Two-Limit-Cycle Piecewise Linear Oscillator

T.-W. Yang, S.Hadjipanteli, R. W. Newcomb, and R. deFigueiredo
Microsystems Laboratory
Electrical Engineering Department
University of Maryland
College Park, MD 20740, USA

Abstract

Here we develop a piecewise linear two-limit-cycle oscillator by specializing the general theory of deFigueiredo for n-limit cycle systems. After a review of deFigueiredo's theory we specialize that theory to the case of piecewise linear nonlinearities to obtain a set of design conditions for n-limit cycles. Using these conditions a two-limit-cycle example is presented along with simulation results from SPICE. The results could be useful for reconfiguring a control system that needs to switch between two limit cycles.

1 Introduction

Liénard equations with autonomous nonlinear oscillations have been widely discussed in the literature [1, 2, 3, 4, 5]. In [1] it was proven that Liénard type of equations can have n-periodic solutions and the criteria for this was discussed. The property of n-periodic solutions is very promising for systems such as multiple valued logic control ones. [6, 7]

In this paper, we adapt the theory in [1] to the case of piecewise linear nonlinearities for which the same design criteria hold. The scaling scheme is also developed to prepare for eventual VLSI implementation. A second order circuit, a pair of capacitors with a pair of linear voltage-controlled current sources and one nonlinear voltage-controlled current source, is presented and shown to have two limit cycles by simulation using SPICE and confirmed by MATLAB simulation.

The design criteria in [1] are briefly described in the following Section. In Section 3, our piecewise linear nonlinearities are presented. The scaling scheme is proposed in Section 4 and the SPICE and MATLAB simulation results for the two-limit-cycle example are presented in Section 5. Conclusions are given in the final Section.

2 Design Criteria for n-Limit-Cycle Oscillations

The Liénard nonlinear differential equation can be represented as

$$\frac{dx}{dt} = y \qquad (1a)$$
$$\frac{dy}{dt} = -F(y) - x, \qquad (1b)$$

where $F(.)$ is a real-valued, continuous and locally Lipschitzian function.

Given $F(y)$ we search for positive numbers, B_{k-1}, M_0, M_k and y_k, that satisfy the criteria to be stated below (see equation(3)). Here M_0 and M_k are the absolute values of slopes of bounding linear curves and the B_{k-1} are the absolute values of their vertical axis intercepts, as illustrated in Fig. 1; the y_k are numbers chosen to satisfy equations (2) below.

For $k = 1, \ldots, 2n-2$, $M_0 < min\{2, M_k\}$ must hold and the y_k are positive constants satisfying

$$y_{k-1} < y_k \, , y_0 = 0 \qquad (2a)$$
$$y_k \geq N_k B_{k-1} \qquad (2b)$$

Here (2b) is the critical equation with the constants N_k, depending upon M_0 and M_k, being calculated according to the appendix.

For equations (1) to posses at least n periodic solutions, one of the following sets of conditions must hold.

If $dF(0)/dy > 0$, then the conditions for n periodic solutions (limit cycles) are

$$(-1)^{k-1}F(y) < B_{k-1} - M_k y \, , \, y_{k-1} < y < y_k \qquad (3a)$$

where $k = 1, \ldots, 2n-1$. In the other case, $dF(0)/dy < 0$, the conditions are

$$(-1)^{k-1}F(y) > -B_{k-1} + M_k y \, , \, y_{k-1} < y < y_k \qquad (3b)$$

where $k = 1, \ldots, 2n-2$. The situation (3b) is illustrated in Fig. 1.

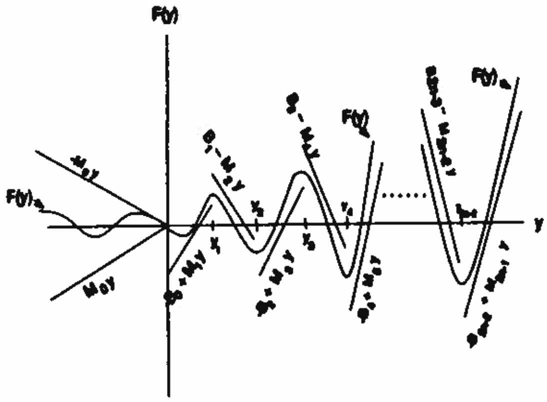

Figure 1: $F(y)$ as bounded by a set of linear segments

3 Piecewise Linear Nonlinearity

In the previous section, to get the n limit cycles, the set of linear curves bounding $F(y)$ is decided. Here we specialize the $F(y)$ to be a continuous piecewise linear function $G(y)$ a typical one being shown in Fig. 2 where the break points are $Y_i, i = -1, 0, 1, \ldots, 2n-2$. For $i \geq 1$, we choose

