pression system and the system in fig. 11 are equivalent. The above restriction implies delay lines with a large number of short delay sections. A formal analysis of the case where this restriction is relaxed is currently under investigation, and if successful will be the subject of a further paper.

The expansion-compression system is still subject to the limitations which Gabor postulated, namely that it can only deal with signals having periodic gaps in their spectrum. Gabor suggested this technique for speech, but a lumped delay line with total delay of 100ms and adequate bandwidth is not a practicable alternative to film scanning. However, television signals also have a spectrum with periodic gaps and the shorter delays required here, albeit at a much increased bandwidth, may possibly justify the use of this technique. A delay network of about the same complexity as for speech compression is required for each line of the television signal that is to be stored.

4. Conclusion

It can be seen that there are a number of novel circuit situations which can arise when the delay line filter is extended to permit it a time-dependent impulse response. At the present time, the time-dependent network is receiving a good deal of attention with the advent of parametric amplifiers. The examples given in this paper by no means exhaust the potential of the time-dependent delay line filter, and one interesting possibility would be its use as a travelling wave parametric amplifier. Of the examples given in the paper, probably the most interesting is the bandwidth compressor and an evaluation of its potential is now being made. The present state of development is that successful video frequency multipliers have been produced as a by-product of current thin film research and further progress is awaiting the design of satisfactory active delay line networks which is being pursued by another research group in the Electrical Engineering Department, University of Western Australia.

Nonreciprocal Transmission-Line n-Port Synthesis

R. W. NEWCOMB, M.I.R.E.E.(AUST.)*

Summary

By the use of the theorem of Richards'
As extended to matrices Z
We consider the synthesis of
Nonreciprocal n-ports with lines
Of the type that are lossless and linked
Through their integer multiple lengths;
Some resistors are there-in contained;
As a maximum n is their number.

1. Introduction

Until the advent of Richards' 1-port transmission-line synthesis[1] of 1948 there was no rigorous means of designing distributed parameter networks. Following the appearance of this synthesis it was modified in various ways to cover specific structures, as well as transfer functions[2, 3, 4], while other useful, but less exact methods also appeared[5]. Somewhat more recently Saito[6] gave a 2-port transmission-line synthesis based upon Bayard's matrix extension of Richards' theorem[7]. However, none of these methods covers the synthesis of nonreciprocal distributed parameter circuits which are of importance in many contexts. In particular this is the case in parametric amplifier design.

With synthesis as the final goal we here develop the theoretical aspects of nonreciprocal resistor-transmission-line n-ports, under the customary assumption of lossless lines with rationally related lengths. We assume linearity, passivity and a finite number of elements, the nonreciprocal ones of which are taken to be gyrators. Transformers will generally be allowed but their number can to a large extent be minimized.

2. Properties of Resistor-Transmission-Line n-Ports

Consider an n-port which is constructed from passive resistors, gyrators, transformers, and lossless transmission-lines whose electrical lengths are all rational multiples of each other. For convenience such a net-

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1. Richards, P. I., "Resistor-transmission-line circuits", *Proc. IRE*, 36, Feb. 1948, 217-220.
2. Ozaki, H. and Ishii, J., "Synthesis of transmission-line networks and the design of UHF filters", *Trans. IRE*, CT-2, Dec. 1955, 325-336.
3. Grayzel, A. I., "A synthesis procedure for transmission line networks", *Trans. IRE*, CT-3, Sept. 1958, 172-181.
4. Ozaki, H. and Ishii, J., "Synthesis of a class of strip-line filters", *Trans. IRE*, CT-5, June 1958, 104-109.
5. Mumford, W. W., "Maximally-flat filters in waveguide", *BSTJ*, 27, Oct. 1948, 684-713.
6. Saito, N., "A coupled transmission line filter", *J. Inst. Elect, Com. Engrs. Japan*, 44, July 1961, 1036-1040.
7. Bayard, M., "Théorie des réseaux de Kirchhoff", *Editions de la Revue d'Optique*, Paris 1954, 170.

work will be called a *distributed n-port*. In a distributed n-port we can then choose a base length, taken to be one-half wavelength at some fixed base frequency f_o, such that all line lengths are integer multiples of this base length. Any one of these lines of length l, if we take ports at opposite ends of the line, can be considered as a 2-port, as shown in Fig. 1. For this 2-port, if the line inductance and capacitance are L henries/meter and C farads/meter, the characteristic impedance is $Z_o = \sqrt{(L/C)}$.

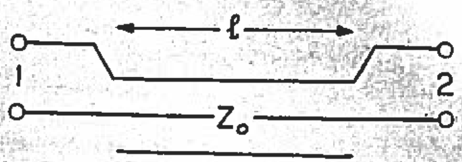

Figure 1.—Lossless line.

If the line is m half-wavelengths long at the base frequency, then $l = m/(2f_o \sqrt{(LC)})$, and the line is described by the impedance matrix $\mathbf{z}_L(s)$ (see Appendix 1)

$$\mathbf{z}_L(s) = Z_o \begin{bmatrix} \text{ctnh } (ms/2f_o) & \text{csch } (ms/2f_o) \\ \text{csch } (ms/2f_o) & \text{ctnh } (ms/2f_o) \end{bmatrix} \quad (1)$$

