SYNTHESIS OF PASSIVE NETWORKS FOR NETWORKS ACTIVE AT p_o-I

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Summary

If a one or two port network has $q_+(p_o) \leq 0$, it is possible to obtain a passive embedding network to obtain a natural frequency at p_o. This will be demonstrated for those networks which possess an admittance matrix Y and for which the skew-symmetric imaginary part of Y is zero at p_o. The method consists of obtaining canonical forms for Y which can be easily treated. To exhaust all possibilities over twenty distinct cases are treated.

Introduction

Problem

Until recently the design of active circuits has been a semi-haphazard process. One usually followed the designs of previous workers, incorporating small improvements which mainly resulted from trial and error. However in 1957, Thornton, [1], initiated a study of the limitations of the natural frequencies of such devices. This was followed in early 1960 by the work of Desoer and Kuh, [2]. Thornton's paper is concerned with determining the possible natural frequencies of an active resistive device with parasitic capacitance which is embedded in a transformer network. Although some special results are presented, a general treatment is only touched upon. In contrast Desoer and Kuh develop a criterion for an arbitrary active device, embedded in a passive network, to possess a natural frequency. Their criterion is that $q_+(p_o) \leq 0$ at the required frequency, p_o, [2, p. 15]. Thus one now knows a restriction on the switching speed of a flip-flop, say, whereas previously this was determined experimentally.

However, more is desired. We would : be able to synthesize a passive network su that, when a given active device is embedd it, a desired, natural frequency results. is the subject of this paper. In particu wish to investigate the following questio

"Given an (active) network for which $q_+(p_o) \leq 0$ for p_o in Re $p \geq 0$ does there e: finite passive embedding network such tha two networks combined support a natural f: quency at p_o? If so, how is the passive network obtained?"

Using this we will solve the above p: by exhibiting a passive network for the o: and a large class of two ports. The solu the remaining class of two ports has been obtained and will be reported at a later d

Review of q_+ and Conventions

Consider an n-port N which is assume possess an admittance matrix Y(p). Now le excited by the voltage vector $v(t) = Ve^{pt}$ is a vector of complex constants and p = c Let a superscript tilde, $\tilde{}$, denote matrix position, a superscript asterisk, *, deno complex conjugation and $Y_H(p)$ denote the Hermitian part of Y.

We now define, for $\sigma \geq 0$,

$$Q_+(V,p) = \begin{cases} \tilde{V}^* Y_H(p)V + (\sigma/|p|)|\tilde{V}Y(p)V| & \text{if } \sigma \\ \tilde{V}^* Y_H(p)V & \text{if } \sigma \end{cases}$$

here $|\ |$ denotes the absolute value of a complex number. Physically, if $\sigma \neq 0$, $\sigma^{-1}e^{2\sigma t}Q_+$ represents the upper limit on energy into N for a given v(t) at a given instant. Desoer and Kuh work with, [2, p. 15],

$$q_+(p) = \min_{||V|| = 1} Q_+(V,p) \qquad (2)$$

where for $\tilde{V}=[V_1,\ldots,V_n]$ we have $||V||^2 = \Sigma|V_i|^2$. From the meaning of Q_+ we see that $\sigma^{-1}e^{2\sigma t}q_+$ represents the smallest of the upper limits on the energy into N at a given instant for all normalized non-zero V. From the meaning attached to q_+ it should be physically clear that q_+ should depend only upon the device and not the mode of description. In other words we should be able to define q_+ even though a Y (or Z) matrix doesn't exist. Such a quantity is clearly obtained from

$$\mathcal{Q}_+(V,I,p) = \begin{cases} (\tfrac{1}{2})[\tilde{V}^*I+\tilde{I}^*V]+(\sigma/|p|)|\tilde{V}I| & \text{if } \omega\neq 0 \\ (\tfrac{1}{2})[\tilde{V}^*I+\tilde{I}^*V] & \text{if } \omega=0 \end{cases} \qquad (3)$$

By the nature of our problem we must base our work on q_+, but it is important to note that if we find some non-zero V for which $Q_+ \leq 0$ then $q_+ \leq 0$. Thus, if in a specific instance, we are only interested in the fact that q_+ is non-positive and not in its exact value, we may profit by using Q_+ and avoid the tedious job of finding a minimum.

Now consider an n-port N which is connected in parallel to a passive n-port N_p. The combined networks form a new n-port N_o whose terminal pairs are taken as the common terminal pairs of N and N_p. We say that N supports a mode $v(t)=Ve^{p_o t}$, $\sigma_o \geq 0$, if the voltage v(t) can appear across N_o when the terminal currents of N_o are zero. Such an N has been called active at p_o, [2, p. 4]. If N and N_p have admittance matrices Y and Y_p then the following results have l established.

1. [2,p.15] If N is active at p_o tl necessarily $q_+(p_o) \leq 0$.
2. [2,p.7] N is active at p_o if an there exists some N_p such that $\det[Y(p_o)+Y_p(p_o)] = 0$.

The second of these results is the key synthesis methods, since it gives an an way of determining if we have solved ou problem. It should be pointed out that definition of natural frequency used he from the usual one which rests upon ini ditions in energy storage elements, sin work for which the determinant is zero such as an open circuit, can support an This introduces a subtlety which is exp upon in Example E-1.

For transforming one network into a will have use for ideal transformer net gyrators. Consider two networks N_C and admittance matrices are related by

$$Y_C = \tilde{T}YT$$

where T is a real matrix. Then N_C is o connecting a transformer network to N, and [4,p.301]. The notation for the gy be clarified. Let

$$E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Then the polarities for the gyrator are by Fig. 1 for which

$$Z = \gamma E \qquad \gamma = \text{gyration resistance}$$

$$Y = -\gamma^{-1}E$$

We will also adhere to the following notation. 1_n will denote the unit matrix of order n, 0_n the corresponding zero matrix and $\dot{+}$ will mean the direct sum of two matrices. Further we will assume, as in [2], that Y(p) has rational elements with real coefficients.

