^*}} \begin{bmatrix} 1 & -k \\ -k^* & 1 \end{bmatrix} \begin{bmatrix} f(z) & 0 \\ 0 & 1 \end{bmatrix}, \quad f(z) = \frac{1-a^*z}{z-a} \tag{16}$$

and the corresponding signal-flow diagram is shown in Fig. 2. Here k is a constant such that $|k| < 1$, the factor $1/\sqrt{1-kk^*}$ is used to make $\theta_N(z)$ J-para-unitary, and $f(z)$ is a delay function.

The inverse transfer scattering matrix is

$$\theta_N^{-1}(z) = \begin{bmatrix} \frac{1}{f(z)} & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{1-kk^*}} \begin{bmatrix} 1 & k \\ k^* & 1 \end{bmatrix}. \tag{17}$$

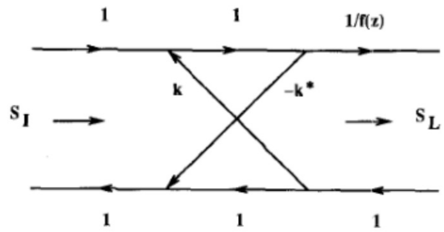

Fig. 2. Signal flow diagram for $|a| > 1$.

The product $\theta_N(z)\theta_N^{-1}(1)$ is then calculated as

$$\theta_N(z)\theta_N^{-1}(1) = \frac{1}{1-kk^*}\begin{bmatrix} \frac{f(z)}{f(1)} - kk^* & k\frac{f(z)}{f(1)} - k \\ -k^*\frac{f(z)}{f(1)} + k^* & -kk^*\frac{f(z)}{f(1)} + 1 \end{bmatrix}. \tag{18}$$

By adding and subtracting 1 from entry (1, 1), adding and subtracting kk^* from entry (2, 2), and separating the matrix into the sum of two matrices, one of which is the identity matrix, the above equation is rewritten as follows: (the standard lattice form)

$$\theta_N(z)\theta_N^{-1}(1) = I_2 + \frac{(z-1)}{(1-a^*)(z-a)}\frac{|a|^2-1}{1-|k|^2}\begin{bmatrix} 1 & -k \\ -k^* & |k|^2 \end{bmatrix}J. \tag{19}$$

By comparing this result with (13), we get the following expressions for x and k

$$\begin{aligned} |x_1|^2 &= \frac{|a|^2-1}{1-|k|^2}, \quad |x_2|^2 = \frac{|k|^2}{1-|k|^2}(|a|^2-1), \\ x_1x_2^* &= -k\frac{|a|^2-1}{1-|k|^2}, \quad k = -\left(\frac{x_2}{x_1}\right)^*. \end{aligned} \tag{20}$$

The above equations guarantee the J-para-unitary property of $\theta(z)$ as $x_*Jx = |x_1|^2 - |x_2|^2 = |a|^2 - 1$. Notice that the condition $|a| > 1$ ensures that $|x_1|^2$ and $|x_2|^2$ are both positive.

Case 2: $|a| < 1$ The transfer scattering matrix $\theta_N(z)$ is almost the same as in the previous case, except that the delay function is located in the lower branch of the lattice. Its expression is given by

$$\theta_N(z) = \frac{1}{\sqrt{1-kk^*}}\begin{bmatrix} 1 & -k \\ -k^* & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & f(z) \end{bmatrix} \tag{21}$$

and the corresponding signal-flow diagram is shown in Fig. 3. The inverse transfer scattering matrix is then

$$\theta_N^{-1}(z) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{f(z)} \end{bmatrix}\frac{1}{\sqrt{1-kk^*}}\begin{bmatrix} 1 & k \\ k^* & 1 \end{bmatrix}. \tag{22}$$

The product $\theta_N(z)\theta_N^{-1}(1)$ is then calculated as

$$\theta_N(z)\theta_N^{-1}(1) = \frac{1}{1-kk^*}\begin{bmatrix} 1 - kk^*\frac{f(z)}{f(1)} & k - k\frac{f(z)}{f(1)} \\ k^*\frac{f(z)}{f(1)} - k^* & -kk^* + \frac{f(z)}{f(1)} \end{bmatrix}. \tag{23}$$

By adding and subtracting kk^* from entry (1, 1), adding and subtracting 1 from entry (2, 2), and separating the matrix into

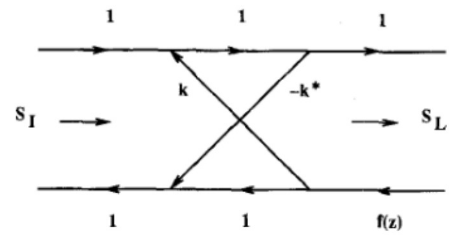

Fig. 3. Signal flow diagram for $|a| < 1$.