The admittance matrix $\mathbf{y}_L(s)$ is the inverse of this, $\mathbf{y}_L = \mathbf{z}_L^{-1}$, and is found by replacing Z_o by $Y_o = Z_o^{-1}$ and multiplying the off diagonal terms by -1. Here $s = \sigma + j\omega$, $\omega = 2\pi f$, is the actual complex frequency. When, as we are assuming for this discussion, m is an integer, the impedance matrix of Eq. (1) is rational in $\exp [s/2f_o]$, and if we let

$$\exp [s/2f_o] = (p + 1)/(p - 1) \quad (2a)$$

which is

$$p = \text{ctnh}(s/4f_o) \quad (2b)$$

$\mathbf{z}_L(s) = Z_L(p)$ becomes rational in p. Then given any distributed n-port the standard descriptions, such as the scattering matrix S(p) or the immittance matrices Z(p) and Y(p), considered as functions of p will be rational in p. This follows because resistor, transformer and gyrator descriptions are frequency independent and only rational operations are used to combine these with matrices $Z_L(p) = \mathbf{z}_L(s)$ which are rational in p. Further, these descriptions satisfy the normal realizability conditions (see Appendix 2), since p, by Eq. (2b), is a positive-real function of s; this would not be the case if the right of Eq. (2a) were replaced by p itself. Consequently, we can state that the necessary and sufficient condition for a given scattering matrix $\mathbf{s}(s)$ to describe a distributed n-port is that there exists a base frequency f_o such that $S(p) = \mathbf{s}(s)$ is a rational bounded-real, or simply BR, matrix. Every distributed n-port has such a scattering matrix, while if Z(p) or Y(p) exist, which they need not, these must be rational positive-real, or simply PR, matrices. The sufficiency of these conditions follow from Section 3 where a synthesis procedure is given.

It is worthwhile noting that Eq. (1) is valid for non-integer m since the quarter wave-length line, for which $m = \frac{1}{2}$, has considerable interest. For this value of m, after arbitrarily choosing the positive branch of $\sqrt{(p^2 - 1)}$, Eq. (1) under Eq. (2) becomes

$$\mathbf{z}_L(s) \Big|_{m = 1/2} = Z_{UE}(p) = \begin{bmatrix} p\, Z_o & \sqrt{p^2 - 1}\, Z_o \\ \sqrt{p^2 - 1}\, Z_o & p Z_o \end{bmatrix} \quad (3)$$

Even though this is not rational and thus not PR, it is positive-real being a positive-real matrix of a positive-real function. This is of considerable importance since it defines the 2-port unit-element. That is the 2-port unit-element is the p plane device having $Z_{UE}(p)$ as its impedance matrix; it is the p plane equivalent of the s plane quarter wavelength line. From this is derived the n-port unit element upon which synthesis is based. For this consider the 2n-port of Fig. (2a) where all lines are a quarter-wavelength long at the base frequency, and initially assumed uncoupled. Defining the diagonal characteristic impedance matrix by

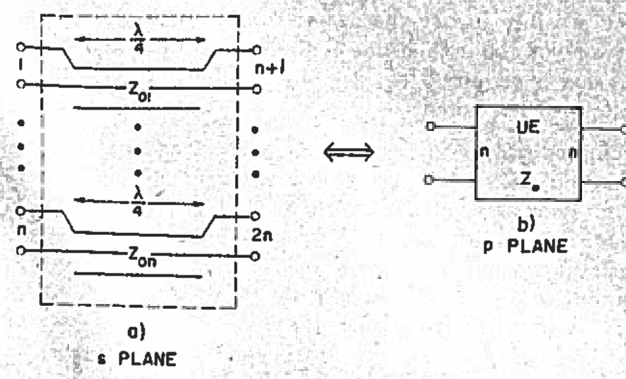

Figure 2.—n-port unit-element.

$$Z_o = Z_{o1} \dotplus \ldots \dotplus Z_{on} \quad (4)$$

where $\dotplus$ denotes the matrix direct sum[8] and Z_{oi}, $i = 1, \ldots, n$, are nonnegative scalars, Eq. (3) shows that Fig. 2 is also described by Eq. (3) with Z_{UE} now a $2n \times 2n$ matrix partitioned into $n \times n$ submatrices. If the lines in Fig. 2 are mutually coupled, then Eq. (3) can still be used to describe the 2n-port when Z_o is taken as a symmetric positive semidefinite matrix. In either case, coupled or not, the 2n-port of Fig. 2, when considered in the p plane, will be called an *n-port unit-element*, or simply unit-element, and denoted as shown in part (b) of the figure. It is of interest to recognize that the admittance matrix of the unit-element has the same form as Eq. (3) except that the (1, 2) and (2, 1) terms are multiplied by -1, and Z_o is replaced by Y_o where $Y_o = Z_o^{-1}$ when the inverse exists. When the admittance form is important the Z_o of Fig. (2b) will be replaced by Y_o.

If now we load an n-port unit-element by another n-port as shown in Fig. 3, where the impedance matrices

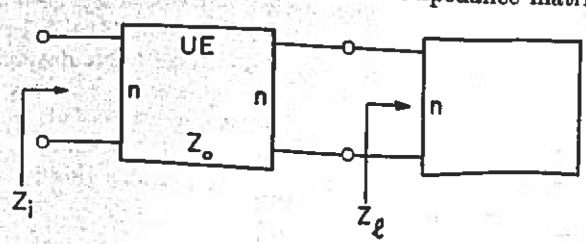

Figure 3.—Loaded unit-element.

8. MacDuffee, C. C., "The theory of matrices", Chelsea, New York, 1956, 81.

Z_i and Z_l are defined, we find (see Appendix 3)

$$Z_i(p) = [pZ_l(p) + Z_o]\,[Z_l(p) + pZ_o]^{-1}\,Z_o \qquad (5a)$$

$$Z_l(p) = Z_o[Z_i(p) - pZ_o]^{-1}\,[Z_o - pZ_i(p)] \qquad (5b)$$

The results in terms of admittances are identical with the appropriate Y's replacing Z's in Eqs. (5). One of the most important characteristics to be seen from Eqs. (5) is that Z_i is rational in p, with Z_l, and vice versa, even though Z_{UE} is not. This, of course, could have been guaranteed by replacing f_o by $2f_o$ in Eq. (2a), but Eq. (5) shows that this is unnecessary when quarter-wavelength lines are loaded as in Fig. 3. Some simple equivalences result from the connection of Fig. 3. For instance, if the load consists of short circuits then Eq. (5a) with $Z_l = 0_n$ gives $Z_i = Z_o/p$ and the configuration acts as a p plane capacitor n-port; here 0_n is the nth order zero matrix. If the load consists of open circuits a dual treatment shows that the input behaves as a p plane inductor n-port, $Y_i = Y_o/p$. These equivalences are illustrated in Fig. 4. Also of some interest are the n-port Kuroda identities illustrated in Fig. 5. The first of these is proven in Appendix 4 while the second follows in several ways from the first, for instance by duality or exchange of input and output. For the figure, Z_o is assumed nonsingular while D and Γ need not be.

