McMILLAN'S RECIPROCAL n-PORT SYNTHESIS
SIMPLIFIED THEORY

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Prologue:

> Come on a visit to my countree
> where McMillan's result is reviewed;
> The treatment is short and concise
> with a structure quite simply imbued.

1. INTRODUCTION

In 1948 McMillan presented [1] and in 1952 [2] he published in full detail one of the first general synthesis methods for finite, linear, time-invariant, passive, reciprocal, n-ports. At about the same time Oono [3] and Tellegen [4] presented syntheses, which, like McMillan's, are based upon the Brune process [5], while recently Belevitch [6] and ourselves [7], [8], have used the same type of ideas to relax the reciprocity constraint. Of these various methods McMillan's is probably conceptually the simplest, however, this simplicity is masked by the somewhat complicated details of the proof. Some simplications of this proof have been given by Bayard [9, p. 388], but the somewhat vague ideas are treated with nowhere near the rigor of McMillan.

In this paper we further simplify the proof of the main step in McMillan's synthesis, called by him the "pièce de résistance" [2, p. 558]. Besides giving a new proof of this step, we give a concrete physical structure which uses uncoupled inductors and capacitors in conjunction with transformers. The preliminary steps are only quickly reviewed, since they are elegently covered in McMillan.

2. SYNTHESIS

Consider an $n \times n$ PR impedance matrix $Z(p)$; that is, $Z(p)$ is 1) symmetric, 2) rational in the complex frequency variable $p = \sigma + j\omega$, 3) analytic in $\sigma > 0$, 4) real when $p = \sigma > 0$, and 5) the Hermitian part of $Z(p)$ is positive semidefinite in $\sigma > 0$. Since Z is PR, any poles on $\sigma = 0$ are simple with positive semidefinite residue matrices. Further, the inverse of any PR matrix is again PR. The first step in the synthesis is to remove any poles on $\sigma = 0$ of Z and then $\sigma = 0$ poles of the inverse of the remainder, repeating until a PR matrix Z_r is obtained for which neither Z_r nor Z_r^{-1} has poles on $\sigma = 0$. Any non-zero, singular matrices met in this process are transformed into nonsingular ones, bordered by zeros, by using real, constant, congruency transformations. Assuming Z_r of order n, if the rank of the real part of $Z_r(j\omega)$ is n its rank is lowered at some ω_o, $0 \leq \omega_o \leq \infty$, by a resistance extraction. McMillan's extraction can be improved upon by using the method descirbed first by Oono [3, p. 168] and later by Tellegen [4, p. 4]. If $\Delta_{11}(\omega)$ and $\Delta(\omega)$ are the one-one minor and the determinant of the real part of $Z_r(j\omega)$, respectively, one determines r = minimum over ω of $[\Delta(\omega)/\Delta_{11}]$ and then one forms the PR matrix $Z_m(p) = Z_r(p)-[r\dotplus 0_{n-1}]$ where $\dotplus$ denotes the direct sum and 0_{n-1} the zero matrix of order $n-1$; the real part of $Z_m(j\omega_o)$ has rank $n-1$. If Z_m^{-1} has any $\sigma = 0$ poles, the above procedure is repeated; this will always be the case if $\omega_o = 0$ or ∞. The mathematical details of the above steps, except for the use of r, and their physical meaning are adequately discussed in McMillan [2, p. 541-588].

The synthesis is then reduced to the realization of a PR impedance matrix, assumed $n \times n$ and written as $Z(p)$, which a) has no poles of it or its inverse on $\sigma = 0$, b) has its real part singular at $p = p_o = j\omega_o$ and c) is nonsingular at every p on $\sigma = 0$ (and hence also in $\sigma > 0$). We now follow McMillan, with slight terminology changes intended to be more physically suggestive. We have $Z(p_o) = R(\omega_o) + jX(\omega_o)$ with R and X real and symmetric. $X(\omega_o)$ is diagonalized by a real congruency transformation to give $X(\omega_o) = \tilde{W}DW$ where the tilde denotes matrix transposition. Then the constant diagonal matrix D is written as $D = D_+ - D_-$ where D_+ and D_- are also diagonal with all entries nonnegative. The following two positive semidefinite matrices are next defined

$$\omega_o L_1 = \tilde{W} D_+ W \tag{1a}$$

$$D_1 = \omega_o \tilde{W} D_- W \tag{1b}$$

and then the following impedance matrix is formed

$$Z^{(2)}(p) = Z(p) + pL_1 + D_1/p \tag{2}$$

Then $Z^{(2)}(p)$, being the sum of PR matrices, is PR. Further $Z^{(2)}(p_o) = R(\omega_o)$, by direct calculation, and hence $Y^{(2)}(p) = [Z^{(2)}(p)]^{-1}$ has a pole at p_o; this inverse exists as is seen by letting $p = 1$. As a result of the real coefficients a pole exists at $-p_o$, and combining residue matrices one writes