$$y_i = Y_i \tag{4}$$

which y_i are shown in Fig. 1. To design a $G(y)$ for n

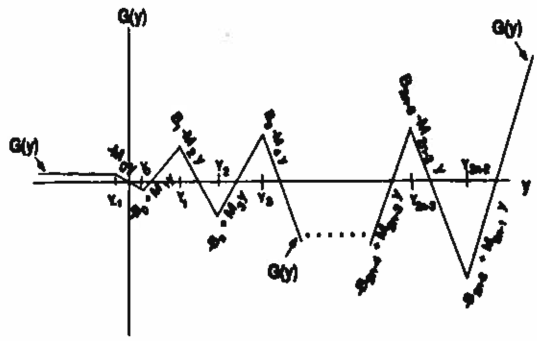

Figure 2: The piecewise linear $G(y)$

limit cycles with the case(3b) the following steps are provided:

1. Choose $0 < M_0 < min\{2, M_k\}$, $M_k > 0$, $Y_{-1} < 0$ and $Y_0 > 0$, where $k = 1, \ldots, 2n-1$. Choose $G(Y_{-1}) > 0$ and for $y \leq Y_{-1}$, the constant value

$$G(y) = G(Y_{-1}) \tag{5}$$

Calculate $N_k(M_0, M_k)$ according to the appendix.

2. For $Y_{-1} < y \leq Y_0$, $G(y)$ passes the origin by the choice

$$G(y) = -M_0 y \tag{6}$$

3. Decide B_0 by

$$B_0 = (M_0 + M_1)Y_0 \tag{7}$$

4. For each $i = 1, \ldots, 2n-2$, in sequence choose Y_i such that

$$Y_i \geq N_i B_{i-1} > Y_{i-1} \tag{8}$$

Since $Y_i = y_i$ for $i > 0$ this decides the y_k along the y'axis. To optimize the design, Y_i can be set equal to $N_i B_{i-1}$. Calculate B_i according to

$$B_i = -B_{i-1} + (M_i + M_{i+1})Y_i \tag{9}$$

and repeat for the next i.

Once the above designing steps are finished, for analytic study and circuit construction, it is useful to represent the piecewise linear function $G(y)$ in the following form

$$G(y) = m_0 y + b_0 + \sum_{i=-1}^{2n-2} a_i |y - Y_i| \tag{10}$$

for which the following three constraints necessarily hold.

1.

$$m_0 = \sum_{i=-1}^{2n-2} a_i \tag{11}$$

2.

$$a_{-1} = -M_0/2 \tag{12a}$$

$$a_i = (-1)^{i+1}(M_{i-1} + M_i)/2, \quad i = 1, \ldots, 2n-1. \tag{12b}$$

3.

$$b_0 = a_{-1}Y_{-1} - \sum_{i=0}^{2n-2} a_i Y_i \tag{13}$$

The case of (3a) is essentially the same. But it should be noted that the origin is stable in the case when $dF(0)/dy > 0$, case (3a), and unstable in case (3b), so that the case (3b) is more useful for an oscillator.

As mentioned in this section, the bounding curves are merged into the actual curve $G(y)$. The resulting $G(y)$ generally results in too large Y_{2n} for practical devices and, hence, scalings are needed.

4 Scaling

To implement the n limit cycles in practical circuits, the following scaling scheme is developed. The variables, x, y and t in equation (1), where $F(y)$ is replaced by $G(y)$, are de-normalized to v_x, v_y and τ by

$$x = c_x v_x \quad (14a)$$
$$y = c_y v_y \quad (14b)$$
$$t = T\tau \quad (14c)$$

Substitution of these de-normalized variables into equation (1) results in

$$\frac{c_x}{T}\frac{dv_x}{d\tau} = c_y v_y \quad (15a)$$
$$\frac{c_y}{T}\frac{dv_y}{d\tau} = -G(c_y v_y) - c_x v_x. \quad (15b)$$

These are convenient for the circuit realization of Fig. 3 which is derived by

$$c_1\frac{dv_1}{d\tau} = g_{12}v_2 \quad (16a)$$
$$c_2\frac{dv_2}{d\tau} = -g(v_2) - g_{21}v_1 \quad (16b)$$

where $v_i s$ are voltages, $g_{ij}s$ are transconductances of linear voltage-controlled current sources, $g(v_2)$ is a nonlinear voltage dependent current sources and $c_i s$ are capacitance.