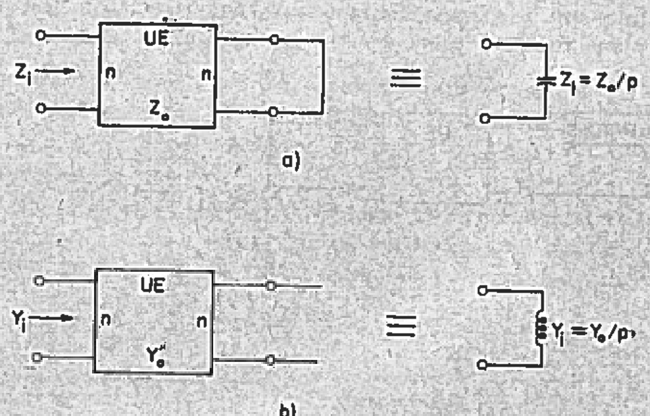

Figure 4.—p plane equivalences.

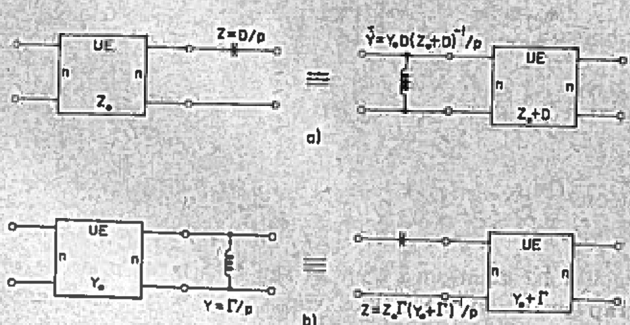

Figure 5.—n-port Kuroda identities.

3. Synthesis

The synthesis of distributed n-ports can proceed in many ways. Here we begin by assuming that a PR impedance matrix Z(p) is given, this being rational by an appropriate choice of the base frequency for Eqs. (2).

Given such an impedance matrix we then apply the ideas of a previous paper[9] and derive from Z an $(n + r) \times (n + r)$ impedance matrix X(p) of a lossless network. That is, with a subscript asterisk denoting replacement of p by $-p$ (called Hurwitz conjugation) and a tilde denoting matrix transposition, $X = -\tilde{X}_*$ with X being PR and r the rank of $Z + \tilde{Z}_*$. A realization for Z then results from one for X by terminating the last r ports by unit resistors, as shown in Fig. 6. Since $r < n$ it is clear that no more resistors than the number of terminal ports need be used.

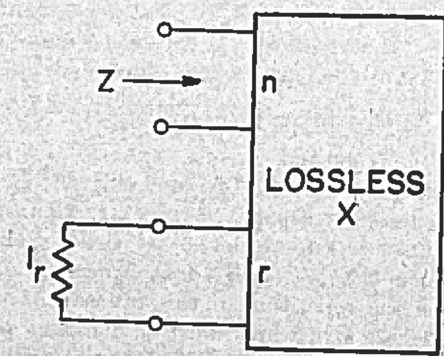

Figure 6.—Terminated lossless (n + r)-port giving Z.

Several methods are available for synthesizing X. The first and, most likely, least useful way is to synthesize X in the standard manner using p plane inductors, capacitors, gyrators and transformers[10]. This method generally uses an excessive number of transformers and the transmission lines do not occur in as useful combinations as in the following cascade synthesis. Nevertheless the use of the Kuroda identities may be of some assistance in making this type of synthesis useful, as illustrated in the special case of Fig. 7.

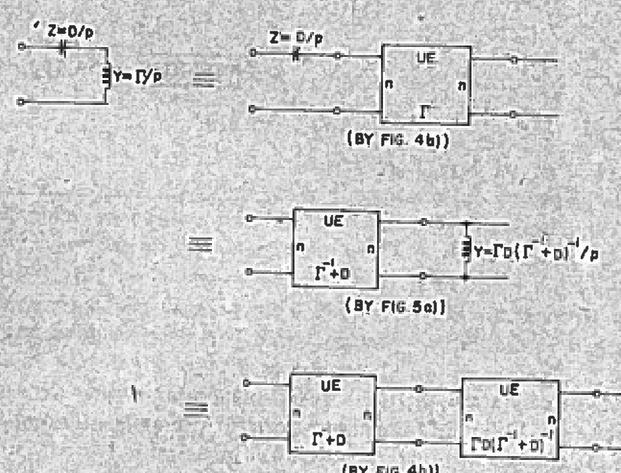

Figure 7.—Application of Kuroda's identity.

9. Newcomb, R. W., "A Bayard-type nonreciprocal n-port synthesis", *Trans. IEEE*, CT-10, March 1963, 85-90.
10. Oono, Y. and Yasuura, K., "Synthesis of finite passive 2n-terminal networks with prescribed scattering matrices", *Memoirs of the Faculty of Engineering Kyushu University*, 14, May 1954, 155-168.