Method

We will assume that a network N is given which possesses an admittance matrix Y at p_o and for which $q_+(p_o) \leq 0$. Clearly a dual situation holds if only an impedance matrix Z exists. In part II we will show how to obtain a Z or Y if neither exists, for n=2. Consequently, for n=2, the assumption on Y is no restriction. At p_o, Y is a matrix of complex numbers and we write

$$Y = Y_{RS} + Y_{RSS} + jY_{IS} + jY_{ISS} \qquad (7)$$

where the subscripts R and I refer to real and imaginary parts and S and SS refer to symmetric and skew-symmetric matrices.

In this paper the synthesis of the passive network N_p will be given for $Y_{ISS}=0$. By the use of transformers and gyrators we will transform N into a canonical network N_C. A passive network N_{PC} will be obtained for N_C; the passive network for N will then consist of N_{PC} and the transforming network as illustrated in Fig. 2.

From the physical interpretation of q_+ it should be clear that if N can be obtained from N_C then N and N_C have the same q_+. This will be justified analytically for the actual transformations that we will use.

As a consequence of the canonical forms actually used, we will have many cases to consider. For some of the cases several synthesis methods are available for N_{PC}. In this paper we will present only the simplest synthesis methods, some of the alternates can be found in [6]. At the very beginning we can assume $Y_{RSS}=0$, since it can be absorbed in N_C. We will usually do this, but in cases where fewer gyrators will be used, Y terms will appear in the canonical forms.

By glancing at Q_+, it can readily be appreciated that we must consider two regi ω_o. If $\omega_o \neq 0$, there is no loss in generali assuming $\omega_o > 0$ since Y(p) has real coeffici

Synthesis of N_p: n=1

Here Y is a scalar which at p_o can be written as

$$Y(p_o) = g + jb$$

Region 1: $\omega_o=0$

As Y is real at p_o, b=0. The conditi $q_+(p_o) \leq 0$ gives $g \leq 0$. We then let $Y_p(p)=-g$.

Region 2: $\omega_o>0$

The condition $q_+(p_o) \leq 0$ now requires

$$g \leq 0; \ (\sigma_o b)^2 \leq (\omega_o g)^2$$

If $\sigma_o=0$ we cancel b by an inductance or a capacitance and g by a positive resistance $\sigma_o>0$ we form

$$Y_p(p)=(\tfrac{1}{2})[(-g/\sigma_o)-(b/\omega_o)]p +$$

$$(\tfrac{1}{2})[(-g/\sigma_o)+(b/\omega_o)](\sigma_o^2+\omega_o^2)/p$$

Y_p is positive real as a result of the con straints of Eq. 8. It should be noticed t the second of Eq. 9 is equivalent to the a constraints for positive real functions, [5,p.114]. Alternative Y_p are easily foun they may not hold for all allowed g and b this one does.

We can now appreciate a difficulty wh occur. It may happen that two different a networks have the same admittance matrix a When N_p is connected to these, the resulti determinant may be identically zero for all one while merely falling to zero at p_o for the other. The latter situation is the one actually desired, but, since we can only assume $Y(p_o)$ known, we can not tell which situation occurs in general. Of course if the properties of a device are known for all p we can actually check to see what happens. This is illustrated by the following example.

E-1: Let N have $Y(p)=-1$ and consider $p_o=(\frac{1}{2})+j(\sqrt{3/2})$. If we choose N_P to have $Y_1(p)=+1$ then $Y+Y_1=0$ for all p. Then N supports $e^{p_1 t}_o$ even though no energy storage elements need to be considered. In contrast let N_P have $Y_2(p)=p+1/p$ then $Y(p_o)+Y_2(p_o)=0$ but this is not true for all p. Now consider another active network N' described by $Y'=-Y_2(p)$. Then at p_o, N' and N are indistinguishable. However, $Y'+Y_1$ has only an isolated zero at p_o.

Synthesis of N_P: $n=2$, $Y_{ISS}=0$

We recall that we will generally assume $Y_{RSS}=0$ as Y_{RSS} can be lumped in N_C. Clearly q_+ remains the same before and after Y_{RSS} is deleted, since Q_+ is independent of Y_{RSS}. With this assumption we will generally transform $Y(p_o)$ to a canonical form $Y_C(p_o)$ through the use of Eq. 4, with T non-singular. This operation also leaves q_+ invariant since V in Q_+ is replaced by TV which assumes all values with V. In two cases the canonical form will require a cascade connection of gyrators in addition to the transformers. For these situations the invariance of q_+ is proven in Appendix 1.

Region 1: $\omega_o=0$

Here $Y(p_o)=Y_{RS}+Y_{RSS}$ and we need use no gyrators. We diagonalize Y_{RS} to obtain $Y_C'(p_o)=[g_1 \dotplus g_2]+gE$. The condition $q_+\leq 0$ requires that at least one of g_1 or g_2 be ≤ 0; through our diagonalization process we can assume it to be g_1. We then short port two to obtain a one port with $Y_C(p_o)=g_1$. A natural mode results by forming, for all p, $Y_{PC}=-g_1$.

Region 2: $\omega_o>0$

If $\sigma_o=0$ we again don't require any gyrators. In this case we diagonalize $Y_C'(p_o)=[g_1 \dotplus g_2]+gE+jY_I'$ with $g_1\leq 0$ by q_+ then short port two to obtain a one port $Y_C(p_o)=g_1+jb$. b is then canceled by an inductance or a capacitance and g by a resistance.

If $\sigma_o>0$ we are apparently forced t the following mutually exclusive cases. Unfortunately there are many subcases, leading to a different canonical form. note that Y_{RS} cannot be positive defini $q_+\leq 0$.

Case 1: Y_{RS} positive semi-definit (rank 0 or 1)

Case 2: Y_{RS} negative definite (ra

Case 3: Y_{RS} negative semi-definit

Case 4: Y_{RS} indefinite (rank 2)

Case 1: Y_{RS} positive semi-definite (ra

Clearly $q_+=0$. If Y_{RS} has rank zer diagonalize Y_{IS} to get $Y_C'(p_o)=gE+j[b_1 \dotplus b$ $q_+=0$ then requires that at least one of be zero; by our transformation we can a to be b_1. Shorting port two then gives and we have used no gyrators.