Identifying the circuit with the equations we have

$$v_x = v_1,\ v_y = v_2 \quad (17a,b)$$
$$\frac{Tc_y}{c_x} = \frac{g_{12}}{c_1},\ \frac{Tc_x}{c_y} = \frac{g_{21}}{c_2} \quad (17c,d)$$
$$\frac{TG(c_y v_y)}{c_y} = \frac{g(v_2)}{c_2} \quad (17e)$$

In (17e), if we follow (10) and let

$$G(c_y v_y) = m_0 c_y v_y + b_0 + \sum_{i=-1}^{2n-2} a_i|c_y v_y - Y_i| \quad (18)$$

then the scaled piecewise linear function for the practical circuit becomes

$$g(v_2) = Tc_2(m_0 v_2 + \frac{b_0}{c_y} + \sum_{i=-1}^{2n-2} a_i|v_2 - \frac{Y_i}{c_y}|) \quad (19)$$

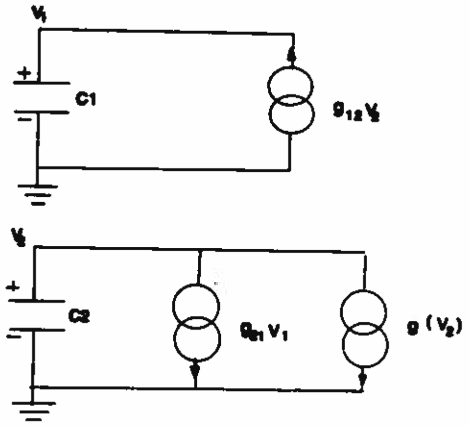

Figure 3: The circuit with $g(v_2)$

Observing (19) we see that we can adjust the break points, Y_i, by the factor of c_y. Tc_2 allows us to also change the amplitude of $g(v_2)$. Therefore by adjusting c_y and Tc_2, $G(y)$ can be freely adjusted for different circuit characteristics, also allowing the size of the limit cycle to be adjusted for practical use.

Next we adjust for circuit parameters by choosing suitable normalization constants T, c_y and c_x to fit practical c_1, c_2, g_{12} and g_{21}. We mulitiply (17c) by (17d) to give

$$T^2 = \frac{g_{12}g_{21}}{c_1 c_2} \quad (20)$$

while dividing (17c) by (17d) gives

$$(\frac{c_y}{c_x})^2 = \frac{g_{12}c_2}{g_{21}c_1} \quad (21)$$

From (20) and (21), the circuit parameters can be decided. For example, if we desire $c_1 = c_2 = c$ and $g_{12} = g_{21} = g$ then $c_x = c_y$ and $T = g/\tau$.

5 Example

As an example we consider a second order circuit with a pair of capacitors and two linear and one nonlinear voltage-controlled current sources, as shown in Fig. 3. To design a two-limit-cycle oscillator, we choose case (3b) and follow the steps mentioned in Section 3. Here $n = 2$

1. We arbitrarily set $M_0 = 0.1$, $M_k = 1.9$, for $k = 1, 2, 3 = 2n-1$, $Y_{-1} = -1$ and $Y_0 = 0.5$. Thus for $y \leq -1$, $G(y) = 0.1$

2. For $Y_{-1} \leq y < Y_0$, $G(y) = -0.1y$ by (6)
3. $B_0 = (M_0 + M_1)Y_0 = 1$ by (7)
4. For each $i = 1, 2, 3$ in sequence, choose

$$Y_i = N_i B_{i-1}$$

where

$$N_1(M_0, M_1) = N_2(M_0, M_2) = N_3(M_0, M_3) = 0.826$$

are calculated from the appendix by applying the case for $M_k < 2$. By (9)

$$B_i = -B_{i-1} + (M_i + M_{i+1})Y_i$$

The following values are obtained in sequence,

$$Y_1 = N_1 B_0 = 0.826$$

and $B_1 = -B_0 + (M_1 + M_2)Y_1 = 2.139$

$$Y_2 = N_2 B_1 = 1.767$$

and $B_2 = -B_1 + (M_2 + M_3)Y_2 = 4.576$

After finishing the above calculations, the piecewise linear function representation of (10) is presented by

$$G(y) = 0.95y - 2.238 - 0.05|y + 1| + |y - 0.5|$$
$$-1.9|y - 0.826| + 1.9|y - 1.767| \qquad (22)$$

and plotted in Fig. 4.