The cascade method proceeds from Fig. 3 in conjunction with a generalization of Richards' theorem to matrices. Observing Eq. (5b) we see that the loaded unit-element gives Z_l in terms of Z_i through the use of Richards' theorem for matrices (see Appendix 5) if $k = 1$ is chosen. In fact, as $k = 1$, this is the justification for the name "unit-element". Noting that Richards' theorem requires the matrix to be symmetric at $p = k = 1$, we first extract series gyrators by writing

$$X_i(p) = X(p) - X_{ss}(1) \tag{6}$$

where $2X_{ss} = X - \tilde{X}$ defines the skew-symmetric part of X. While X_i remains PR, $X_{ss}(1)$ can be realized with the use of gyrators since it is a constant skew-symmetric matrix. A unit-element is then extracted by forming

$$X_l(p) = X_i(1)[X_i(p) - pX_i(1)]^{-1}[X_i(1) - pX_i(p)] \tag{7}$$

whenever the required inverse exists. By the cited Richards' theorem X_l is PR, and Fig. 3 shows that forming X_l does correspond to the extraction of a unit-element. Further, if we let $\delta[\ ]$ denote McMillan's degree[11], then $\delta[X_l] < \delta[X_i]$ (see Appendix 5). As a consequence the repetition of this procedure leads to a matrix of degree zero which must be constant and hence skew-symmetric since $X = -\tilde{X}_*$. This matrix is realized by gyrators and terminates the synthesis, if the inverses needed for Eq. (7) exist at all steps.

From this we can conclude that the singularity of $X_i(p) - pX_i(1)$ implies that of $X'_i(p)$ (the converse is also true). For this conclusion we note that when $X_i(p) - pX_i(1)$ is singular there exists a nonzero vector v such that $X'_i(p)v = pX'_i(1)v$. Now if $X'_i(p)$ is nonsingular then so is $X'_i(1)$ and vice versa, since a PR matrix which is singular at any one point in the right half plane must be identically singular. Thus, if $X'_i(p)$ is nonsingular, $[X'_i(1)]^{-1}X'_i(p)v = pv$ results, which is impossible unless v is zero since the left side has no poles at infinity. We then conclude that if $X'_i(p)$ is nonsingular then so is $X_i(p) - pX_i(1)$. Letting $X'_i(p)$ be PR of rank m, then there exists a real constant $m \times n$ matrix T such that[12]

$$X'_i(p) = \tilde{T}X_{io}(p)\,T \tag{9}$$

where X_{io} is PR and nonsingular of order m. Eq. (7) can now be applied to X_{io}, as the required inverse exists, and the process can therefore be continued. Physically Eq. (8a) represents a series extraction of a p plane inductor n-port and Eq. (9) states that X'_i results by terminating a transformer (n + m)-port in the m-port[13] for X_{io}. This completes the synthesis which takes the form illustrated in Fig. 8. Part (a) of this figure shows the overall form when no singular matrices are met, while part (b) shows a typical singular section. The actual transmission lines are, of course, obtained by converting from the p plane to the s plane by the use of Figs. 2 and 4. It

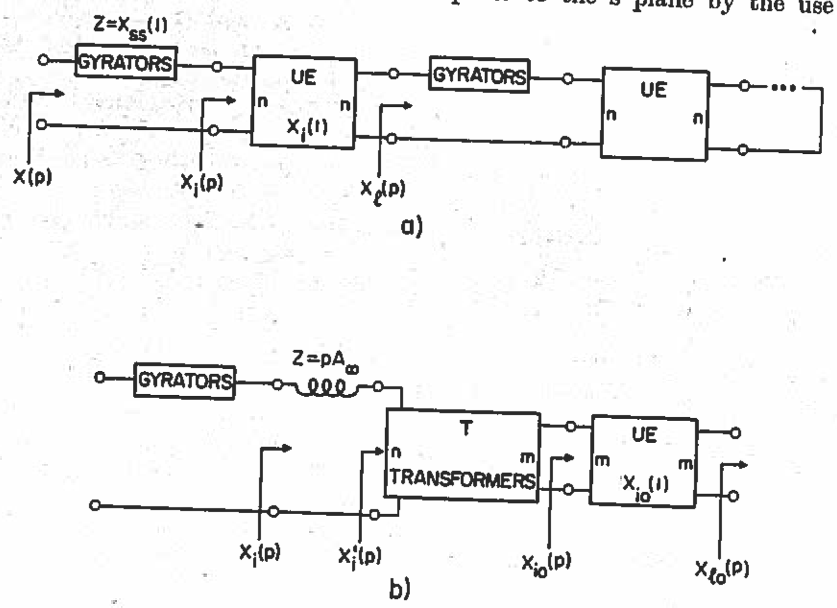

Figure 8.—(a) Nonsingular canonical realization of X.
(b) Singular section.

If at any stage $X_i(p) - pX_i(1)$ is singular then we can make the following considerations. Any pole at infinity of $X_i(p)$ can be extracted by writing

$$X_i(p) = X'_i(p) + pA_\infty \tag{8a}$$

where A_∞ is the positive semi-definite, symmetric, residue matrix at infinity and X'_i is again PR but now analytic at infinity. Then

$$X_i(p) - pX_i(1) = X'_i(p) - pX'_i(1) \tag{8b}$$

should be mentioned that in the matrix case Richards' transformation sometimes yields a reduction in degree when applied directly to Z. In such cases the extra steps involved in finding X can be avoided. However, as yet, the conditions on Z for this degree reduction are not known.

11. McMillan, B., "Introduction to formal realizability theory-II", *BSTJ*, 31, May 1952, 580.

12. Newcomb, R. W., "Properties of positive-real matrices", Stanford Electronics Labs., TR No. 2254-4, Dec. 1962, 19-21.

13. Belevitch, V., "Synthèse des réseaux électriques passifs a n paires de bornes de matrice de répartition prédéterminée", *Annales de Télécommunications*, 6, Nov. 1951, 307.