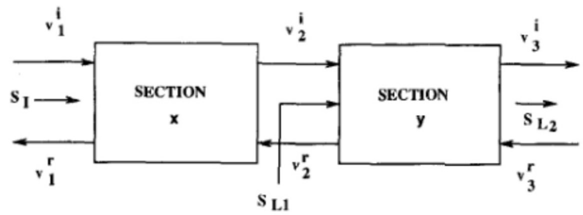

Fig. 4. The first two lattice sections.

the sum of two matrices one of which is the identity matrix, the above equation is rewritten as (the standard lattice form)

$$\theta_N(z)\theta_N^{-1}(1)p = I_2 + \frac{(z-1)}{(1-a^*)(z-a)}\frac{1-|a|^2}{1-|k|^2}\begin{bmatrix} |k|^2 & -k \\ -k^* & 1 \end{bmatrix}J. \tag{24}$$

By comparing this result with (13), we get the following expressions for x and k

$$\begin{aligned} |x_1|^2 &= \frac{|k|^2}{1-|k|^2}(1-|a|^2), \quad |x_2|^2 = \frac{1-|a|^2}{1-|k|^2}, \\ x_1x_2^* &= -k\frac{1-|a|^2}{1-|k|^2}, \quad k = -\frac{x_1}{x_2}. \end{aligned} \tag{25}$$

The above solution guarantees the J-para-unitary property for $\theta(z)$ because $x_*Jx = |x_1|^2 - |x_2|^2 = |a|^2 - 1$. Notice, in this configuration, the condition $|a| < 1$ ensures that $|x_1|^2$ and $|x_2|^2$ are both positive.

IV. Real Transfer Scattering Matrix of Degree Two

To obtain a real lattice of degree two when the zeros of transmission are complex, we cascade two complex standard lattices of degree one that realize a complex conjugate pair of zeros of transmission and determine conditions (refer to the proposition below) for which the product of their transfer scattering matrices is real [1].

Proposition: Given two lattice sections of degree one with complex coefficients, described by their transfer scattering matrices $\theta_x(z)$ and $\theta_y(z)$, realizing a complex conjugate pair of zeros of transmission (a, a^*) (refer to (13)), such that

$$\theta_x(z) = I_2 + \frac{(z-1)}{(1-a^*)(z-a)}xx_*J \tag{26}$$

and

$$\theta_y(z) = I_2 + \frac{(z-1)}{(1-a)(z-a^*)}yy_*J \tag{27}$$

where x and y are complex vectors satisfying $x_* Jx = |a|^2 - 1$ and $y_* Jy = |a^*|^2 - 1 = |a|^2 - 1$, then

$$\theta(z) = \theta_x(z)\theta_y(z) = I_2 + \frac{(z-1)}{|K|^2} \times \left(\frac{uv_*}{(1-a^*)(z-a)} + \frac{u^* v^T}{(1-a)(z-a^*)}\right) J \quad (28)$$

is the transfer scattering matrix of a degree two real lattice if the following conditions are satisfied

$$u = x \quad (29)$$

$$v = Ky^* \quad (30)$$

$$|K|^2 = 1 - |\gamma|^2 (r^2 + 4d^2)^{-1} \quad (31)$$

$$\gamma = u^T Ju \quad (32)$$

$$r = |a|^2 - 1 \quad (33)$$

$$d = \frac{a - a^*}{2j} = Im(a) \quad (34)$$

$$v = u - u^* \gamma (r - 2jd)^{-1}. \quad (35)$$

Proof: Here we outline the proof without going into the details. Multiplying the expressions of $\theta_x(z)$ and $\theta_y(z)$ and reducing the factors of x and y to the same denominator yields the following expression for $\theta(z)$

$$\theta(z) = I_2 + \frac{(z-1)}{(1-a^*)(1-a)(z-a)(z-a^*)} \times [xx_*(1-a)(z-a^*) + yy_*(1-a^*)(z-a) + (z-1)xx_* Jyy_*]J. \quad (36)$$

