ξ and ζ are the independent variables In practice, therefore, the two pistons of an E–H tuner should never be adjusted separately but always simultaneously, either in the same sanse (ζ) or in opposite senses (ξ) A mechanical linkage achieving this objective could easily be devised A good match would then be obtained in a few rapidly convergent adjustments, whatever the initial mismatch

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USEFUL TIME-VARIABLE CIRCUIT-ELEMENT EQUIVALENCES†

Because time variable circuit elements have properties not possessed by time invariant ones, such as the ability to modulate or amplify with low noise, it is useful to have different ways of looking at them Here we present general equi valences for time variable inductors, capacitors, resistors and gyrators, using time invariant elements and time variable transformers, which allow various pro perties to be determined and interpreted on physical grounds

Because time-variable circuit elements have properties not possessed by time-invariant ones, such as the ability to modulate or amplify with low noise, it is useful to have different ways of looking at them Here we present general equivalences for time-variable inductors, capacitors, resistors and gyrators, using time-invariant elements and time-variable transformers,[1] which allow various properties to be determined and interpreted on physical grounds

We first consider the inductor of inductance $l(t)$, whose defining constraint is

$$v = d[li]/dt \quad (1a)$$

$$= l'i + li' \quad (1b)$$

where v is the voltage across and i is the current through the inductor, a prime denotes differentiation Since no physical measurement can prove otherwise, we assume l to be an infinitely differentiable real-valued function of time t Consequently, we can reasonably assume that, over any finite interval $[a, b]$, where $a \leqslant t \leqslant b$, l is a function of bounded variation [2] Hence we write, over $[a, b]$,

$$l(t) = l_+(t) - l_-(t) \quad (2a)$$

where l_+ and l_- are the positive and negative variations,[2] except that $l(a)$ is contained in l_+ if $l(a) \geqslant 0$ or in l_- if $l(a) < 0$ to obtain two nonnegative, nondecreasing functions which are, in fact, differentiable, i e

$$l_+ \geqslant 0,\ l_+' \geqslant 0,\ l_- \geqslant 0,\ l_-' \geqslant 0 \quad (2b)$$

Moreover, it is possible for eqns 2 to take $b = +\infty$ and often $a = -\infty$ [but not always $a = -\infty$, as $l(t) = \cos t$, and $l(t) = \varepsilon^{-t}$ show] In any event, we can choose a (dependent on i) such that $v = i = 0$ for $t < a$, which rigorously[3] must be possible for every i (since we assume all physical i to be zero before a finite time)

With these preliminaries, consider the circuit of Fig 1a, which represents a time-variable transformer[1] of turns ratios $t_{ij}(t)$ loaded in passive and active time-invariant inductors and resistors Using the definition of a time-variable transformer, we easily find

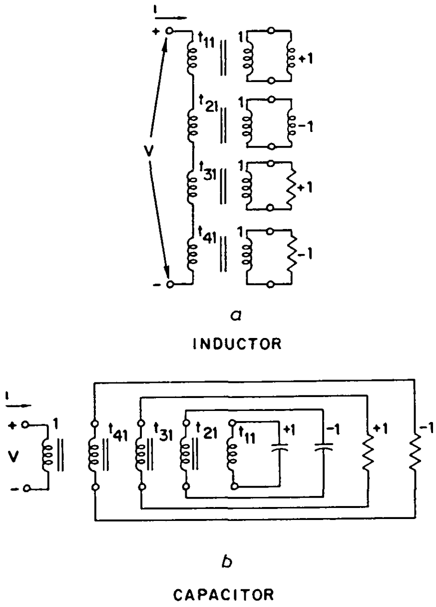

Fig. 1 *Time-variable reactive elements*

$$v = (t_{11}t_{11}' + t_{31}^2 - t_{21}t_{21}' - t_{41}^2)\,i + (t_{11}^2 - t_{21}^2)\,i' \quad (3)$$

Equating coefficients of eqns 3 with eqn 1, and using eqns 2, we arrive at

$t_{11}^2 = l_+$	$t_{21}^2 = l_-$	(4)
$2t_{31}^2 = l_+'$	$2t_{41}^2 = l_-'$	

We conclude that over the interval $[a, \infty]$, for every finite a, any time-variable inductor is equivalent to the circuit of Fig 1a with parameters chosen by eqn 4

At this point, one can apply physical reasoning to deduce the consequences of various properties If an inductor is passive, the negative elements of Fig 1a are necessarily absent, requiring

$$t_{21} = t_{41} \equiv 0,$$

or

$$l_- \equiv 0 \text{ (passive } l) \quad . \quad . \quad . \quad (5)$$

If an inductor is passive and lossless, then, besides the negative elements being absent, the positive resistor must not be present, forcing also t_{31} to zero and hence l_+' to zero, or

$$l = l_+ \equiv \text{constant (lossless } l) \quad (6)$$

From this one concludes that the time variation must be absent from all lossless inductors The equivalence also shows that one is naturally led to the consideration of time-variable transformers for synthesis of time-variable networks [4]