Although distributed n-ports can be specified through the impedance matrix Z(p), the most general specification is through the scattering matrix S(p), which by a suitable choice of base frequency must be BR. For such a specification we can revert to the impedance matrix, which will be PR, by

$$Z(p) = (1_n + S(p))\,(1_n - S(p))^{-1} \tag{10a}$$

whenever the inverse exists, and then apply the theory given above to Z; here 1_n is the nth order identity matrix. If the inverse of $1_n - S$ does not exist then there exists a real constant orthogonal matrix T such that[14]

$$S = \tilde{T}[S' \dotplus 1_{n-m}]T \tag{10b}$$

where S′ is an m × m BR matrix with $1_n - S'$ nonsingular. S′ can be used in Eq. (10a), with n replaced by m, to obtain a PR impedance matrix Z′ which is in turn synthesized as above. The realization of S results by terminating a 2n-port transformer of turns ratio matrix T at its last n − m output ports by open circuits and at its first m output ports by the realization of Z′. If the admittance matrix Y is given, then it can be synthesized directly by a process dual to that used for Z. Alternatively Y can be made nonsingular by a transformation similar to that in Eq. (9) and the above impedance synthesis used on the inverse with transformers realizing the transformation.

The following example illustrates almost all of the interesting points of the synthesis method.

Example

Suppose it is desired to design a 2-port one way, matched, second order, Butterworth transmission-line filter, perhaps for an antenna feed, as described by the scattering matrix

$$S(p) = \begin{bmatrix} 0 & 0 \\ \dfrac{\sqrt{2}\,p}{p^2 + \sqrt{2}\,p + 1} & 0 \end{bmatrix} \tag{11a}$$

If we let $d = p^2 + \sqrt{2}\,p + 1$ with even and odd ports $\epsilon v\, d = p^2 + 1$, $Od\, d = \sqrt{2}\,p$, then the (2, 1) term is $s_{21} = (Od\, d)/d$ and Eq. (10a) gives

$$Z(p) = \begin{bmatrix} 1 & 0 \\ \dfrac{2\,Od\,d}{d} & 1 \end{bmatrix} \tag{11b}$$

Using the Case 1 method of a previous paper[9] gives

$$X = \begin{bmatrix} U & V \\ -\tilde{V}_* & W \end{bmatrix} \tag{12a}$$

which is obtained by factoring

$$Z'' = \tfrac{1}{2}(Z + \tilde{Z}_*) = N(dL)^{-1}(dL)_*^{-1}N_* \tag{12b}$$

$$= \begin{bmatrix} d & O \\ Od\,d & \epsilon v\,d \end{bmatrix} \begin{bmatrix} d & O \\ O & d \end{bmatrix}^{-1} \begin{bmatrix} d_* & O \\ O & d_* \end{bmatrix}^{-1} \begin{bmatrix} d_* & Od\,d_* \\ O & \epsilon v\,d_* \end{bmatrix} \tag{12c}$$

which defines $L = 1_2$ and N, and forming

$$V = N[Od(dL)^{-1}] \tag{12d}$$

$$W = [\epsilon v(dL)]\,[Od(dL)^{-1}] \tag{12e}$$

$$-U = Z - V(W + 1_2)^{-1}\tilde{V}_* \tag{12f}$$

Performing these operations gives

$$X(p) = \frac{1}{Od\,d}\left[\begin{array}{cc:cc} \epsilon v\,d & -Od\,d & d & O \\ Od\,d & \epsilon v\,d & Od\,d & \epsilon v\,d \\ \hdashline d_* & -Od\,d & \epsilon v\,d & O \\ O & \epsilon v\,d & O & \epsilon v\,d \end{array}\right] \tag{12g}$$

Then

$$X_{ss}(1) = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{12h}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$= \tilde{T}_g Z_g T_g \tag{12i}$$

Letting $x(p) = (\epsilon v\,d)/(Od\,d)$ which is a reactance function we have

$$X_t(p) = X(p) - X_{ss}(1) = x(p)\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \tag{12j}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(p) & 0 \\ 0 & x(p) \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$= \tilde{T}X_{to}(p)T \tag{12k}$$

In our case $X_{to}(p) = x(p)\,1_2 = \dfrac{p^2+1}{\sqrt{2}\,p}\,1_2$ with $X_{to}(1) = \sqrt{2}\,1_2$. Applying Eq. (7) to X_{to} gives

$$X_{to}(p) = \sqrt{2}\,p1_2 \tag{12l}$$

Transformers can be used to realize T and T_g but those for T can be omitted since they merely connect port three to one and port four to two. Most turns for T_g can also be omitted since the + 1 entries, which give unit turns ratios, represent direct connections; however, the 6-port transformer realization of T_g is given to illustrate the general case. This is given on the left of Fig. 9 which shows the final realization of the given S(p). By closely observing the calculations, the results of this example up to Eq. (12e) are seen to be valid for any Hurwitz polynomial d. For arbitrary d the only change in the realization of Fig. 9 is the addition of more cascade sections on the right.