The conditions listed in the proposition are obtained by setting $\theta(z) = \theta^*(z^*)$, which represents the realness condition of $\theta(z)$. By substituting the expressions of x and y in terms of the new vectors u and v into (36), we get the expression in (28) which shows that $\theta(z)$ is real since the terms

$$\frac{uv_*}{(1-a^*)(z-a)}, \quad \text{and} \quad \frac{u^* v^T}{(1-a)(z-a^*)} \quad (37)$$

form a complex conjugate pair and their sum is, therefore, real. □

V. The Lattice Synthesis Technique

Given the overall input reflection coefficient, $S_I(z)$, we proceed to extract lossless cascade lattices, by reducing the degree of $S_I(z)$ by one at each extraction. As above, we derive the technique for zeros of transmission outside and inside the unit circle.

A. Complex Zeros

Case 1: $|a| > 1$ Consider the first two lattice sections, shown in Fig. 4, that realize the conjugate pair (a, a^*) of zeros of transmission. Both lattices have degree one and are described by their transfer scattering matrices $\theta_x(z)$, for a, and $\theta_y(z)$, for a^*, respectively. The entries of $\theta_x(z)$ are

$$\theta_{11}^x = \frac{(1-a^*)(z-a) + (z-1)|x_1|^2}{(1-a^*)(z-a)} \quad (38)$$

$$\theta_{12}^x = \frac{-(z-1)x_1 x_2*}{(1-a^*)(z-a)} \quad (39)$$

$$\theta_{21}^x = \frac{(z-1)x_1^* x_2}{(1-a^*)(z-a)} \quad (40)$$

$$\theta_{22}^x = \frac{(1-a^*)(z-a) - (z-1)|x_2|^2}{(1-a^*)(z-a).} \quad (41)$$

The output reflection coefficient, S_{L1}, of the first lattice is calculated according to (6).

$$S_{L1}(z) = [\theta_{22}^x - S_I(z)\theta_{12}^x]^{-1}[S_I(z)\theta_{11}^x - \theta_{21}^x]. \quad (42)$$

Substituting the expressions of θ_{11}^x, θ_{12}^x, θ_{21}^x, and θ_{22}^x, and canceling the common denominator yields

$$S_{L1}(z) = \frac{S_I(z)[(1-a^*)(z-a) + (z-1)|x_1|^2] - (z-1)x_1^* x_2}{(1-a^*)(z-a) + (z-1)[S_I(z)x_1 x_2^* - |x_2|^2]} = \frac{N_{L1}(z)}{D_{L1}(z)}. \quad (43)$$

At first glance, $S_{L1}(z)$ appears to be one degree higher than $S_I(z)$; therefore, to reduce its degree by two, we choose the complex parameters x_1 and x_2 so that the numerator and the denominator of $S_{L1}(z)$ have $(z-a)$ and $(z-1/a^*)$ factors in common, i.e., $N_{L1}(z) = 0$ and $D_{L1}(z) = 0$ at $z = a$ and $z = 1/a^*$. For $z = a$,

$$N_{L1}(a) = (a-1)[S_I(a)x_1 - x_2]x_1^* = 0. \quad (44)$$

Since $a \neq 1$, and $x_1^* \neq 0$ will be chosen later at (55), we have the following expressions for x_2 and the cross-arm gain, k, of the lattice of (20) as a result

$$x_2 = S_I(a)x_1, \quad k = -S_I^*(a) \quad (45)$$

We also have

$$D_{L1}(a) = (a-1)[-x_2 + S_I(a)x_1]x_2^* = 0. \quad (46)$$

The value of x_2 that satisfies (46) is the one given in (45). Therefore, by choosing $x_2 = S_I(a)x_1$, we guarantee that both $N_{L1}(z)$ and $D_{L1}(z)$ have $(z-a)$ in common. Now for $z = 1/a^*$, we have

$$N_{L1}\left(\frac{1}{a^*}\right) = S_I\left(\frac{1}{a^*}\right) \times \left[(1-a^*)\left(\frac{1}{a^*} - a\right) + \left(\frac{1}{a^*} - 1\right)|x_1|^2\right] - \left(\frac{1}{a^*} - 1\right)x_2 x_1^* = 0 \quad (47)$$

and

$$D_{L1}\left(\frac{1}{a^*}\right) = (1-a^*)\left(\frac{1}{a^*} - a\right) + \left(\frac{1}{a^*} - 1\right) \times \left[S_I\left(\frac{1}{a^*}\right)x_1 x_2^* - |x_2|^2\right] = 0. \quad (48)$$