$$Y^{(2)}(p) = [Z^{(2)}]^{-1} = \frac{2pG}{p^2+\omega_o^2} + Y^{(3)}(p) \tag{3}$$

where $Y^{(3)}$ is finite at p_o, PR, and nonsingular in p; G is real, symmetric and positive semidefinite. Then we can write

$$Z^{(3)}(p) = [Y^{(3)}]^{-1} = Z^{(4)}(p) + pL_3 + D_3/p \tag{4}$$

where $Z^{(4)}$ is PR and finite at 0 and ∞. L_3 and D_3 are nonzero with L_1 and D_1 and, besides being positive semidefinite, satisfy an important constraint which is seen by multiplying (3) on the left by $Z^{(2)}$ and on the right by $Z^{(3)}$ to get

$$[Z(p) - Z^{(4)}(p)] + p[L_1 - L_3] + (1/p)[D_1 - D_3] = \frac{2pZ^{(2)}(p)GZ^{(3)}(p)}{p^2 + \omega_o^2} \tag{5}$$

Multiplying this by p and $1/p$ and letting p tend to 0 and ∞ respectively, gives

$$D_1 - \frac{2D_1GD_3}{\omega_o^2} - D_3 = 0_n \tag{6a}$$

$$L_1 - 2L_1GL_3 - L_3 = 0_n \tag{6b}$$

From this we see that $D_1 = D_3[1_n - 2GD_3/\omega_o^2]^{-1}$ and $L_1 = L_3[1_n - 2GL_3]^{-1}$, and, thus, D_1 and D_3 have the same rank, r_c, and L_1 and L_3 have the same rank, r_ℓ; here 1_n is the $n \times n$ identity matrix.

A circuit for obtaining Z in terms of $Z^{(4)}$ is shown in Fig. 1 where we define

$$\Gamma_2 = 2G \tag{7a}$$

$$C_2 = 2G/\omega_o^2 \tag{7b}$$

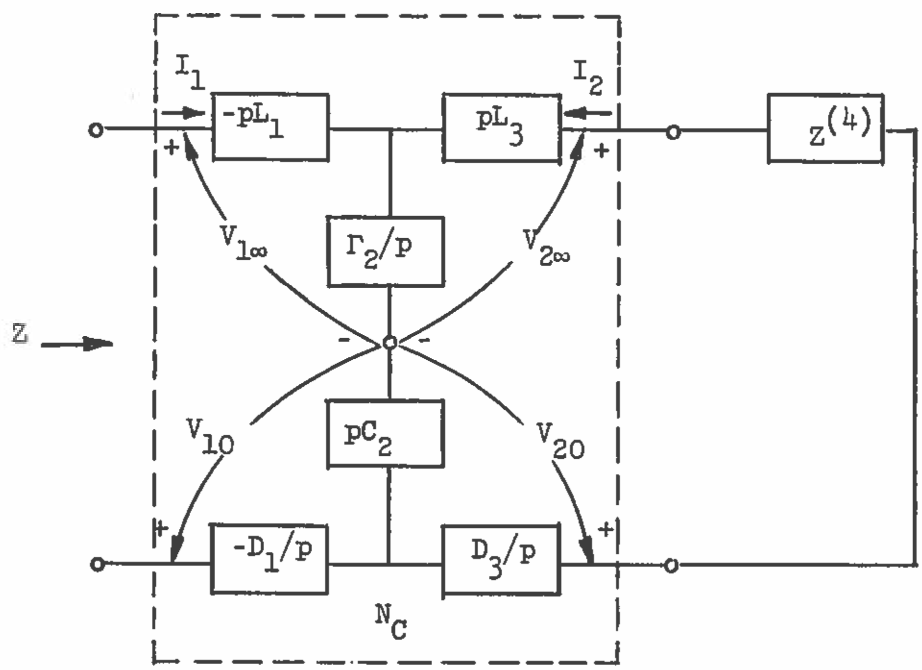

Figure 1. Initial Realization of $Z(p)$.

In the first figure, the series arms represent series connections of n-ports described by their impedance matrices, while the shunt n-ports are described by their admittance matrices. The coupling network N_C corresponds to McMillan's M_{AD} [2, p. 563].