4. Conclusions

In this paper a synthesis of nonreciprocal resistor-transmission-line n-ports has been given. Although several alternatives exist, the method using the matrix Richards' transformation seems the most useful, and

14. Oono, Y., "Applications of scattering matrices to the synthesis of n-ports", *Trans. IRE*, CT-3, June 1956, 112.

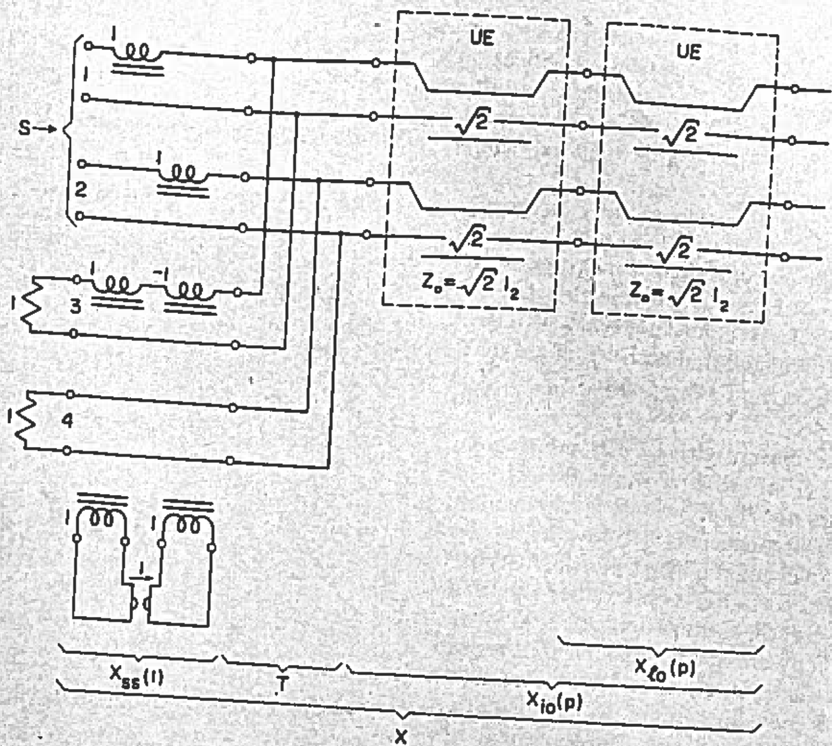

Figure 9.—Second order one-way filter example.

therefore, has been presented in some detail, since a cascade type realization is obtained. Transformers may be required at several places, such as when converting S to Z or at more bothersome points, as in preparing to use Richards' theorem or in realizing the coupling among lines for a unit-element. However, at higher frequencies transformers are not such a nuisance as at lower frequencies. As yet, as with general resistive networks, the conditions for realization without transformers are unknown, but a congruency transformation allows any symmetric positive semi-definite Z_o to be realized from a diagonal one by the use of transformers (Appendix 1 shows how L and C are affected by such a transformation in the positive definite case). The use of Kuroda's identity as mentioned in Section III and illustrated in Fig. 7 will sometimes lead to a more useful structure than obtained directly, but singular matrices may hamper its application. As seen by the example, singular matrices may very well occur in the procedure, requiring the considerations of Eqs. (8) and (9). However, as also seen by the example, the term at infinity, pA_∞, may not always have to be separately extracted. It would be useful if all the gyrators could be removed at an initial or final step, as actually occurred in the example, since then the unit-elements shown in Fig. (8a) could be more conveniently physically connected. However, series and shunt transmission-line connections can often be physically realized by the use of Grayzel's connections[3] generalized to n-ports where possible.

At Eq. (2b), in order to get the simplest rational expressions, we used the transformation $p = \text{ctnh}(s/4f_o)$. However, the transformation $p = \text{tnh}(s/4f_o)$ is possible and common[1]. Since for $p = j\omega$ both transformations are periodic, identical responses over a given finite frequency band can be obtained from either theory, perhaps by the use of appropriate normalizations. A recent related work by Matsumoto* discusses the physical parameters of multiwire reciprocal transmission lines.

Acknowledgments

The author wishes to acknowledge the National Science Foundation, under Grants NSF G-18945 and NSF GP-520, for support of this research and the Fulbright Commission and the University of New South Wales for their support during the paper's preparation. As acknowledgment of her most wonderful assistance with this and other matters during our absence, we dedicate this work to Barbara Serrano.

Appendix 1

Lossless Transmission Lines

The lossless transmission line of Fig. 1 is described by the two dual equations

$$\frac{\partial v(x, t)}{\partial x} = -\frac{L\partial i(x, t)}{\partial t} \qquad \text{(A-1a)}$$

$$\frac{\partial i(x, t)}{\partial x} = -\frac{C\partial v(x, t)}{\partial t} \qquad \text{(A-1b)}$$

where v and i are the voltage and current on the line, t is time and x is distance down the line measured with

*Matsumoto, A., "Network Synthesis with Multiwire Lines", Monograph No. 11, The Research Institute of Applied Electricity, Hokkaido University, Sapporo, Japan, 1963.

respect to port one. Taking the bilateral Laplace transform with respect to time yields

$$\frac{d\,V(x, s)}{dx} = -\,s\,LI(x, s) \qquad \text{(A-1c)}$$

and its dual. Here we have replaced the partial derivatives by total derivatives, which allows the time differentiation to go into multiplication by s, and is justified by x being independent of t. Differentiating Eq. (A-1c) with respect to x and using the dual equation yields

$$\frac{d^2\,V(x, s)}{dx^2} = s^2\,LC\,V(x, s) \qquad \text{(A-1d)}$$

This can be solved using the unilateral Laplace transform in conjunction with Eq. (A-1c) evaluated at x = 0 to give

$$V(x, s) = \cosh(s\sqrt{LC}\,x)V(o, s) - \sqrt{LC^{-1}}\sinh(s\sqrt{LC}\,x)I(o, s) \qquad \text{(A-2a)}$$

$$I(x, s) = \cosh(s\sqrt{LC}\,x)I(o, s) - \sqrt{CL^{-1}}\sinh(s\sqrt{LC}\,x)V(o, s) \qquad \text{(A-2b)}$$

where the second follows by duality from the first. Defining

$$Z_o = \sqrt{LC^{-1}} \qquad \text{(A-2c)}$$

letting $V_2(s) = V(l, s)$, $V_1(s) = V(o, s)$, $I_2(s) = -I(l, s)$, $I_1(s) = -I(o, s)$, and solving Eqs. (A-2a, b) for the voltages in terms of the currents yields

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = Z_o \begin{bmatrix} \operatorname{ctnh}(s\sqrt{LC}\,l) & \operatorname{csch}(s\sqrt{LC}\,l) \\ \operatorname{csch}(s\sqrt{LC}\,l) & \operatorname{ctnh}(s\sqrt{LC}\,l) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \qquad \text{(A-2d)}$$

which gives the impedance matrix for the line. A wavelength is defined at a given frequency f by $\lambda = 1/(f\sqrt{LC})$, and, thus, if the line is m half-wavelengths long at the frequency f_o, then $l = m/(2f_o\sqrt{LC})$.