Substituting (45) in (47) and (48) yields

$$N_{L1}\left(\frac{1}{a^*}\right) = |x_1|^2\left(\frac{1}{a^*} - 1\right)\left[S_I\left(\frac{1}{a^*}\right) - S_I(a)\right] - S_I\left(\frac{1}{a^*}\right)(1-a^*)\left(\frac{1}{a^*} - a\right) = 0 \quad (49)$$

and

$$D_{L1}\left(\frac{1}{a^*}\right) = |x_1|^2\left(\frac{1}{a^*}-1\right)\left[S_I\left(\frac{1}{a^*}\right)S_I^*(a) - |S_I(a)|^2\right] - (1-a^*)\left(\frac{1}{a^*}-a\right) = 0. \quad (50)$$

The solution x_1 obtained from (49) is

$$|x_1|^2 = \frac{S_I(\frac{1}{a^*})(aa^*-1)}{S_I(\frac{1}{a^*}) - S_I(a)} \quad (51)$$

and the one obtained from (50), is

$$|x_1|^2 = \frac{(aa^*-1)}{S_I(a^*)\left[S_I(\frac{1}{a^*}) - S_I(a)\right]}. \quad (52)$$

The expressions in (51) and (52) are the same, except that the term $S_I(1/a^*)$ in the numerator of (51) is replaced by $1/S_I^*(a)$ in (52). But since a is a zero of transmission, i.e., $S_{21}(a) = 0$, it follows from the para-unitary property (7) that

$$S_I(a)S_I\left(\frac{1}{a}\right) = 1 \quad \text{or} \quad S_I^*(a)S_I^*\left(\frac{1}{a}\right) = 1 \quad (53)$$

which, because $S_I(z)$ has real coefficients, implies that

$$S_I\left(\frac{1}{a^*}\right) = S_I^*\left(\frac{1}{a}\right) = \frac{1}{S_I^*(a)} \quad (54)$$

in which case the two expressions (51) and (52) are identical. Using (53) and (54), we rewrite this solution in the following form

$$|x_1|^2 = \frac{|a|^2-1}{1-|S_I(a)|^2}. \quad (55)$$

Notice that x_1 is not unique since its phase is arbitrary. As expected, $|x_1|^2$ is a positive real number since $|a| > 1$ and $|S_I(a)| < 1$ from the boundedness property of $S_I(z)$ outside the unit circle. Furthermore, $x_* Jx = |x_1|^2 - |x_2|^2 = |a|^2 - 1$ which implies that the solution obtained by reducing the degree of $S_I(z)$ guarantees that $\theta_x(z)$ is J-para-unitary. It is important to have checked the J-para-unitary condition of $\theta_x(z)$ because it was not inserted in the degree reduction process. In conclusion, in order to guarantee degree reduction of $S_I(z)$ by one, we choose the complex vector x such that its components, x_1 and x_2, are given by (45) and (55). As we have demonstrated earlier, the para-unitary property of S, which involves a given zero of transmission and the conjugate of its reciprocal, is crucial to the degree reduction process and it is the main reason why we choose to cancel the factors $(z-a)$ and $(z-1/a^*)$ and not $(z-a)$ and $(z-a^*)$.

Now we check the properties of $S_{L1}(z)$ and show that, except for realness, they are the same as those of $S_I(z)$, except that the degree has decreased by one, in which case the same degree reduction process can be repeated on S_{L1}, via section y, as if it were S_I, and S_{L1} has complex coefficients when a is complex. First, we observe from (43) that $S_{L1}(z)$ is rational in z since $S_I(z)$ is rational in z. Second, since $(z-a)$ and $(z-1/a^*)$ are eliminated by the choice of x_1 and x_2, it follows that $S_{L1}(z)$ is analytic and bounded where $S_I(z)$ is analytic and bounded, i.e., in $|z| > 1$.

