RELATION BETWEEN THE DOA MATRIX METHOD AND THE ESPRIT METHOD

Qin-Ye Yin** Robert W. Newcomb* Sunita Munjal* Li-He Zou*

*Microsystems Laboratory, Electrical Engineering Department
University of Maryland, College Park, MD 20742, USA
(301) 454-6869

*Information and Control Engineering Department
Xi'an Jiaotong University, Xi'an, P. R. China

ABSTRACT

There have been many algorithms dealing with the estimation of the direction-of-arrival (DOA) for multiple narrowband sources. Recently the most well-known of these is undoubtedly the ESPRIT method. Moreover, a new 2-D eigenstructure approach called the DOA matrix method has been proposed. Unlike the ESPRIT method, it estimates the 2-D directions from both nonzero eigenvalues and corresponding eigenvectors of a DOA matrix. In this paper the relation between them is theoretically analyzed. It is shown that the ESPRIT method may be regarded as a special case of the DOA matrix method, and that the DOA matrix method may give more generalized and more perfect results.

I. Introduction

Estimation of the direction of arrival (DOA) from noisy sensor array data has attracted tremendous research attention for several decades because of its application in radar, sonar, seismic, and radio signal processing. There have been a variety of techniques and algorithms proposed for dealing with this issue. The sensor arrays may not be uniformly spaced and linear. But, with the uniformly spaced linear sensor array systems, a very important array system in practice, we can save a lot of computations and storages. Here, we discuss the linear sensor array system.

In recent years, there has been a growing interest in eigenstructure based methods. These methods, pioneered by Pisarenko [1], Schmidt [2] and Kumaresan [3], are known to yield high resolution and asymptotically unbiased estimates. Furthermore, Paulraj, et. al., [4,5,6] have proposed a subspace rotation approach called ESPRIT. Like the MUSIC method [2], it exploits correctly the underlying signal model to generate asymptotically unbiased estimates. But it estimates the DOA by using the eigenvalues of a matrix pair rather than the eigenvectors of the covariance matrix, which are used by most of the eigenstructure methods including the MUSIC method. Without the one-dimensional (1-D) spectral peak search, the ESPRIT method has better performance and less computations than previous ones. But these mentioned above are 1-D methods. When applying them to linear sensor arrays, we have to assume that all radiating sources are located in the same plane. On the other hand, most available two-dimensional (2-D) methods need a large rectangular plane sensor array and have to perform 2-D spectral peak searches with large computations.

Recently a new 2-D eigenstructure approach called the DOA matrix method was proposed in [7,8] and extended to the 3-D case [9] with good performance in resolution and computation. Unlike the ESPRIT method, the DOA matrix method estimates the 2-D angles of arrival by simultaneously using the nonzero eigenvalues and corresponding eigenvectors of a DOA matrix. It seems that there is some relation between these two methods.

In this paper the ESPRIT method and the DOA matrix method are briefly introduced. The relation between them is theoretically analyzed. It is pointed out that the ESPRIT method is just a special case of the DOA matrix method, and the DOA matrix method gives more generalized and more perfect results. Both should have the same performance for 1-D cases. But the DOA matrix method can be used in the 2-D case. Furthermore, it even can be extended to solve the 3-D problem, as we did in [9], while the ESPRIT method can only be used in the 1-D case.

II. Problem Formulation

Consider an array system consisting of two uniformly spaced linear subarrays of p sensors spaced D apart as shown in Fig. 1. The two subarrays, the subarray X_a and the subarray Y_a, are parallel with subarray X_a lying on the X axis and starting at the origin; d is the distance between them as measured by the Y axis intercept of subarray Y_a. Assume that plane waves with known center frequency Ω_0 emitted by K narrowband sources impinge on this array system and that the DOA of the

* Supported by the Science Fund of China under Contract Ji-85847.
* Supported in part by NSF Grant MIP 87-19886.

sources are $\{\theta_1, \theta_2, \ldots, \theta_K\}$, where $\theta_k=(\alpha_k, \beta_k)$, α_k is the DOA of the k-th source relative to the X axis and β_k relative to the Y axis, as shown in Fig. 1. The sampled vectors of subarrays X_a and Y_a can be written as [7]

$$X(t) = As(t) + N_x(t) \quad (1)$$
$$Y(t) = A\Phi s(t) + N_y(t) \quad (2)$$

where

$$X(t) = [x_1(t), x_2(t), \ldots, x_p(t)]^T \quad (3)$$
$$Y(t) = [y_1(t), y_2(t), \ldots, y_p(t)]^T \quad (4)$$
$$s(t) = [s_1(t), s_2(t), \ldots, s_K(t)]^T \quad (5)$$
$$N_x(t) = [n_{x1}(t), n_{x2}(t), \ldots, n_{xp}(t)]^T \quad (6)$$
$$N_y(t) = [n_{y1}(t), n_{y2}(t), \ldots, n_{yp}(t)]^T \quad (7)$$