If we have a 2n-port of coupled lines, then Eqs. (A-1) remain valid with v and i n-vectors and L and C coefficient matrices. Assuming these latter positive definite L and C^{-1} can be simultaneously diagonalized by properly choosing T in the following equations[15]

$$v_d = \tilde{T}v \qquad \text{(A-3a)}$$

$$i = Ti_d \qquad \text{(A-3b)}$$

$$L_d = \tilde{T}LT \qquad \text{(A-3c)}$$

$$C_d = T^{-1}C\tilde{T}^{-1} \qquad \text{(A-3d)}$$

We also define the diagonal characteristic impedance matrix

$$Z_{od} = [L_d C_d^{-1}]^{1/2} \qquad \text{(A-3e)}$$

which is formed by taking the square root of each element in $L_d C_d^{-1}$. If all lines are a quarter wavelength long at some frequency f_o we then obtain, in partitioned form,

$$\begin{bmatrix} V_{1d} \\ V_{2d} \end{bmatrix} = \begin{bmatrix} \operatorname{ctnh}(s/4f_o)Z_{od} & \operatorname{csch}(s/4f_o)Z_{od} \\ \operatorname{csch}(s/4f_o)Z_{od} & \operatorname{ctnh}(s/4f_o)Z_{od} \end{bmatrix} \begin{bmatrix} I_{1d} \\ I_{2d} \end{bmatrix} \qquad \text{(A-4a)}$$

where the functional multipliers of Z_{od} can be taken as scalars. Defining

$$Z_o = \tilde{T}^{-1}Z_{od}T^{-1} \qquad \text{(A-4b)}$$

shows that, in this case, the coupled lines can be described by the impedance matrix in Eq. (A-4a) when the subscripts d are omitted. Note that here

$$Z_o = \tilde{T}^{-1}[\tilde{T}\,L\,T\,\tilde{T}\,C^{-1}T]^{1/2}\,T^{-1}$$

which is not Eq. (A-2c) with scalars replaced by matrices; this latter replacement causes trouble in previous works[16, 17].

Appendix 2

BR and PR Matrices

An n × n matrix S(p) is called *bounded-real*[18] if in Re p > o : (1) S(p) is analytic, (2) $S(p) = S^*(p)^*$, (3) $1_n - \tilde{S}^*(p)S(p)$ is positive semi-definite; here the superscript asterisk denotes complex conjugation. A rational bounded-real matrix is conveniently called BR and then condition (1) follows from (2) and (3) with analyticity even holding on the ω axis. Every BR scattering matrix coincides with a passive network[10] and vice versa. The passivity conditions on the impedance matrix Z(p) are that in Re p > o : (1) Z(p) is analytic, (2) $Z(p) = Z^*(p^*)$, (3) $Z(p) + \tilde{Z}^*(p)$ is positive semi-definite. Such an n × n matrix is called *positive-real*[18], or PR if Z is rational in which case condition (1) is again a consequence of (2) and (3). Given a PR matrix Z(p) it can be synthesized by a passive network[9], but every passive network need not possess a PR impedance matrix, as shown by the transformer. Eq. (1) defines a positive-real matrix in s which through Eq. (2) becomes PR in p.

Given an n × n matrix S(p) of rational functions with real coefficients the following conditions are necessary and sufficient for it to be BR: (1) S(p) is analytic in Re p ⩾ o, (2) $1_n - \tilde{S}^*(j\omega)S(j\omega) = D(j\omega)$ is positive semi-definite. These conditions are relatively easy to apply, since D(jω) need only be considered on the ω axis; they follow by applying the maximum modulus theorem to the BR definition. A similar test holds for PR matrices[19], but it is often convenient to recognize that Z(p) is PR if and only if $S = (Z + 1_n)^{-1}(Z - 1_n)$ is BR.

Appendix 3

Loaded Unit-Element

Here we derive Eqs. (5). For this define voltage and current variables for the network of Fig. 3 such that

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} p\,Z_o & \sqrt{p^2 - 1}\,Z_o \\ \sqrt{p^2 - 1}\,Z_o & p\,Z_o \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \qquad \text{(A-5a)}$$

$$V_2 = -Z_l I_2 \qquad \text{(A-5b)}$$

15. Bellman, R., "Introduction to Matrix Analysis", McGraw-Hill, New York, 1960, 58.

16. Rice, S. O., "Steady state solutions of transmission line equations", *BSTJ*, 20, April 1941, 135-137.

17. Pipes, L. A., "Matrix theory of multiconductor transmission lines", *Philosophical Magazine*, 24 (7th series), July 1937, 102.

18. Youla, D. C., Castriota, L. J. and Carlin, H. J., "Bounded real scattering matrices and the foundations of linear passive network theory", *Trans. IRE*, CT-6, March 1959, 116 & 122.