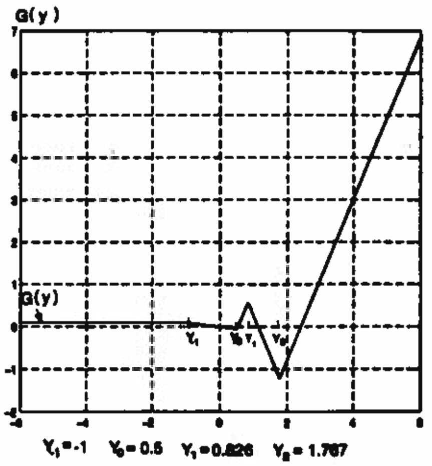

Figure 4: $G(y)$ for the example

To design the oscillator we choose $c_1 = c_2 = c = 10pFd$ and $g_{12} = g_{21} = g = 10^{-4}Mho$ as practical parameters for the VLSI process. Then $T = g/c = 10^7$ and $c_x = c_y$ are to be chooosen as per the scaling scheme in Section 4. Noting that the largest Y_i is $Y_2 = 1.767$, if we desire the largest breakpoint no bigger than 0.2, we choose $c_y = 10 = c_x$ which gives

$$g(v_2) = 10^{-4}(0.95v_2 - 0.2238 - 0.05|v_2 + 0.1| + |v_2 - 0.05|$$
$$-1.9|v_2 - 0.0826| + 1.9|v_2 - 0.1767|) \qquad (23)$$

This has all previous Y_k scaled by the factor of 0.1 and the amplitude brought to the tenth of milliamp range. After scaling, the two scaled limit cycles are shown in Fig. 5 as obtained by SPICE simulation.

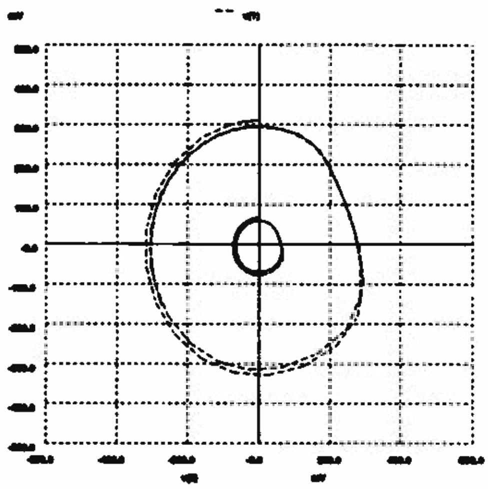

Figure 5: SPICE simulation results for the two-limit-cycle scaled oscillator

Furthermore, to check our results, MATLAB is applied. Figure 6 shows the $G(y)$ and two limit cycles obtained from MATLAB simulation before applying the scaling scheme. The scaled limit cycles and $g(v_2)$ are also plotted in Fig. 7.

6 Conclusions

In this paper, we show how to design a piecewise linear nonlinearity $G(y)$ for a n -limit-cycle oscillator. Also, the scaling scheme for adjusting the piecewise linear nonlinearity to obtain practical circuits is proposed. A design example for the two-limit-cycle case is presented and the scaling scheme is applied. The results by SPICE and MATLAB simulation are shown and seen to be as we predict. As a result, to implement a VLSI system with n limit cycle oscillator becomes feasible for the near future.