If Y_{RS} has rank one, we can first Y_{RS} to $1 \dotplus 0$. The requirement $q_+=0$ the that the (2,2) term of the transformed zero. If the (1,2) term of the new Y_{IS} zero we have $Y_C(p_o)=[(1+jb)\dotplus 0]$ which has determinant. If the (1,2) term is not further transform by adding the second column to the first (with a proper mult to have the (1,1) term zero in the new Thus we have arrived at

$$Y'(p_o) = \begin{bmatrix} 1 & jb \\ jb & 0 \end{bmatrix}$$

We now connect a gyrator as shown in Fig. 3 to obtain (see Appendix 2)

$$Y_C(p_o) = \begin{bmatrix} 1 & jb \\ -jb & b^2 \end{bmatrix}$$

which has a zero determinant.

Case 2: Y_{RS} negative definite (rank 2)

We simultaneously diagonalize Y_{RS} and Y_{IS} to get, [7,p.107],

$$Y_C(p_o)=-1_2+gE+j[b_1\dot{+}b_2] \qquad (11)$$

Such an admittance always has $q_+<0$ since we can find a non-zero V such that $|\tilde{V}Y_CV|=0$, Appendix 3. Consequently there are no constraints on b_1 and b_2. We have three subcases to consider.

Case 2_a: $b_1b_2=0$

There is no loss in generality in assuming $b_1=0$. Then we let, for all p,

$$Y_{PC}=[1\dot{+}0]-gE$$

to obtain a zero determinant.

Case 2_b: $b_1b_2>0$

We let, for all p,

$$Y_{PC}=\{[1+(b_1/b_2)]\dot{+}0\} + \{[(b_1/b_2)(b_2^2+1)]^{\frac{1}{2}}-g\}E$$

which yields a zero determinant.

Case 2_c: $b_1b_2<0$

We derive a new canonical form from Eq. 11 by normlaizing the imaginary (2,2) term to $-b_1$.

$$Y_C'(p_o)=[-1\dot{+}(b_1/b_2)]+g'E+j[b_1\dot{+}(-b_1)]$$

where $g'=(-b_1/b_2)^{\frac{1}{2}}g$. We first add, for all p,

$$Y_{PC}'=1\dot{+}(-b_1/b_2)$$

We then add all elements of the resulting to obtain a zero input admittance. This corresponds to connecting port one to port is illustrated in Fig. 4.

It should be noted that if, in Eq. 11 and $0<|b_i|\leq(\omega_o/\sigma_o)$ for i=1 or 2 then we ca passive network to port i to get a zero de ant. This would then avoid the gyrator in 2_b. The following will exhibit a simple C synthesis while clarifying the general pro to be used.

E-2: Let N be the network so denoted in F for which

$$Y(p) = \begin{bmatrix} -1+p & 1 \\ 1 & -4+p+(1/p) \end{bmatrix}$$

Let $p_o=(1/2)+j(\sqrt{3}/2)$, then

$$Y(p_o)= \begin{bmatrix} (-1/2)+j(\sqrt{3}/2) & 1 \\ 1 & -3 \end{bmatrix}$$

We find, using

$$T = \begin{bmatrix} 0 & \sqrt{6} \\ 1/\sqrt{3} & \sqrt{2/3} \end{bmatrix}$$

that $Y_C(p_o)=\tilde{T}Y(p_o)T=-1_2+j[0\dot{+}3\sqrt{3}]$. The Case synthesis then gives Fig. 5.

Case 3: Y_{RS} negative semi-definite (r

We first diagonalize Y_{RS} to obtain

$$Y'(p_o)=[(-1)\dot{+}0]+g'E+j\begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}$$

From this we obtain three canonical forms de ing upon the vanishing or non-vanishing of t and b_{12}.

Case 3_a: $b_{22}\neq 0$

In Eq. 12 we can add the second row and column to the first to eliminate the (1,2) term. If the (1,1) term of the new imaginary part is non-zero, we can normalize the (2,2) term to equal the (1,1), except possibly for sign. We have then

$$Y_C(p_o)=[(-1)\dot{+}0]+gE+j[b_1\dot{+}b_2] \quad \text{where } b_1=\pm b_2 \text{ or } 0$$

Here $g=g'$ if $b_1=0$ or $g=\sqrt{\pm b_1/b_2}\,g'$ otherwise. Because we can find a V with $V_1\neq 0$ such that $|\tilde{V}Y_C V|=0$, this case always has $q_+<0$. Thus there is no restriction on b_1. However, we have two further cases to consider as far as synthesis is concerned.

Case 3_{a1}: $b_1 b_2\geq 0$

We here add, for all p,

$$Y_{PC}=[1\dot{+}0]+(b_1-g)E$$

to obtain a zero determinant.

Case 3_{a2}: $b_1 b_2<0$

We here add, for all p,

$$Y'_{PC}=1\dot{+}0$$

and then apply feedback by connecting port one to port two to obtain a zero input admittance (compare with Case 2_c).

Case 3_b: $b_{22}=0$ [in Eq. 12]

Here we again have two further subcases, this time depending on b_{12}.

Case 3_{b1}: $b_{12}=0$

Assuming that $g'=0$, Eq. 12 then takes the form

$$Y_C(p_o)=[(-1+jb)\dot{+}0] \quad \text{where } b=b_{11}$$

Here Y_C already has a zero determinant which corresponds to $Q_+=0$ with $V_1=0$. Consequently b is not constrained if $q_+=0$. However, if $q_+<0$, Eq. 9 shows that $b^2<(\omega_o/\sigma_o)^2$ and synthesis c results from Eq. 10.

Case 3_{b2}: $b_{12}\neq 0$

We now add the second row and colu Eq. 12 to the first and then normalize term to obtain

$$Y_C(p_o)=[(-1)\dot{+}0]+gE+j\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

where $g=g'/b_{12}$. Again $q_+<0$, since ther zero V such that $|\tilde{V}Y_C V|=0$. We force th minant to zero by adding, for all p,

$$Y_{PC}=[0\dot{+}(1+g^2)]$$

The following example will serve t illustrate a Case 3 synthesis.