An entirely dual procedure yields the equivalence of Fig 1b for the linear time-variable capacitor of capacitance $c(t)$ We have

$$i = c\,v + cv' \quad (7a)$$

$$c = c_+ - c_- \quad (7b)$$

$t_{11}^2 = c_+$	$t_{21}^2 = c_-$	. (8)
$2t_{31}^2 = c_+'$	$2t_{41}^2 = c_-'$	

An equivalence for the time-variable resistor should now be clear Denoting the resistance by $r(t)$ with r_+ and r_- nonnegative functions, we have

$$v = ri \quad (9a)$$

$$r = r_+ - r_- \quad (9b)$$

from which Fig 2a results, with

$t_{11}^2 = r_+$	(10)
$t_{21}^2 = r_-$	

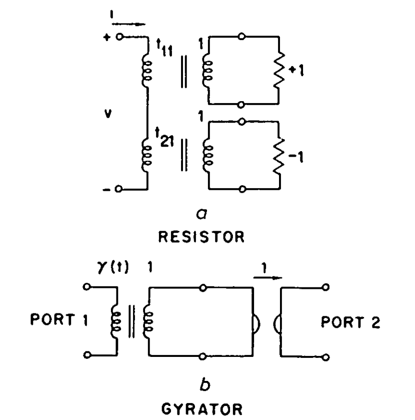

Fig. 2 *Resistive equivalents*

However, this equivalence is not unique, the monotonicity of r_+ and r_- is not required As with l and c, we can actually add any positive nondecreasing $f(t)$, which is infinitely differentiable, to r_+ and r_-, to obtain a nonunique decomposition In contrast to the case with l and c, arbitrary infinitely differentiable $f(t)$ can be added to r_+ anc r_- to obtain nonuniqueness A dual result holds for conductance, while Fig 2b shows the

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equivalence for the gyrator, which follows from the time-variable impedance matrix[5]

$$z(t, \tau) = \delta(t - \tau)\begin{bmatrix} 0 & \gamma(t) \\ -\gamma(t) & 0 \end{bmatrix}$$

$$= \delta\begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix} \quad . \; . \; (11)$$

Here $\gamma(t)$ is the gyration resistance, and δ is the unit impulse.

As in the time-invariant case, the inductor and capacitor can be interrelated through the gyrator. Such an equivalence is shown in Fig. 3, for which we have

$$v = \gamma \frac{d[\gamma c i]}{dt} = T \frac{d[Tli]}{dt} \quad . \; . \; (12)$$

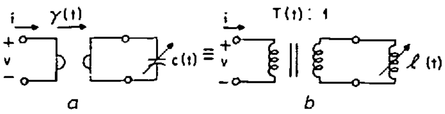

Fig. 3 *Capacitor-inductor equivalent*

To summarise, linear time-variable inductors, capacitors, resistors and gyrators have the equivalences given. Thus any connection of a finite number of such elements can be described by a circuit having transformers as the only time-variable elements. This allows us to consider all networks of this class to be looked upon as a transformer network loaded by a time-invariant inductor, capacitor, resistor, gyrator network to which various physical properties can be ascribed. Because the turns ratios of Figs. 1 and 2 are related to the positive and negative variations of element values, various constraints on the individual elements can be easily obtained by physical reasoning on the equivalent circuits.

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ENERGY TRANSPORT AND THE SCATTERING MATRIX

An energy-transport equation is obtained for the n-port scattering matrix of a general lossless anisotropic dispersive medium. This yields a theorem that in a lossless nonreciprocal 2-port the sum of the phase derivatives of the two transmission scattering coefficients must be nonnegative.

An appropriate definition of group velocity to describe energy transport of a wave packet was formulated and discussed by Brillouin[1] for an isotropic dispersive lossless medium. This has been extended by Hines[2] and Auer *et al.*[3] to anisotropic (nonreciprocal) plasma media. In general, the delay per unit length (inverse of group velocity) is shown to be proportional to average stored energy density per unit incident power density. The purpose of this note is to show how, for a very general lossless medium, Brillouin's definition of energy-transport delay is related to the S matrix used in the theory of electromagnetic scattering and in linear networks. The starting point is to extend a frequency-variation theorem derived by Dicke,[4] for an isotropic, homogeneous nondispersive lossless medium, to a dissipative medium with a dispersive tensor dielectric and tensor permeability. It may be shown that Dicke's frequency-variation integral for the more general medium becomes

$$\int_S (\boldsymbol{E}^* \times \boldsymbol{H}' + \boldsymbol{E}' \times \boldsymbol{H}^*) \, . \, d\boldsymbol{S}$$
$$= j\int_\tau [\boldsymbol{E}^* . (\omega\epsilon)' \boldsymbol{E} + \boldsymbol{H}^* . (\omega\mu)' \boldsymbol{H}] \, d\tau$$
$$- j\omega\int_\tau [\boldsymbol{H}^* \, . \, (\mu^+ - \mu) \boldsymbol{H}' + \boldsymbol{E}^* \, . \, (\epsilon^+ - \epsilon) \boldsymbol{E}'] \, d\tau \quad . \; . \; (1)$$