A is a pxK Vandermonde matrix

$$A = [a(\alpha_1), a(\alpha_2), \ldots, a(\alpha_p)] \quad (8)$$

with k-th column

$$a(\alpha_k) = [1, \ldots, \exp(j\Omega_0(p-1)\tau_{Dk})]^T \quad (9)$$

τ_{Dk} is the inter-element path delay of the plane wave from the k-th source, $\tau_{dk}=(D/c)\cos\alpha_k$, where c is the wave propagation velocity. Φ is a KxK diagonal matrix

$$\Phi = \mathrm{diag}[\exp(j\Omega_0\tau_{d1}), \ldots, \exp(j\Omega_0\tau_{dK})]$$
$$= \mathrm{diag}[\phi(\beta_1), \ldots, \phi(\beta_K)] \quad (10)$$

τ_{dk} is the path delay of the plane wave of the k-th source between the two subarrays, $\tau_{dk}=(d/c)\cos\beta_k$.

Note that the k-th column of the matrix A, $a(\alpha_k)$, and the k-th element, $\phi(\beta_k)$, of the diagonal matrix Φ are associated with the DOA, i.e. (α_k, β_k), of the k-th source. They are referred to as the signal vector and signal element, respectively.

The auto-covariance matrix of X(t) is given by

$$R_{xx} = E[X(t)X(t)^H] = ASA^H + \sigma^2 I \quad (11)$$

where ^{H} denotes the conjugate transpose and E the expectation operator, $S=E[s(t)s(t)^H]$ is the covariance matrix of source signals, I is the identity matrix, and σ^2 is the variance of the additive noise. The cross-covariance matrices of Y(t) and X(t) are given by

$$R_{yx} = E[Y(t)X(t)^H] = A\Phi SA^H + E[A\Phi s(t)N_x(t)^H] + E[N_y(t)s(t)^H A^H] + E[N_y(t)N_x(t)^H] \quad (12)$$

and

$$R_{xy} = E[X(t)Y(t)^H] = R_{yx}^H \quad (13)$$

We assume that the additive noises are uncorrelated with signals and with each other, in which case Eqs. (12) and (13) can be written as

$$R_{yx} = A\Phi SA^H \quad (14)$$
$$R_{xy} = R_{yx}^H = AS\Phi^H A^H \quad (15)$$

III. The ESPRIT Method

In Eq. (11), the auto-covariance matrix R_{xx} has two terms. The one associated with source signals is denoted by

$$R_{xx0} = R_{xx} - \sigma^2 I = ASA^H \quad (16)$$

Obviously, the rank of R_{xx0} is equal to K, the number of sources if there are no fully correlated (coherent) sources.

The subspace rotation method (ESPRIT) for DOA estimation relies on determining Φ from the estimated covariance matrices R_{xx0} and R_{xy}. It is shown in [4] if S is nonsingular, the K nonzero generalized eigenvalues of the matrix pair $\{R_{xx0}, R_{xy}\}$ are equal to $\phi(\beta_k)$. Thus, if $\{\mu_i\}$, i=1,2,...,K, are the nonzero eigenvalues of the matrix pair $\{R_{xx0}, R_{xy}\}$, the estimated angles $\{\beta_i\}$, i=1,2,...,K, are given by

$$\beta_i = \cos^{-1}\{\frac{c}{\Omega_0 d} \arg(\mu_i)\} \quad (17)$$

The ESPRIT method is a 1-D method, it merely estimates the 1-D angles, $\{\beta_i\}$, i=1,2,...,K, from the generalized eigenvalues of the matrix pair $\{R_{xx0}, R_{xy}\}$.

IV. The DOA Matrix Method

In [7], a pxp matrix R referred to as the DOA matrix is defined

$$R = R_{yx}R_{xx0}^{\#} \quad (18)$$

where $R_{xx0}^{\#}$ denotes the pseudoinverse of R_{xx0} and is constructed by the nonzero eigenvalues, σ_i, and corresponding eigenvectors, v_i, of R_{xx0} via

$$R_{xx0}^{\#} = \sum_{i=1}^{K} \sigma_i^{-1} v_i v_i^H \quad (19)$$

It is shown in [7] that if A and S are nonsingular, the DOA matrix R has its K nonzero eigenvalues equal to the K diagonal elements of Φ and corresponding eigenvectors equal to the K column vectors of matrix A, i.e.