E-3: Let N be as illustrated in Fig. 6.

$$Y(p)=\begin{bmatrix}-2+p & 3-p\\ -3-p & -3+2p\end{bmatrix}$$

Let $p_o=1+j1$, then

$$Y(p_o)=-\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}+j\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}+3E$$

A Case 3_{a1} synthesis is required. Using with

$$T=(1/5)\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}$$

we obtain

$$Y_C(p_o)=[(-1)\dot{+}0]+j(1/5)1_2+(3/5)E$$

The final network is shown in Fig. 6. It should be noted that we could replace the gyrator-resistor network by a gyrator L-C network in this case.

Case 4: Y_{RS} indefinite (rank 2)

Depending upon the rank of Y_{IS} we now have several cases.

Case 4_a: Y_{IS} of rank zero

We can diagonalize Y_{RS} to obtain

$$Y_C(p_o)=[1\dot{+}(-1)]+gE$$

We then add rows and columns (connect port one to port two) to get a zero input admittance. It should be noted that here we always have $q_+<0$.

Case 4_b: Y_{IS} of rank one

We begin by diagonalizing Y_{IS}

$$Y'(p_o)=\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} +j[b_{11}\dot{+}0]+g'E \qquad (13)$$

From this will be derived three canonical forms depending upon the value of g_{22}.

Case 4_{b1}: $g_{22}>0$

After assuming $g'=0$ we now use elementary transformations to add the second row and column of Eq. 13 to the first and normalize to obtain

$$Y_C(p_o)=[(-1)\dot{+}1]+j[b\dot{+}0]$$

The requirement $q_+\leq 0$ yields $b^2\leq(\omega_o/\sigma_o)^2$, as is seen by choosing $V_2=0$. Using Eq. 10 we add a passive network to port one of N_C to obtain a zero determinant.

Case 4_{b2}: $g_{22}<0$

Using the same procedure as in the previous case we obtain

$$Y_C(p_o)=[1\dot{+}(-1)]+j[b\dot{+}0]$$

This clearly has $q_+<0$ (choose $V_1=0$), and for all p,

$$Y_{PC}=0\dot{+}1$$

to obtain a zero determinant.

Case 4_{b3}: $g_{22}=0$

Since Y_{RS} is indefinite we have $g_{12}\neq$ Using the transformation method of the pr two cases we can arrive at

$$Y_C(p_o) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} +j[b\dot{+}0]+gE$$

Here we always have $q_+<0$ as is shown in A 3. This is substantiated by the fact tha normalize any non-zero b to ± 1. We here for all p,

$$Y_{PC}=-(g+1)E$$

to obtain a zero determinant.

Case 4_c: Y_{IS} of rank two

We must divide this case into two fu cases depending upon whether Y_{IS} is defin or indefinite. The second of these calls rather elaborate synthesis methods.

Case 4_{c1}: Y_{IS} definite

To obtain a canonical form we simult diagonalize Y_{RS} and Y_{IS} and then normaliz to obtain

$$Y_C(p_o)=[1\dot{+}(-1)]+j[b_1\dot{+}b_2]+gE$$

We have two regions for b_2 which are of i

Case $4_{c1\alpha}$: $b_2^2<(\omega_o/\sigma_o)^2$

Here we always have $q_+<0$, as is seen choosing $V_1=0$. After cancelling gE by a g we synthesize a passive network by Eq. 10

Case $4_{c1\beta}$: $b_2^2 \geq (\omega_o/\sigma_o)^2$

If $q_+=0$, Appendix 3 shows that we require $b_2^2=(\omega_o/\sigma_o)^2$. As a consequence, a synthesis for $q_+=0$ follows that of the preceding case.

If $q_+<0$, Appendix 3 shows that we require $b_1^2>b_2^2$ (Appendix 3 also shows that $\sigma_o[(1+b_1^2)^{1/2}+(1+b_2^2)^{1/2}]>2|p_o|$ and that there exists a non-zero V such that $|\tilde{V}Y_C V|=0$). Because Y_{IS} is definite we have $(b_1/b_2)>0$ and we can force the determinant to zero by adding, for all p,

$$Y_{PC}=[(1/b_2)(b_1-b_2)\dot{+}0]+\{[(b_1/b_2)(b_2^2+1)]^{1/2}-g\}E$$

Case 4_{c2}: Y_{IS} indefinite

We begin by diagonalizing Y_{RS} to get

$$Y'(p_o)=[1\dot{+}(-1)]+j\begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}+gE \qquad (15)$$

Let $B=[b_{ij}]$ be the second matrix on the right. We have three cases depending upon the form into which B can be brought by the congruency transformations of Theorem 1 of Appendix 4 (the subscripts on the following B's refer to the corresponding matrices in Appendix 4).

Case $4_{c2\alpha}$: $B=B_I=[b_{11}\dot{+}b_{22}]$

Letting $b_{11}=b_1$, $b_{22}=b_2$, Eq. 15 is identical to Eq. 14 except that instead of $(b_1/b_2)>0$ we now have $(b_1/b_2)<0$. The same subcases occur that were present in Case 4_{c1}.

Case $4_{c2\alpha 1}$: $b_2^2<(\omega_o/\sigma_o)^2$

The properties and the synthesis method are identical to that of Case $4_{c1\alpha}$.

Case $4_{c2\alpha 2}$: $b_2^2 \geq (\omega_o/\sigma_o)^2$

We have the same properties as in Case $4_{c1\beta}$ except that $(b_1/b_2)<0$. We now revert to that case, after first assuming g=0 in Eq. 15. To obtain this result we connect a gyrator in cascade with port one (as we did in Fig. 3). The new Y matrix is obtained by using the results of Appendix 2, and then multiplying the first r[ow] and column by $1+b_1^2$; we get

$$Y_C(p_o)=[1\dot{+}(-1)]+j[(-b_1)\dot{+}b_2]$$

which now is of the form required for Case 4[...]. It should be noticed that in general this me[thod] uses three gyrators. However, after pulling [the] shunt resistor through a gyrator, the final [...] these are in cascade and can be replaced by [a] transformer. This procedure is illustrated [in] example E-4 which follows the remaining case.