In eqn. 1, $\boldsymbol{E}(\omega, \boldsymbol{r})$ and $\boldsymbol{H}(\omega, \boldsymbol{r})$ are the electric and magnetic complex vector field functions of position $\boldsymbol{r}$, under the assumption that the fields at any point in the medium have harmonic time dependence of the form $\varepsilon^{j\omega t}$.

The notations $[\;]^*$, $[\;]^+$ and $[\;]'$ indicate, respectively, conjugate complex, adjoint and derivative with respect to the radian frequency ω. In the surface integral, the positive normal to $d\boldsymbol{S}$ is into the medium, and $d\tau$ is the volume element. All field magnitudes are expressed as r.m.s. values. The quantities $\epsilon(\omega, \boldsymbol{r})$, $\mu(\omega, \boldsymbol{r})$ are the tensor dielectric and permeability parameters. In the case of a lossless medium, μ and ϵ are Hermitian,[5] $\mu^+ = \mu$, $\epsilon^+ = \epsilon$, and the second volume integral in eqn. 1 vanishes. The first volume integral is twice the total integrated stored energy W averaged in time. For the loss-free medium considered here, a simple thermodynamic argument indicates that this total energy, consisting of a contribution of the electrostatic type as well as kinetic energy due to motion of charged carriers and magnetic dipoles under the influence of the alternating field, is nonnegative.[5]

Therefore

$$W \geqslant 0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (2)$$

$$\left.\begin{array}{l} (\omega\epsilon)' \geqslant 0 \\ (\omega\mu)' \geqslant 0 \end{array}\right\} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (3)$$

where the inequalities imply nonnegative Hermitian tensors.

The final form of eqn. 1 is

$$\int_S (\boldsymbol{E}^* \times \boldsymbol{H}' + \boldsymbol{E}' \times \boldsymbol{H}^*) \, . \, d\boldsymbol{S} = 2jW \quad . \; . \; . \; . \; (4)$$

for a lossless medium.

We may presume that any modes (including the evanescent variety) may exist in τ, but access to this region is via a set of n lengths of uniform lossless guide, each of which supports only one propagating mode at the frequency in question. This is no restriction, since the existence of a multiplicity of propagating modes merely increases the effective number of ports. Other than these guides, the region is surrounded by metallic walls. Suppose we fix a transverse reference plane in each of the guides. Then the surface integral in eqn. 4 is zero, except over these reference planes, and the result may be written as

$$\sum_n (v_n{}^* i_n{}' + v_n{}' i^*) = 2jW \; . \; . \; . \; (5)$$

or

$$v^+ i' + i^+ v' = j2W \; . \; . \; . \; . \; . \; (6)$$

In eqns. 5 and 6, v_n and i_n are the normalised voltage and current mode coefficients in the reference planes or ports, and v and i are column vectors of these voltages and currents. If the scattering representation is used,

$$v = a + b \; . \; . \; . \; . \; . \; . \; . \; . \; (7a)$$

$$i = a - b \; . \; . \; . \; . \; . \; . \; . \; . \; (7b)$$

where a and b are column vectors of incident and reflected complex wave amplitudes at the ports. If S is the scattering matrix normalised to real positive normalising numbers,

$$b = Sa \; . \; . \; . \; . \; . \; . \; . \; . \; (8a)$$

$$S(\omega) = [s_{ij}(\omega)] = \{S_{ij}(\omega) \exp [i\phi_{ij}(\omega)]\} \; . \; . \; (8b)$$

where $S_{ij}(\omega)$ and $\phi_{ij}(\omega)$ are real amplitude and phase functions, respectively. Since the medium is lossless, S is unitary.[6]

$$S^+S = SS^+ = I \; . \; . \; . \; . \; . \; . \; . \; (9)$$

I is the identity matrix. Substituting eqns. 7, 8a and 9 in eqn. 6, one obtains

$$ja^+S^+S'a = W \; . \; . \; . \; . \; . \; . \; . \; (10)$$

This result was originally derived by Dicke for an isotropic nondispersive medium. It may also be noted that eqn. 6 immediately leads to the general statement of Foster's reactance theorem:

$$v^+ Y' v = i^+ Z' i = 2jW \; . \; . \; . \; . \; (11)$$

where Z and Y are the impedance and admittance matrixes, respectively, and are not necessarily symmetric.

Suppose the region τ has only two waveguide ports of access. Terminate port 2 in a nonreflecting absorber, and