To extract the second section, we use $S_{L1}(z)$ as the input reflection coefficient and the conditions given in the proposition to reduce the degree of $S_{L1}(z)$ by one and to ensure that the cascade of the two degree-one complex sections extracted yields a real degree-two section. The factors being extracted by section y in this second step are $(z-a^*)$ and $(z-1/a)$ and the complex vector y [for (27)] is

$$y = \frac{x^*}{K^*} - x\frac{\gamma^*}{K^*(r+2jd)} \quad (56)$$

where K, γ, r, and d are calculated according to the proposition. The value of y is then used to determine the entries of $\theta_y(z)$ which, in turn, are used to evaluate the output reflection coefficient $S_{L2}(z)$ from (43) in which $S_I(z)$ is replaced by $S_{L1}(z)$, $S_{L1}(z)$ by $S_{L2}(z)$, x by y, and a by a^*. With these conditions, $\theta(z) = \theta_x(z)\theta_y(z)$ is real and of degree two and $S_{L2}(z)$ is readily shown to be real of degree two less than that of $S_I(z)$. The same procedure is repeated to extract additional lattice sections until the degree of the output reflection coefficient becomes zero. The terminating section is a one port system, characterized by a constant input reflection coefficient, which could be lossless or lossy depending on whether the ARMA filter is lossless or lossy. The order in which the zeros of transmission are extracted does not affect the realness character of the realization as long as complex conjugate pairs are extracted sequentially.

Case 2: $|a| < 1$ As mentioned earlier, in the case $|a| < 1$ the structure of the delay lattice is different in the sense that the delay is located in the lower branch of the lattice instead of the upper one. This difference in structure does not affect the degree reduction procedure which yields the same solutions for x_1 and x_2 as given in (45) and (55). It does however affect the value of the cross-arm gain, k, of the lattice, which, in this case, is given by

$$k = -\frac{1}{S_I(a)}. \quad (57)$$

To justify $|x_1|^2 > 0$, we use the boundedness of $S_I(z)$ in $|z| > 1$, which implies that for $|1/a| > 1$, $|S_I(1/a)| < 1$, and the para-unitary property of S as follows

$$S_I(a)S_I\left(\frac{1}{a}\right) = 1 \Longrightarrow |S_I(a)| = \frac{1}{|S_I(\frac{1}{a})|} > 1. \quad (58)$$

B. Real Zeros

For real zeros we extract real degree one sections each of which realizes one real zero of transmission. The extraction procedure is the same as the one outlined in (38) through (58) excluding (56).

VI. Applications to Simple Examples

To illustrate how the degree reduction procedure works, we treat two examples, one with zeros of transmission outside the unit circle and an even number of lattice sections, and the other with zeros of transmission inside the unit circle and an odd number of lattices. Although in both examples we synthesize an ARMA (n, m) filter for which the degree, n, of the numerator of the transfer function is chosen equal to the degree, m, of its denominator the technique applies for any n and m.

Example 1: In this example, we synthesize an ARMA(6,6) filter as a cascade of three real lattice sections, each of degree two, from the following voltage transfer function

$$H(z) = \frac{z^6 - 8z^5 + 37z^4 - 108z^3 + 196z^2 - 200z + 100}{z^6 - 3.73z^5 + 6.07z^4 - 5.47z^3 + 2.89z^2 - 0.85z + 0.11} \tag{59}$$

which has the following zeros, a_i and poles, p_i

$$\begin{aligned} &a_1 = 1 + j, \quad a_2 = a_1^*, \quad a_3 = 2 - j, \\ &a_4 = a_3^*, \quad a_5 = 1 + 3j, \quad a_6 = a_5^* \end{aligned} \tag{60}$$

$$\begin{aligned} &p_1 = \frac{1}{2} + \frac{1}{3}j, \quad p_2 = p_1^*, \quad p_3 = \frac{2}{3} - \frac{1}{3}j, \\ &p_4 = p_3^*, \quad p_5 = 0.7 + 0.25j, \quad p_6 = p_5^*. \end{aligned} \tag{61}$$

We obtain $S_{21}(z)$ by normalizing $H(z)$ as indicated in (11), where M is chosen to be 1600, the smallest value that satisfies (11). To evaluate $S_I(z)$, we factor (7) by choosing its denominator, $D_I(z)$ to be the same as that of $S_{21}(z)$ and by solving for its numerator $N_I(z)$ from

$$N_I(z)N_{I*}(z) = D_{21}(z)D_{21*}(z) + N_{21}(z)N_{21*}(z). \tag{62}$$

The above equation has many solutions depending on the choice of the zeros. We choose the ones that are inside the unit circle to form $N_I(z)$. The zeros are

$$\begin{aligned} &z_1 = 0.50 - 0.33j, \quad z_2 = z_1^*, \quad z_3 = 0.62 - 0.32j, \\ &z_4 = z_3^*, \quad z_5 = 0.9 - 0.23j, \quad z_6 = z_5^* \end{aligned} . \tag{63}$$