Case $4_{c2\beta}$: $B \Rightarrow B_{II}=\begin{bmatrix} b_{11}-b_{22} & \sqrt{-\Delta_b} \\ \sqrt{-\Delta_b} & 0 \end{bmatrix}$; Δ_b=det

Appendix 4 shows that there is a non-si[ngu]lar real T such that

$$Y''(p_o)=\tilde{T}Y'(p_o)T=[1\dot{+}(-1)]+jB_{II}+g''E$$

From Eq. 16 we will obtain two canonical for[ms] depending upon $b_{11}-b_{22}$, by elementary transformations.

Case $4_{c2\beta 1}$: $b_{11}-b_{22}\neq 0$

We first add $-\sqrt{-\Delta_b}/(b_{11}-b_{22})$ times the row and column of Eq. 16 to the second. We normalize by multiplying the second row and column by $[(b_{11}-b_{22})/\sqrt{-\Delta_b}]$ to get

$$Y_C(p_o)=\begin{bmatrix} 1 & -1 \\ -1 & 1-\{(b_{11}-b_{22})^2/(-\Delta_b)\} \end{bmatrix}+$$

$$j(b_{11}-b_{22})[1\dot{+}(-1)]+g_cE$$

Recalling that $\Delta_b<0$ by assumption, we now ad[d] for all p,

$Y'_{PC}=[\{(b_{11}-b_{22})^2/(-\Delta_b)\}\dot{+}0]$

and then add rows and columns (connect port one to port two) to obtain a zero input admittance.

Case $4_{c2\beta 2}$: $b_{11}-b_{22}=0$

We first add 1/2 of the second row and column of Eq. 16 to the first. Following this we subtract the first row and column from the second to get

$$Y_c(p_o)=\begin{bmatrix}3/4 & -5/4\\ -5/4 & 3/4\end{bmatrix}+j\sqrt{-\Delta_b}\,[1\dot{+}(-1)]+g_cE \qquad (18)$$

We now add, for all p,

$Y'_{PC}=[1\dot{+}0]$

and then add rows and columns (connect port one to port two) to obtain a zero input admittance.

It should be noticed that the $b_{11}-b_{22}=0$ and $\neq 0$ cases can be taken care of by a single case. This results from adding $[(1/2\sqrt{-\Delta_b})(b_{22}-b_{11}+\sqrt{-\Delta_b})]$ times the second row and column of Eq. 16 to the first and then subtracting the first row and column from the second. However, the canonical form is not as neat as those of Eqs. 17 and 18. Further, since we have always found an N_p when $B\Rightarrow B_{II}$, the first result of Desoer and Kuh quoted above shows that q_+ is always ≤ 0 here. In fact choosing $V_1=V_2$ shows that $q_+<0$. An example of Case $4_{c2\beta}$ will be given as part of an example in part II.

Case $4_{c2\gamma}$: $B\Rightarrow B_{III}=\begin{bmatrix}0 & -\sqrt{-\Delta_b}\\ -\sqrt{-\Delta_b} & b_{22}-b_{11}\end{bmatrix}$; $\Delta_b=\det. B$

This is the final and worst case. By Appendix 4 we find a real, non-singular T such that Eq. 15 becomes

$$Y''(p_o)=\tilde{T}Y'(p_o)T=[1\dot{+}(-1)]+jB_{III}+g''E \qquad (19)$$

Here we can assume that $b_{22}-b_{11}\neq 0$, as otherwise this is covered by the treatment for B_{II}. We then have two subcases.

Case $4_{c2\gamma 1}$: $(b_{22}-b_{11})^2\leq -4\Delta_b$ (recall $\Delta_b<0$ by assumption)

We follow the procedure used to obtain Eq. 17. Thus we first add $\sqrt{-\Delta_b}/(b_{22}-b_{11})$ times the second row and column to the first and then normalize by multiplying the first row and column by $[(b_{22}-b_{11})/\sqrt{-\Delta_b}]$. This gives

$$Y_c(p_o)=j(b_{22}-b_{11})[(-1)\dot{+}1]+gE$$

$$+\begin{bmatrix}-1+[(b_{22}-b_{11})^2/(-\Delta_b)] & -1\\ -1 & -1\end{bmatrix}$$

We now add, for all p,

$Y'_{PC}=[\{4-[(b_{22}-b_{11})^2/(-\Delta_b)]\}\dot{+}0]$

and then add rows and columns to obtain a zero input admittance. Note that again we always have $q_+\leq 0$, since we have found an N_p.

Case $4_{c2\gamma 2}$: $(b_{22}-b_{11})^2>-4\Delta_b$

We will reduce this to Case $4_{c2\alpha}$. We apply Theorem 2 of Appendix 4, which shows that there is a real, non-singular T_c such that Eq. 19 takes the form

$$Y_c(p_o)=\tilde{T}_cY''(p_o)T_c=[1\dot{+}(-1)]+j[b_1\dot{+}b_2]+g_cE \qquad (20)$$

In fact we have

$$b_1\ (\text{or } b_2)=[-2t\sqrt{-\Delta_b}+t^2(b_{22}-b_{11})]/(1-t^2) \qquad (21)$$

$$b_2\ (\text{or } b_1)=[(b_{22}-b_{11})-2t\sqrt{-\Delta_b}]/(1-t^2)$$

$$t=(1/2)[(b_{22}-b_{11})\pm\sqrt{(b_{22}-b_{11})^2+4\Delta_b}]/\sqrt{-\Delta_b}$$

Eq. 20 now falls under the description of Case $4_{c2\alpha}$. Thus if $q_+ \leq 0$ that case applies and gives a synthesis. However, it has not yet been determined under what constraints on the b_{ij}, satisfying $(b_{22}-b_{11})^2 > -4\Delta_b$, $q_+ \leq 0$. Example E-6 shows that q_+ may be >0, and E-5 shows that $q_+ \leq 0$ can also occur.

The following instructive examples illustrate the Case 4 synthesis.