Since the terms $-pL_1$ and $-D_1/p$ in N_C describe active networks, the main problem of the theory is to show that N_C can be realized by passive circuit elements. At this point we deviate from the somewhat complex ideas of McMillan [2, p. 562-580]. Consider the upper tee sub-network of N_C, this can be described by its $2n \times 2n$ chain matrix $\mathcal{Y}_\ell$ defined as the coefficient matrix in

$$\begin{bmatrix} V_{1\infty} \\ I_1 \end{bmatrix} = \begin{bmatrix} 1_n - L_1\Gamma_2 & 0_n \\ \Gamma_2/p & 1_n + \Gamma_2 L_3 \end{bmatrix} \begin{bmatrix} V_{2\infty} \\ -I_2 \end{bmatrix} \tag{8}$$

again 1_n is the $n \times n$ identity matrix. Here (8) can be easily obtained by multiplying the individual chain matrices for the three sub-parts and using (6b) to obtain 0_n for the (1,2) term. Now multiply (6b) on the right by $-\Gamma_2$ and add 1_n to both sides to get $(-L_1\Gamma_2)[1_n + L_3\Gamma_2] + 1_n[1_n + L_3\Gamma_2] = 1_n$ which gives, using the symmetry of L_3 and Γ_2,

$$1_n + \Gamma_2 L_3 = [1_n - \Gamma_2 L_1]^{-1} \tag{9}$$

This also shows that $1_n - \Gamma_2 L_1$ is nonsingular. Inserting this in the (2,2) terms of (8) and factoring the result, gives again writing a tilde for the transpose,

$$\mathcal{Y}_\ell = \begin{bmatrix} \widetilde{[1_n - \Gamma_2 L_1]} & 0_n \\ \Gamma_2/p & [1_n - \Gamma_2 L_1]^{-1} \end{bmatrix} \tag{10a}$$

$$= \begin{bmatrix} 1_n & 0_n \\ \Gamma_2 + \Gamma_2 L_3 \Gamma_2 / p & 1_n \end{bmatrix} \begin{bmatrix} \widetilde{[1_n - \Gamma_2 L_1]} & 0_n \\ 0_n & [1_n - \Gamma_2 L_1]^{-1} \end{bmatrix} \quad (10b)$$

By exactly similar reasoning one gets for the lower tee subnetwork of N_C

$$\mathcal{Y}_c = \begin{bmatrix} 1_n & 0_n \\ p[C_2 + C_2 D_3 C_2] & 1_n \end{bmatrix} \begin{bmatrix} \widetilde{[1_n - C_2 D_1]} & 0_n \\ 0_n & [1_n - C_2 D_1]^{-1} \end{bmatrix} \quad (10c)$$

Now an ideal transformer 2n-port is described by $V_1 = \tilde{T} V_2$, $I_2 = -T I_1$, where T is the turns-ratio matrix [10, p. 233] and thus the right-hand terms of (10b) and (10c) describe ideal transformers with

$$T_\ell = 1_n - \Gamma_2 L_1 \quad (11a)$$

$$T_c = 1_n - C_2 D_1 \quad (11b)$$

The left-hand terms of (10b) and (10c) describe shunt inductive and capacitive n-ports which are passive since their admittance matrices

$$Y_\ell(p) = [\Gamma_2 + \Gamma_2 L_3 \Gamma_2]/p \quad (12a)$$

$$Y_c(p) = p[C_2 + C_2 D_3 C_2] \quad (12b)$$

are PR, the residue matrices being positive semidefinite with L_3 and D_3. Note that, since $Y_\ell = [1_n + \Gamma_2 L_3]\Gamma_2/p$, (9) shows that the rank of Y_ℓ is that, m, of Γ_2. Further, $Y_c = pC_2[1_n + D_3 C_2]$ which also has rank m, by (7). As a consequence of these considerations the realization of Fig. 1 can be replaced by the one using purely passive elements of Fig. 2.

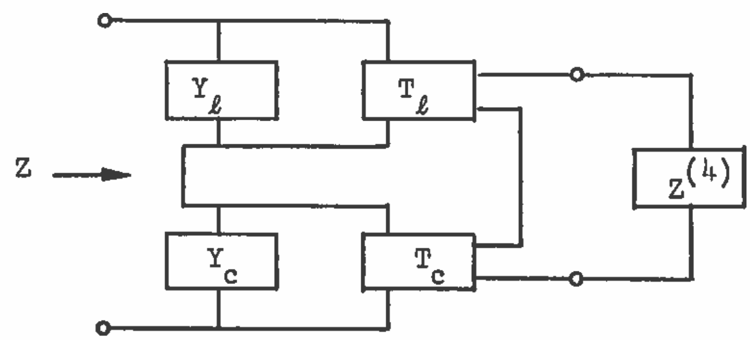

Figure 2. Passive Realization of Z.