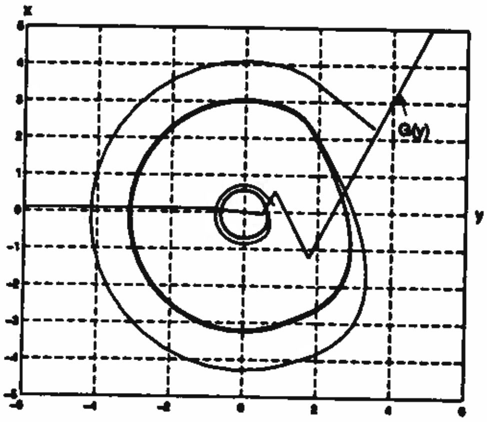

Figure 6: $G(y)$ and unscaled two limit cycles using MATLAB

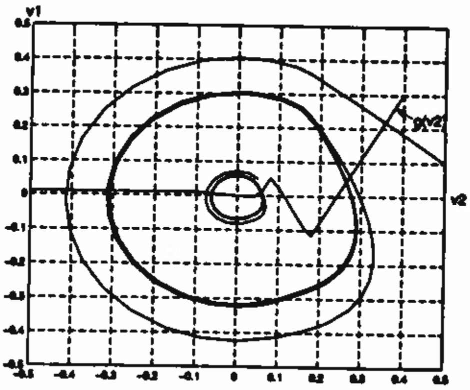

Figure 7: Scaled two limit cycles using MATLAB

A Appendix-Calculation of N_k

First define

$$\omega_0 = (1 - \frac{M_0^2}{4})^{\frac{1}{2}} \qquad (A.1)$$

and then calculate

$$\rho_0 = exp(-\frac{\pi M_0}{2\omega_0})^{\frac{1}{2}} \qquad (A.2)$$

According to the size of M_k we have three cases for the determination of N_k:

If $M_k < 2$,

$$\omega_k = (1 - \frac{M_k^2}{4})^{\frac{1}{2}} \qquad (A.3)$$

$$\rho_k = exp(-\frac{\pi M_k}{2\omega_k})^{\frac{1}{2}} \qquad (A.4)$$

$$N_k(M_0, M_k) = \frac{(\rho_0 + 1)\rho_k}{\rho_0 - \rho_k} exp[\frac{M_k}{2\omega_k}(\pi - \phi)], \qquad (A.5)$$

where

$$\phi = \arctan(\frac{2\omega_k}{M_k}), (0 < \phi < \frac{\pi}{2}) \qquad (A.6)$$

If $M_k = 2$,

$$N_k(M_0, M_k) = \frac{(\rho_0 + 1)}{\rho_0} \exp^{-1} \qquad (A.7)$$

If $M_k > 2$,

$$N_k(M_0, M_k) = \frac{(\rho_0 + 1)}{\rho_0} [\frac{r_1^{r_1}}{r_2^{r_2}}]^{\frac{1}{r_1 - r_2}} \qquad (A.8)$$

where,

$$r_1, r_2 = \frac{M_k}{2} \pm (\frac{M_k^2}{4} - 1)^{\frac{1}{2}} \qquad (A.9)$$

References

[1] R. J. P. deFigueiredo, "On the Existence of N Periodic Solution of Liénard's Equation," *Nonlinear Analysis: Theory, Methods and Applications*, Vol. 7, No. 5, pp. 483-499, May, 1983.

[2] S. Lefshetz, *Differential Equations:. Geometric Theory*. Interscience, New York, 1957.

[3] P. Le Corbeiller, "Two-Stroke Oscillators," *IRE Transaction on Circuit Theory*, Vol. 7, No. 12, pp. 387-398, December, 1960.

[4] N. Wax, "On Some Periodic Solutions of the Liénard's Equation," *IEEE Transcations on Circuit Theory*, Vol. 13, No. 4, pp.419-423, December, 1966.

[5] R. J. P. deFigueiredo, "Existence and Uniqueness Results for Liénard's Equation," *IEEE Transcation on Circuit Theory*, Vol. CT-17, No.3, pp.313-321, August, 1970.

[6] S. L. Hurst, "Two Decades of Multiple-Valued Logic - An Invited Tutorial," *Proceedings The Eighteenth International Symposium on Multiple-Valued Logic*, Palma de Mallorca, Spain, pp.164-175, May, 1988.

[7] K. C. Smith, "The Prospects for Multiple-Valued Logic: A Technology and applications View," *IEEE Transaction on Computer*, Vol. C-30, No. 9, pp. 619-632, September, 1981.

Proceedings of

IEEE Singapore International Conference on Intelligent Control and Instrumentation (IEEE SICICI '95)

1 – 8 July 1995
Shangri-La Hotel, Singapore

Organised by:

Control Chapter, IEEE Singapore

Technical Co-Sponsor:

IEEE Control Systems Society (CCS)