E-4: Let N be as shown in Fig. 7. Then

$$Y(p)=[(5-4p)\dot{+}(-3+2p)]$$

Let $p_o=1+j1$ then

$$Y(p_o)=[(1-j4)\dot{+}(-1+j2)]$$

which requires a Case $4_{c2\alpha 2}$ synthesis. Connecting a gyrator in cascade and multiplying the first row and column of the resulting matrix by $\sqrt{17}$ we get

$$Y_C(p_o)=[(1+j4)\dot{+}(-1+j2)]$$

Connecting

$$Y_{PC}=[1\dot{+}0]+\sqrt{10}E$$

in parallel yields a zero determinant. After pulling the resistor through the gyrator, Fig. 7 shows the final realization. Here the transformer and the two gyrators in cascade have also been replaced by their transformer equivalent.

E-5: Let N be the network shown in Fig. 8. Then

$$Y(p)=\begin{bmatrix}1 & (2/5)+(2p/5)\\ (2/5)+(2p/5) & -2+p\end{bmatrix}$$

and for $p_o=1+j5$ we have

$$Y(p_o)=[1\dot{+}(-1)]+j\begin{bmatrix}0 & -2\\ -2 & 5\end{bmatrix} \qquad (22)$$

We have Case $4_{c2\gamma 2}$ with t=2 or 1/2. Let t=2 then with

$$T=(1/\sqrt{3})\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

we get

$$Y'=\tilde{T}YT=[(1-j1)\dot{+}(-1+j4)]$$

which is treated by Case $4_{c2\alpha 1}$. The fina[l] is shown in Fig. 8. Consequently N is ac[...] p_o and Case $4_{c2\gamma 2}$ actually exists.

E-6: Consider the network of Fig. 9. Th[e]

$$Y(p)=\begin{bmatrix}1 & 2-2p\\ 2-2p & -6+5p\end{bmatrix}$$

Let $p_o=1+j1$ then $Y(p_o)$ is the same as giv[en] Eq. 22. Using the same transformation, E[...] valid. However, now $4=b_2>\omega_o/\sigma_o=1$ and $b_1^2=1<16=b_2^2$. Consequently $q_+>0$ at p_o and [...] passive network exists.

Conclusions

We have shown how a one or two port [...] which possesses an admittance matrix at p_o [...] $q_+(p_o) \leq 0$ and $Y_{ISS}=0$ can be embedded in a [...] passive network to yield a natural frequen[cy] p_o. The general philosophy has been to si[multan]eously put Y_{RS} and Y_{IS} into a canonical fo[rm] the use of transformers. This then led to [...] cases, each of which must apparently be co[nsider]ed separately. However, the synthesis met[hods] are in general combinations of the followi[ng]

1) Connection of a shunt L-C circuit
2) Connection of a shunt resistor.
3) Connection of a cascade gyrator.
4) Connection of port one to port tw[o] (feedback).
5) Shorting one of the ports (to sav[e a] gyrator).

In part II we will show how to remove [the] restrictions of $Y_{ISS}=0$ and the existence o[f] $Y(p_o)$ for n=2. Further most solutions for [...] but $Y_{ISS}=0$, can be obtained. The remainin[g] for n>2 are at present under study.

Acknowledgements

The author is indebted to Professors E. S. Kuh and C. A. Desoer who proposed the problem and have contributed many of the ideas through stimulating discussions. Also the assistance of B. Colosman in pointing out numerical errors in the examples of [6] has been appreciated. Further the support of the National Science Foundation is gratefully acknowledged.

Appendices

1: Invariance of q_+ for the Cascade Gyrator

Let a gyrator of gyration resistance γ be connected in cascade with port one of a network N' as shown in Fig. 3. For Eq. 3 we find

$$\mathcal{Q}_+(V_c,I_c,p)=\mathcal{Q}_+(V',I',p)$$

since $V_{c1}I_{c1}=(-\gamma I_1')(-V_1'/\gamma)=V_1'I_1'$. Assuming N' to have a Y matrix, Appendix 2 shows that N_C also has a Y_C matrix (if $y_{11}'\neq 0$). As a consequence q_+ is the same for both N' and N_C.

2. Determination of Y for the Cascade Gyrator

We wish to find Y_C for N_C of Fig. 3. Here we can write the chain matrix

$$\begin{bmatrix} V_c \\ I_c \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V' \\ I' \end{bmatrix}, \quad I'=Y'V'$$

where A, B, C, D are 2x2 matrices. Solving this we can obtain $Y_C=[C+DY'][A+BY']^{-1}$

Applying this to Fig. 3 we find, assuming a gyration resistance γ, $A=D=0\dot{+}1$; $B=-\gamma\dot{+}0$; $C=(-1/\gamma)\dot{+}0$

If $y_{11}'\neq 0$, these equations are easily evaluated to give

$$Y_C=\begin{bmatrix} 1/(\gamma^2 y_{11}') & y_{12}'/(\gamma y_{11}') \\ -y_{21}'/(\gamma y_{11}') & (y_{22}'y_{11}'-y_{21}'y_{12}')/y_{11}' \end{bmatrix}$$

These results are extended to arbitrary n in [6].

3: Q_+ for Various Cases

Here we will prove some of the statements made about Q_+ and q_+.

a) Case 2: We have $Y_C=-1_2+j[b_1\dot{+}b_2]+gE$ or

$$Q_+=-|V_1|^2-|V_2|^2+(\sigma_o/|p_o|)|V_1^2(-1+jb_1)+V_2^2(-1+jb_2)|$$

Choosing

$$(V_1/V_2)^2=-(-1+jb_2)/(-1+jb_1)$$

gives $|\tilde{V}Y_CV|=0$. As a consequence $q_+=-1$, no matte[r] what values b_1 and b_2 assume.

b) Case 4_{b3}: We have $Y_C=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}+j[b\dot{+}0]+gE$ or

$$Q_+=2\mathrm{Re}(V_1^*V_2)+(\sigma_o/|p_o|)|jbV_1^2+2V_1V_2|$$

$$=|V_1|^2\{2\mathrm{Re}(V_2/V_1)^*+(\sigma_o/|p_o|)|jb+2(V_2/V_1)|\}$$

if $V_1\neq 0$

We now choose

$(V_2/V_1)=u-j(b/2)$ with u arbitrary but <0

Then

$$Q_+=(-2u)[-1+(\sigma_o/|p_o|)]<0$$

As a consequence we know $q_+<0$, independently of b

c) Case 4_{c1}: We have $Y_C=[1\dot{+}(-1)]+j[b_1\dot{+}b_2]+gE$.