With Fig. 2 the validity of McMillan's method is proven, since the process can be repeated on $Z^{(4)}$. Further, $\delta(Z)-\delta(Z^{(4)})$ reactive elements (inductors and capacitors) are used in realising N_C, where $\delta(\ )$ is McMillan's degree [2, p. 543]. To justify this last statement we use the properties of $\delta(\ )$ stated by McMillan [2, p. 543] to write

$$\delta(Z)-\delta(Z^{(4)}) = [\delta(Z^{(2)})-r_\ell-r_c]-[\delta(Z^{(3)})-r_\ell-r_c] \tag{13}$$

$$= [\delta(Z^{(3)})+2m-r_\ell-r_c]-[\delta(Z^{(3)})-r_\ell-r_c]$$

$$= 2m = \delta(Y_\ell)+\delta(Y_c)$$

But we know Y_ℓ and Y_c can be realized by using only $\delta(Y_\ell)$ inductors and $\delta(Y_c)$ capacitors [2, p. 548].

We can bring the realization of Fig. 2 into a form somewhat more familar to those acquainted with the one-port Brune synthesis. For conciseness, we only outline the steps involved, since their validity is easily justified. At the input we insert in cascade two transformer networks of turns ratio matrices T_1 and T_1^{-1}; similarly we insert transformers T_2^{-1}, T_2 in cascade between N_c and $Z^{(4)}$. This step leaves the external behavior unchanged; T_1^{-1} and T_2^{-1} can both be split into two equal transformers again of turns ratio T_1^{-1} and T_2^{-1} such that Y_ℓ and Y_c each have T_1^{-1} in cascade on the left and T_ℓ and T_c have T_2^{-1} cascaded on the right. The cascade of T_ℓ and T_2^{-1} can be combined into one transformer 2n-port of turns ratio $T_2^{-1}T_\ell$ and similarly T_c and T_2^{-1} go into $T_2^{-1}T_c$. At this stage the realization is as in Fig. 3a. The cascade connection of T_1^{-1} and Y_ℓ is now reversed to give a shunt n-port of admittance

$$Y_{\ell d} = T_1 Y \tilde{T}_1 = T_1[\Gamma_2 + \Gamma_2 L_3 \Gamma_2]\tilde{T}_1/p \qquad (14a)$$

again in cascade with T_1^{-1}. T_1^{-1} is now combined with $T_2 T_\ell$ to get a transformer of turns ratio

$$T_{\ell d} = T_2 T_\ell T_1^{-1} \qquad (15a)$$

A similar process on Y_c yields

$$Y_{cd} = T_1 Y_c \tilde{T}_1 = pT_1[C_2 + C_2 D_3 C_2]\tilde{T}_1 \qquad (14b)$$

$$T_{cd} = T_2 T_c T_1^{-1} \qquad (15b)$$

We now simultaneously diagonalize Y_ℓ and Y_c by a proper choice of T_1 in (14a) and (14b); this can always be done since the residue matrices are positive semidefinite [8]. We then choose T_2 such that $T_{cd} = 1_n$, that is, T_{cd} can now be omitted. The final realization then takes the form of Fig. 3b where the inductors and capacitors are "uncoupled" and can include open circuits.