The choice of $D_I(z)$ and $N_I(z)$ guarantees that $S_I(z)$ is bounded real. Under these considerations, $S_I(z)$ is given by (64) shown at the bottom of the page. We start the extraction with $S_I(z)$ as the overall reflection coefficient of the system. After the extraction of the first two complex standard lattices each of degree one, the output reflection coefficient $S_{L2}(z)$ is given by

$$S_{L2}(z) = \frac{120.43z^4 - 273.33z^3 + 249.61z^2 - 108.50z + 19.50}{150.74z^4 - 287.29z^3 + 234.26z^2 - 94.04z + 15.91} \tag{65}$$

which has real coefficients and a degree two less than that of $S_I(z)$. The transfer scattering matrix $\theta_1(z)$, also real and of degree two, and the parameters of the lattices are given by

$$\theta_1(z) = \begin{bmatrix} \frac{3.74-10.02z+7.28z^2}{2-2z+z^2} & \frac{6.04-10.22z+4.18z^2}{2-2z+z^2} \\ \frac{4.18-10.22z+6.04z^2}{2-2z+z^2} & \frac{7.28-10.02z+3.74z^2}{2-2z+z^2} \end{bmatrix} \tag{66}$$

$$\begin{aligned} &x = [1.73, 1.33 + 0.45j]^T, \\ &y = [0.72 + 1.00j, 0.29 - 0.66j]^T, \\ &k_1 = -0.77 + 0.26j, \quad k_2 = 0.29 - 0.51j. \end{aligned} \tag{67}$$

A second extraction yields $S_{L4}(z)$, $\theta_2(z)$, and the lattice parameters as follows:

$$S_{L4}(z) = \frac{17.74z^2 - 13.35z + 3.32}{27.80z^2 - 11.77z + 3.56} \tag{68}$$

$$\theta_2(z) = \begin{bmatrix} \frac{4.26-19.29z+19.02z^2}{10-8z+2z^2} & \frac{14.61-18.79z+4.18z^2}{10-8z+2z^2} \\ \frac{4.18-18.79z+14.61z^2}{10-8z+2z^2} & \frac{19.02-19.29z+4.26z^2}{10-8z+2z^2} \end{bmatrix} \tag{69}$$

$$\begin{aligned} &x = [2.69, 1.79 - 0.22j]^T, \\ &y = [0.75 - 1.88j, 0.34 - 0.07j]^T, \\ &k_1 = -0.66 - 0.08j, \quad k_2 = -0.09 + 0.14j. \end{aligned} \tag{70}$$

After the extraction of the third section, the reflection coefficient of the terminating section is $S_{L6}(z) = 0.39$ and has degree zero, an indication that no more extractions are possible. The extracted section is characterized by

$$\theta_3(z) = \begin{bmatrix} \frac{-16.76-31.90z+129.67z^2}{90-18z+9z^2} & \frac{79.97-42.66z-37.31z^2}{90-18z+9z^2} \\ \frac{-37.31-42.66z+79.97z^2}{90-18z+9z^2} & \frac{129.67-31.90z-16.76z^2}{90-18z+9z^2} \end{bmatrix} \tag{71}$$

$$\begin{aligned} &x = [3.84, 2.38 + 0.25j]^T, \\ &y = [1.72 + 2.74j, 0.88 + 0.84j]^T, \\ &k_1 = -0.62 + 0.07j, \quad k_2 = -0.36 - 0.09j \end{aligned} \tag{72}$$

Example 2: Here we synthesize an ARMA (5,5) as a cascade of two degree-two real sections and a real degree-one lattice from the following voltage transfer function

$$H(z) = \frac{z^5 - 1.05z^4 + 0.15z^3 + 0.525z^2 - 0.35z + 0.075}{z^5 - 2.08z^4 + 1.67z^3 - 0.47z^2 - 0.06z + 0.05} \tag{73}$$

whose zeros, a_i, and poles, p_i, are

$$\begin{aligned} &a_1 = 0.5 - 0.5j, \quad a_2 = a_1^*, \quad a_3 = 0.4 + 0.2j, \\ &a_4 = a_3^*, \quad a_5 = -0.75 \end{aligned} \tag{74}$$

$$\begin{aligned} &p_1 = 0.5 + \frac{1}{3}j, \quad p_2 = p_1^*, \quad p_3 = \frac{2}{3} - \frac{1}{3}j, \\ &p_4 = p_3^*, \quad p_5 = -0.25. \end{aligned} \tag{75}$$