We wish to find the constraints put on b_1 and b_2 by $q_+\leq 0$.

$$Q_+=|V_1|^2-|V_2|^2+(\sigma_o/|p_o|)|V_1^2(1+jb_1)+V_2^2(-1+jb_2)|$$

Let $|V_1|^2=\epsilon|V_2|^2$; $\emptyset_i$=phase of $V_i^2(1+jb_i)$, i=1 and

Since we wish to minimize Q_+ we require $0\leq\epsilon<1$, $\emptyset_1=\emptyset_2+\pi$. Then restricting Q_+ to $|V_1|^2+|V_2|^2=1$, we have

$$Q_+=\frac{1}{1+\epsilon}\{-1+\epsilon+(\sigma_o/|p_o|)|(1+b_2^2)^{\frac{1}{2}}-\epsilon(1+b_1^2)^{\frac{1}{2}}|\}$$

We have two cases to consider.

Case I: $b_1^2\leq b_2^2$: Then

$$Q_+ = \frac{1}{1+\epsilon}\left[-1+\frac{\sigma_o}{|p_o|}(1+b_2^2)^{\frac{1}{2}}\right]-\epsilon\left[-1+\frac{\sigma_o}{|p_o|}(1+b_1^2)^{\frac{1}{2}}\right]$$

$$\geq[(1-\epsilon)/(1+\epsilon)][-1+(\sigma_o/|p_o|)(1+b_2^2)^{\frac{1}{2}}]$$

Consequently for $q_+\leq 0$ we clearly require $b_2^2\leq(\omega_o/\sigma_o)^2$. Further, this inequality on b_2 is sufficient to insure $q_+\leq 0$ as is seen by taking $\epsilon=0$.

Case II: $b_1^2>b_2^2$: Letting $\epsilon=[(1+b_2^2)/(1+b_1^2)]^{\frac{1}{2}}$, then

$$(1+\epsilon)Q_+ =$$

$$\begin{cases}\{[-1+\frac{\sigma_o}{|p_o|}(1+b_2^2)^{\frac{1}{2}}]+\epsilon[1-\frac{\sigma_o}{|p_o|}(1+b_1^2)^{\frac{1}{2}}]\} & \text{for } \epsilon<\epsilon_o \\ \{[-1-\frac{\sigma_o}{|p_o|}(1+b_2^2)^{\frac{1}{2}}]+\epsilon[1+\frac{\sigma_o}{|p_o|}(1+b_1^2)^{\frac{1}{2}}]\} & \text{for } \epsilon\geq\epsilon_o\end{cases}$$

On differentiating we find

$$(1+\epsilon)^2[dQ_+/d\epsilon] =$$

$$\begin{cases}2-(\sigma_o/|p_o|)[(1+b_1^2)^{\frac{1}{2}}+(1+b_2^2)^{\frac{1}{2}}] & \text{for } \epsilon<\epsilon_o \\ 2+(\sigma_o/|p_o|)[(1+b_1^2)^{\frac{1}{2}}+(1+b_2^2)^{\frac{1}{2}}] & \text{for } \epsilon\geq\epsilon_o\end{cases}$$

Two situations can occur.

Case II_α: $\sigma_o[(1+b_1^2)^{\frac{1}{2}}+(1+b_2^2)^{\frac{1}{2}}]\leq 2|p_o|$

In this case $(dQ_+/d\epsilon)\geq 0$ for all ϵ and hence the minimum occurs at $\epsilon=0$. This requires $b_2^2\leq(\omega_o/\sigma_o)^2$ for $q_+\leq 0$.

Case II_β: $\sigma_o[(1+b_1^2)^{\frac{1}{2}}+(1+b_2^2)^{\frac{1}{2}}]>2|p_o|$

Here $dQ_+/d\epsilon$ changes from negative to positive (as ϵ increases) at $\epsilon=\epsilon_o$. The minimum is then at ϵ_o This always has $q_+<0$, since at $\epsilon=\epsilon_o$, $(1+\epsilon_o)Q_+=-1+\epsilon_o<0$.

From these results we conclude that $q_+<0$ and $b_2^2\geq(\omega_o/\sigma_o)^2$ then $b_1^2>b_2^2$.

4: Canonical Forms for Two Indefinite Ma

Theorem 1: Let $G=[1\dot{+}(-1)]$ and let $B=[b_{ij}$ real, symmetric, indefinite matrix. Then exists a real, non-singular matrix T such $\tilde{T}GT=G$ and $\tilde{T}BT$ is one of the following mat

$$B_I=[b_{11}\dot{+}b_{22}],\ B_{II}=\begin{bmatrix}b_{11}-b_{22} & \sqrt{-\Delta_b}\\ \sqrt{-\Delta_b} & 0\end{bmatrix},\ B_{III}=\begin{bmatrix}0 & -\sqrt{-\Delta_b}\\ -\sqrt{-\Delta_b} & b_{22}-b_{11}\end{bmatrix}$$

where Δ_b=det. B.

Proof: Consider the two matrices

$$T_1=(1/\sqrt{1-t^2})\begin{bmatrix}1 & t\\ t & 1\end{bmatrix} \text{ if } t^2<1$$

$$T_2=(1/\sqrt{t^2-1})\begin{bmatrix}t & 1\\ 1 & t\end{bmatrix} \text{ if } t^2>1$$

where t will later be defined. Then we h $\tilde{T}_1GT_1=\tilde{T}_2GT_2=G$ and

$$(1-t^2)\tilde{T}_1BT_1=$$

$$\begin{bmatrix}b_{11}+2tb_{12}+t^2b_{22} & b_{12}(1+t^2)+(b_{11}+b_2\\ b_{12}(1+t^2)+(b_{11}+b_{22})t & b_{11}+2tb_{12}+t^2b_{22}\end{bmatrix}$$

$$(t^2-1)\tilde{T}_2BT_2=$$

$$\begin{bmatrix}b_{22}+2tb_{12}+t^2b_{11} & b_{12}(1+t^2)+(b_{11}+b_2\\ b_{12}(1+t^2)+(b_{11}+b_{22})t & b_{11}+2tb_{12}+t^2b_{22}\end{bmatrix}$$