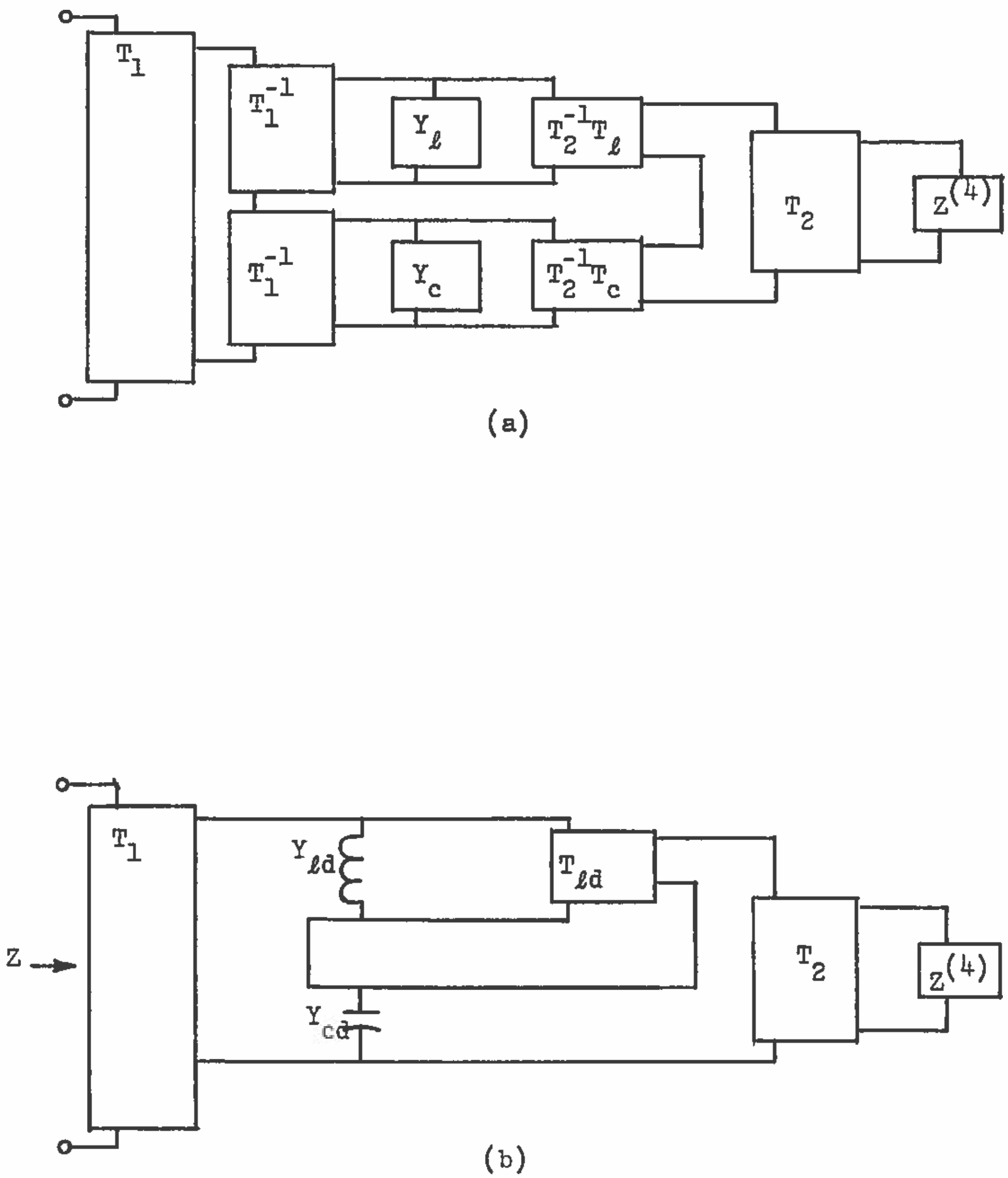

Figure 3. Development of Canonical Realization.

3. DISCUSSION

Here we have followed the synthesis procedure of McMillan to show that any symmetric, rational, positive-real impedance matrix corresponds to a finite, passive, reciprocal n-port. Besides greatly shortening McMillan's proof of the validity of the method, we have given a canonical structure, Fig. 3b, in terms of uncoupled reactive elements. Since this canonical structure reduces to the normal Brune circuit in the one-port case, we believe some physical insight is gained into McMillan's n-port extension. This result is in good agreement with the different type of extension due to Belevitch which relies upon complex resistances [6] and is a special case of our nonreciprocal synthesis [7], [8]. Although Belevitch discusses a structure somewhat like Fig. 3b, he only treats the uncoupled case of $m = 1$ [6, p. 291]. The synthesis of Tellegen [4] also deals largely with the $m = 1$ case, but, as pointed out by Oono and Yasuura [12, p. 150], his synthesis does not seem to cover all cases. Also, in contrast to the title of [4], a Brune type synthesis need not yield a network with the minimum total number of resistors, capacitors and inductors. This is shown by specific examples [12, p. 174]. In contrast to prevalent notions we believe this paper shows that McMillan's result is complete.

In the synthesis it should be observed that, at (2), McMillan extracts X completely. It is also possible to extract only part of X or even add to X. Doing this, one can guarantee that (2) has rank one at $p = p_o$, and, consequently, the section of Fig. 3b can be reduced to one containing only one inductor and one capacitor, if so desired. Further, the method used to go from Fig. 2 to Fig. 3b can be applied to the synthesis of Oono [3] to also simplify that method.

Although many other types of n-port synthesis methods exist and the reciprocity restraint can be relaxed [7], [12], as yet no general method exists to cover the nonrational case.

Epilogue:

I know you have been to my countree
    though I never saw you there;
I know you have loved all things I loved,
    flowery and sweet and fair.

Shaw Neilson [13]

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