We follow the procedure outlined in the previous example to construct $S_{21}(z)$ and $S_I(z)$, and the results are given below

$$N_{21}(z) = N(z), \qquad D_{21}(z) = 4D(z) \tag{76}$$

$$\begin{aligned} &z_1 = 0.51 + 0.30j, \quad z_2 = z_1^*, \quad z_3 = 0.60 + 0.37j, \\ &z_4 = z_3^*, \quad z_5 = -0.26 \end{aligned} \tag{77}$$

$$S_I(z) = \frac{1361.44z^6 - 5505.66z^5 + 9529.84z^4 - 9058.06z^3 + 5012.12z^2 - 1540.32z + 208.35}{1600(z^6 - 3.73z^5 + 6.07z^4 - 5.47z^3 + 2.89z^2 - 0.85z + 0.11)}. \tag{64}$$

$$S_I(z) = \frac{0.20z^5 - 0.30z^4 - 1.69z^3 + 6.5z^2 - 8.58z + 4.315\,52}{4z^5 - 8.34z^4 + 6.67z^3 - 1.89z^2 - 0.23z + 0.20}. \tag{78}$$

From $S_I(z)$ we extract the first section of degree two. The output reflection coefficient, $S_{L2}(z)$, the transfer scattering matrix, $\theta_1(z)$, and the parameters for this section are

$$S_{L2}(z) = \frac{0.42z^3 + 0.07z^2 - 2.11z + 2.15}{1.99z^3 - 2.18z^2 + 0.15z + 0.43} \tag{79}$$

$$\theta_1(z) = \begin{bmatrix} \frac{0.25-0.50z+0.501z^2}{0.25-0.5z+0.5z^2} & \frac{-0.03+0.03z-0.002z^2}{0.25-0.5z+0.5z^2} \\ \frac{-0.002+0.03z-0.026z^2}{0.25-0.5z+0.5z^2} & \frac{0.501-0.50z+0.25z^2}{0.25-0.5z+0.5z^2} \end{bmatrix} \tag{80}$$

$$\begin{aligned} x &= [0.046, -0.66 + 0.25j]^T, \\ y &= [0.03 + 0.0087j, -0.48 - 0.52j]^T, \\ k_1 &= 0.061 + 0.023j, \quad k_2 = 0.038 - 0.023j. \end{aligned} \tag{81}$$

The extraction of the second section yields the following expressions for $S_{L4}(z)$, $\theta_2(z)$, and the parameters of the lattices

$$S_{L4}(z) = \frac{-0.32z + 0.85}{0.78z - 0.38} \tag{82}$$

$$\theta_2(z) = \begin{bmatrix} \frac{0.09-0.35z+0.42z^2}{.08-0.32z+0.4z^2} & \frac{0.12-0.16z+0.044z^2}{.08-0.32z+0.4z^2} \\ \frac{0.044-0.16z+0.12z^2}{.08-0.32z+0.4z^2} & \frac{0.42-0.35z+0.088z^2}{.08-0.32z+0.4z^2} \end{bmatrix} \tag{83}$$

$$\begin{aligned} x &= [0.2, 0.89 + 0.19j]^T, \\ y &= [0.046 + 0.001j, 0.09 - 0.38j]^T, \\ k_1 &= -0.21 + 0.047j, \quad k_2 = -0.015 - 0.049j. \end{aligned} \tag{84}$$

After the third extraction, the output reflection coefficient is $S_{L5}(z) = 0.75$, and $\theta_3(z)$, and the parameters of the lattice are

$$\theta_3(z) = \begin{bmatrix} \frac{-0.39+3.46z}{1.31+1.75z} & \frac{-1.91+1.91z}{1.31+1.75z} \\ \frac{3.46-0.39z}{1.31+1.75z} & \frac{3.46-0.39z}{1.31+1.75z} \end{bmatrix} \tag{85}$$

$$x = [1.31, -1.46]^T, \quad k = 0.89. \tag{86}$$

In this example, we extracted the complex zeros of transmission first, then the real zero. The order of extraction is not imposed by the synthesis technique, however the zeros have to be extracted in such a way that the realization is real.