19. Newcomb, R. W., "On network realizability conditions", *Proc. IRE*, 50, Sept. 1962, 1955.

Eliminating V_2 from these two gives

$$[-Z_l - pZ_o] I_2 = \sqrt{p^2-1}\, Z_o I_1 \quad \text{(A-5c)}$$

Substituting this in the first of Eqs. (A-5a) gives

$$V_1 = [pZ_o + \sqrt{p^2-1}\, Z_o[-Z_l - pZ_o]^{-1} \sqrt{p^2-1}\, Z_o] I_1 \quad \text{(A-5d)}$$

$$= [p(Z_l + pZ_o) - (p^2-1)Z_o] [Z_l + pZ_o]^{-1} Z_o I_1 \quad \text{(A-5e)}$$

$$= [pZ_l(p) + Z_o]\,[Z_l(p) + pZ_o]^{-1} Z_o I_1 \quad \text{(A-5f)}$$

which is Eq. (5a). Setting the coefficient matrix of I_1 equal to Z_i gives

$$Z_i Y_o [Z_l + pZ_o] = pZ_l + Z_o \quad \text{(A-5g)}$$

or

$$[Z_i Y_o - p1_n] Z_l = Z_o - pZ_i \quad \text{(A-5h)}$$

which is Eq. (5b) after multiplying by the appropriate inverse.

Appendix 4

n-Port Kuroda Identity

Consider the left hand network of Fig. (5a). This is described by the impedance matrix

$$Z(p) = \begin{bmatrix} pZ_o & \sqrt{p^2-1}\, Z_o \\ \sqrt{p^2-1}\, Z_o & pZ_o + D/p \end{bmatrix} \quad \text{(A-6a)}$$

the inverse of which is

$$Y(p) = \begin{bmatrix} (Z_o + D)^{-1}(p\,1_n + DY_o/p) & -\sqrt{p^2-1}\,(Z_o+D)^{-1} \\ -\sqrt{p^2-1}\,(Z_o+D)^{-1} & p(Z_o+D)^{-1} \end{bmatrix} \quad \text{(A-6b)}$$

Since Z, Z_o, and D are symmetric, we see that the residue matrix at $p = 0$ of the (1, 1) term of this is also equal to $Y_o D(Z_o + D)^{-1}$. This matrix is easily seen to describe the right hand network of Fig. (5a).

Appendix 5

Richards' Theorem for Matrices

Here we state and prove a generalization of Richards' theorem for PR, but not necessarily symmetric, matrices. Given a PR Z(p) we make it symmetric at the point $p = k$ by defining.

$$Z_i(p) = Z(p) - \tfrac{1}{2}[Z(k) - \tilde{Z}(k)] \quad \text{(A-7a)}$$

To simplify the proof we first normalize. At $p = k$, Z_i is symmetric and positive semi-definite and hence can be factored to

$$Z_i(k) = \tilde{K}K \quad \text{(A-7b)}$$

If $Z_i(p)$ is nonsingular then so is K and we form

$$Z_n(p) = \tilde{K}^{-1} Z_i(p) K^{-1} \quad \text{(A-7c)}$$

This normalized matrix is still PR and has $Z_n(k) = 1_n$. For the theorem we assume that $pZ_n(p) - k1_n$ is non-singular, the singular case being handled by Eqs. (8) and (9) of the text.

Richards' Theorem:

If Z(p) is PR then

$$Z_R(p) = [kZ_n(p) - p1_n]^{-1}[k1_n - pZ_n(p)] \quad \text{(A-8)}$$

is PR for any $k > 0$; $p - k$ cancels from the two terms, as does $p + k$ if $Z(k) = -\tilde{Z}(-k)$.

Proof: Clearly Z_R is rational with real coefficients when Z is. We form the scattering matrix S_R for Z_R.

$$S_R = (Z_R + 1_n)^{-1}(Z_R - 1_n) \quad \text{(A-9a)}$$

$$= \left[\frac{p+k}{p-k}\right](Z_n + 1_n)^{-1}(Z_n - 1_n) \quad \text{(A-9b)}$$

$$= \left[\frac{p+k}{p-k}\right] S_n \quad \text{(A-9c)}$$

where S_n is the scattering matrix for Z_n. To this we apply the BR test of Appendix 2. S_R is analytic in Re $p > 0$ since S_n is and $p - k$ cancels into $Z_n(p) - 1_n$ by the rationality and the fact that this is zero at $p = k > 0$. Further, for $p = j\omega$, $\tilde{S}^*_R S_R = \tilde{S}^*_n S_n$ since $|(j\omega + k)/(j\omega - k)| = 1$. Therefore, S_R is BR and thus Z_R is PR. Clearly $p - k$ cancels from Eq. (A-8) since both factors have a zero at $p = k$; the same is true at $p = -k$ if $Z(k) = -\tilde{Z}(-k)$ since $Z_n(-k) = -1_n$ then. Q.E.D.

Denormalizing Eq. (A-8) gives

$$Z_l(p) = \tilde{K}Z_n(p)K = Z_i(k)\,[kZ_i(p) - pZ_i(k)]^{-1}\,[kZ_i(k) - pZ_i(p)] \quad \text{(A-10)}$$

where Z_l has the properties of Z_R given in the theorem. The choice $k = 1$ is special and corresponds to the result used in the text. Since $\delta[Z_R] = \delta[S_R]$, Eq. (A-9c) shows that $\delta[Z_l] = \delta[Z_R] < \delta[Z_n] = \delta[Z_i] = \delta[Z]$ because $p - k$ cancels and S_R can have no pole at infinity. When $p + k$ cancels, consideration of McMillan's canonical form[11] shows that the degree is lowered and thus $\delta[Z_l] < \delta[Z]$. In particular the degree is lowered in the lossless case where $Z = X = -\tilde{X}_*$.

The result is valid for positive-real matrices and a bridge realization of Z_i in terms of Z_l, analogous to that used in the Bott-Duffin synthesis, can be given[20].

20. Newcomb, R. W., "Richards' theorem for matrices", Stanford Electronics Labs., TR No. 2254-1, July, 1962.