Now if $b_{12}=0$, $B=B_I$ already and if $b_{22}=0$ then $B=B_{II}$ already. Thus assume that $b_{12}\neq 0$ and $b_{22}\neq 0$, then we choose t such that one of the diagonal members in Eq. a.2 is zero, i.e., choose

$$t=-(1/b_{22})[b_{12}\pm\sqrt{-\Delta_b}]$$

Since $b_{12}\neq 0$ and $\Delta_b<0$, we can choose the $\pm$ sign in t such that $t^2\neq 1$. We then choose the T of the theorem to be one of T_1, T_2 depending upon the value of t^2. After perhaps normalizing the (1,2) elements by -1, B_{II} or B_{III} results as is seen by calculating det. B=det. $\tilde{T}B\tilde{T}$. Q.E.D.

Theorem 2: Let G and $B=B_{III}$ be given, as in Theorem 1, and let $(b_{22}-b_{11})^2>-4\Delta_b$. Then there exists a real, non-singular matrix T such that $\tilde{T}GT=G$ and $\tilde{T}B_{III}T$ is diagonal.

Proof: Consider Eqs. a.1 and a.2 where in this latter we make the replacement

$$b_{11}\Rightarrow 0 \qquad b_{12}\Rightarrow -\sqrt{-\Delta_b} \qquad b_{22}\Rightarrow b_{22}-b_{11}$$

We then choose t in Eq. a.2 to make the (1,2) term zero. Such a t is

$$t=(1/2)[(b_{22}-b_{11})\pm\sqrt{(b_{22}-b_{11})^2+4\Delta_b}]/\sqrt{-\Delta_b}$$

t is real and $\neq 1$ by the assumption made on $b_{22}-b_{11}$. Q.E.D.

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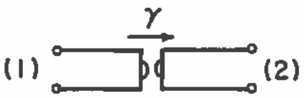

Fig. 1. Gyrator notation.

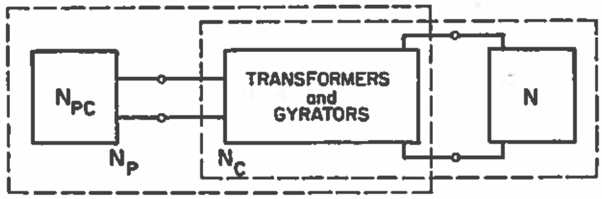

Fig. 2. Construction of N_C and N_P.

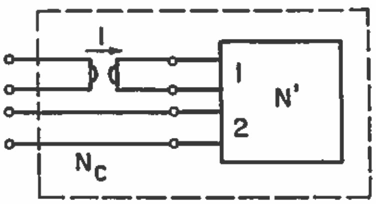

Fig. 3. Gyrator connection to obtain Y_C.

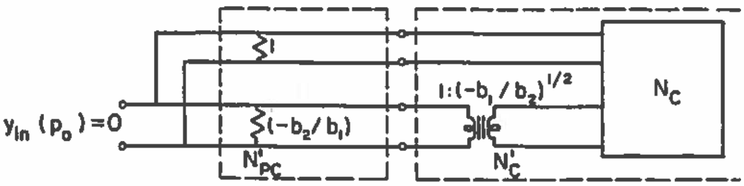

Fig. 4. Illustration of Case 2_c synthesis.

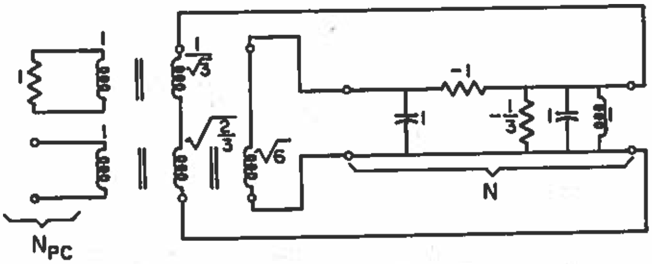

Fig. 5. Networks for E-2.

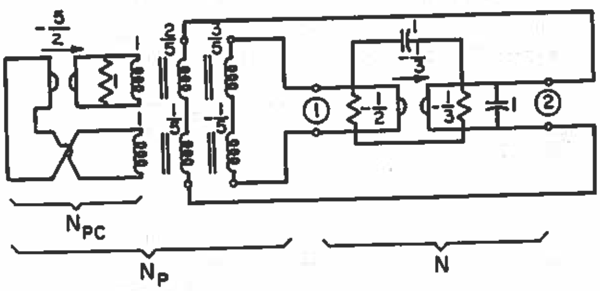

Fig. 6. Networks for E-3.

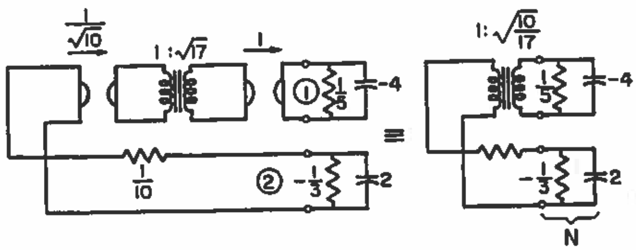

Fig. 7. Networks for E-4.

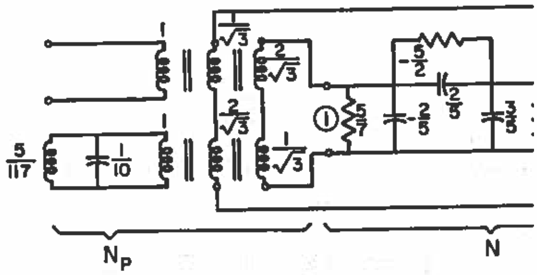

Fig. 8. Networks for E-5.

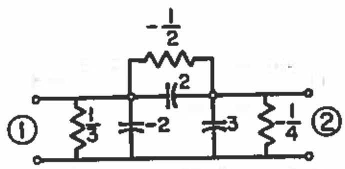

Fig. 9. Network for E-6.

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