VII. Discussion

In this paper we developed a new method to obtain degree-one and real degree-two transfer scattering matrices of two-port lossless lattice filters through the use of complex Richard's function extractions for the minimum degree cascade synthesis of real, stable, single-input, single-output ARMA (n, m) filters, for any finite non-negative integers n and m, from the transfer function or the input reflection coefficient. The lattice realization is terminated on a constant one-port system. The technique relies on a four-step complex Richard's function extraction to calculate the transfer scattering matrices that characterize the lattices. Two of the steps are used to reduce the degree of the transfer function or the reflection coefficient of the filter through the extraction of the zeros of transmission and the other two steps are used to obtain a real realization. The synthesis is minimum, the number of lattice sections being dictated by the degree of the transfer function. Each possibly complex standard lattice section has degree one and is realized as a delay lattice of nonstandard type, followed by a constant lattice. The nonstandard lattice is composed of a cascade of a cross-arm constant gain lattice and a simple delay lattice whose structure differs depending on whether the zero of transmission being realized is located inside or outside the unit circle. This dependence is related to the degree reduction procedure. The advantage of the chosen representation is that the cascade of two complex standard lattices of degree one will always result in a real section of degree two and of the same structure, whereas the same result cannot be obtained by simply cascading two lattices of nonstandard type. The technique applies to cases where the zeros of transmission are inside or outside the unit circle. The technique, however, cannot be used when the zeros of transmission are located on the unit circle and this is because of the structure of the lattice itself. Therefore, other synthesis methods, such as the ones described in [3], that employ a different lattice structure, must be used to handle such a case. It also applies to both stable and unstable filters.

Although the transfer function of the filter is unique, its cascade synthesis through our technique is not. First, the S_{21} coefficient is not unique since it is determined through normalization of the transfer function (12), which is not unique. Secondly, the reflection coefficient S_I is not unique as a result of a nonuniqueness in the selection of the zeros of transmission (10). Thirdly, the order in which the zeros of transmission are extracted is not unique. In some practical applications, such as the synthesis of the scattering model of the human cochlea as a cascade of digital lattice filters [8], the order in which the zeros of transmission are extracted is crucial and their pre-determination through other means, such as mathematical modeling or experimental data, is necessary since the proposed technique does not impose this order in any way, except that complex conjugate pairs must be extracted one after the other. Fourthly, the coefficients of the transfer scattering matrix of each lattice depends on the choice of the vector parameter x, whose components, obtained in (45) and (55), can be chosen real or complex, and if complex can take on many values since the phase is arbitrary.

In cases where the filter being realized models a lossy system, the realization generates lossless lattices and puts the loss term in the terminating section. But because of the type of lattice structure used, the realization can be converted to a lossy one by distributing the loss term among the lattices of the realization, which makes the technique suitable for both lossy and lossless realizations. The technique can be generalized to nonlinear realizations, by suitably inserting nonlinear factors between lattices, as well as to multiport systems.

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Louiza Sellami received the Ingeniorat d'Etat from the Ecole Polytechnique d'Alger, Algeria, in 1984, and the M.S and the Ph.D. degrees from the University of Maryland, College Park, in 1988 and 1992, respectively.

Currently, she is an Assistant Professor with the Department of Electrical Engineering of the U.S Naval Academy. She is also affiliated with the Microsystems Laboratory, University of Maryland. Her research interests include Cochlea modeling and characterization, haircell modeling, system identification, lattice filter synthesis, semistate theory, neural networks, and fuzzy logic.

Robert W. Newcomb was born in Glendale, CA, in June 1933. He received the B.S.E.E. degree from Purdue University, West Lafayette, IN, in 1955, the M.S. degree in electrical engineering from Stanford University, Stanford, CA, in 1957, and the Ph.D. degree from the University of California, Berkeley, in 1960.

After serving on the tenured faculty at Stanford University, he joined the University of Maryland to update the graduate program in electrical engineering. He has held visiting appointments in Australia, Belgium, Malaysia, Spain, and most recently, Singapore. He directs the Microsystems Laboratory of the University of Maryland where emphasis is placed upon VLSI realization techniques. His present research is in the areas of biologically motivated neural networks, their VLSI realization, DSP for the ear, and general systems theory. For more information consult the world wide web entry: http://www.enee.umd.edu/newcomb/mslab.htm.