$$RA = A\Phi \quad (20)$$

Therefore, if $\{\epsilon_i\}_{i=1}^{K}$ and $\{u_i\}_{i=1}^{K}$ are the K nonzero eigenvalues and corresponding eigenvectors of the DOA matrix R, the estimated α_i and β_i can be given by

$$\alpha_i = \cos^{-1}\{\frac{c}{\Omega_0 D(p-1)} \sum_{n=2}^{p} \frac{1}{n-1} \arg[\frac{u_i(n)}{u_i(1)}]\} \tag{21}$$

and

$$\beta_i = \cos^{-1}\{\frac{c}{\Omega_0 d} \arg(\epsilon_i)\} \tag{22}$$

Note the DOA matrix method estimates both angles, α_k and β_k, simultaneously. Clearly, in the special case that all the sources are located in the same plane, it can also be used by simply calculating the α_k or β_k, as with the ESPRIT method.

V. Relation Between the Two Methods

Let $\{\mu_i\}_{i=1}^{K}$ and $\{q_i\}_{i=1}^{K}$ be the nonzero generalized eigenvalues and corresponding eigenvectors of the matrix pair $\{R_{xx0}, R_{xy}\}$ in the ESPRIT method. We have

$$R_{xx0} q_i = \mu_i R_{xy} q_i \qquad i=1,2,\dots,K \tag{23}$$

Substitute R_{xy} with R_{yx} according to (15)

$$R_{xx0} q_i = \mu_i R_{yx}^H q_i \tag{24}$$

Take the complex conjugate transpose of both sides of (24)

$$q_i^H R_{xx0} = \mu_i^* q_i^H R_{yx} \tag{25}$$

where * denotes the complex conjugate. Then

$$q_i^H R_{xx0} R_{xx0}^{\#} = \mu_i^* q_i^H R_{xy} R_{xx0}^{\#}$$

$$q_i^H [\sum_{i=1}^{K} \sigma_i v_i v_i^H][\sum_{i=1}^{K} \sigma_i^{-1} v_i v_i^H] = \mu_i^* q_i^H R_{yx} R_{xx0}^{\#}$$

$$q_i^H [\sum_{i=1}^{K} v_i v_i^H] = \mu_i^* q_i^H R_{yx} R_{xx0}^{\#} \tag{26}$$

According to the orthogonality between the signal subspace and the noise subspace,

$$q_i^H \perp \{v_{K+1}, v_{K+2}, \dots, v_p\}$$

and

$$q_i^H v_k = 0 \qquad k=K+1, K+2, \dots, p \tag{27}$$

Thus, Eq. (26) can be rewritten as

$$q_i^H [\sum_{i=1}^{K} v_i v_i^H + \sum_{i=K+1}^{p} v_i v_i^H] = \mu_i^* q_i^H R_{yx} R_{xx0}^{\#} \tag{28}$$

Here

$$[\sum_{i=1}^{K} v_i v_i^H + \sum_{i=K+1}^{K} v_i v_i^H] = I \tag{29}$$

Therefore, we have

$$q_i^H R_{yx} R_{xx0}^{\#} = (\mu_i^*)^{-1} q_i^H \tag{30}$$

Note that $R_{yx} R_{xx0}^{\#}$ is just the DOA matrix R in (18)

$$q_i^H R = (\mu_i^*)^{-1} q_i^H \tag{31}$$

We know that the nonzero eigenvalues of the DOA matrix have a unit magnitude, i.e. $|\mu_i| = 1$. Therefore, Eq. (31) can be simplified to

$$q_i^H R = \mu_i q_i^H \tag{32}$$

where q_i^H is the left eigenvector of R rather than the right eigenvector which is used by the DOA matrix method. Furthermore, q_i^H is not the signal vector in (10). Eq. (32) implies that the eigenvalues $\{\mu_i\}_{i=1}^{K}$ of the matrix pair in (23) for the ESPRIT method are also the eigenvalues of the DOA matrix R for the DOA matrix method. In the ESPRIT method, only the eigenvalues are used so that almost half of the information is lost. But in the DOA matrix method, both eigenvalues and eigenvectors are fully utilized to estimate the 2-D bearing angles. From these arguments we know that the ESPRIT method is just a special case of the DOA matrix method, and the DOA matrix method is the generalized ESPRIT method. Therefore, both should have the same performance for the 1-D case. But the DOA matrix method can be used in the 2-D case. Furthermore, it can be even extended to solve the 3-D problem, as we did in [9], while the ESPRIT method can only be used in the 1-D case. Moreover, as both the covariance matrices R_{xx0} and R_{xy} are not of full rank, it causes trouble to decompose the generalized eigenvalues of the matrix pair in the ESPRIT method, which has been discussed in [10]. But in the DOA matrix method we have no such problem.

It is interesting to note that in almost all of the eigenstructure methods the eigendecomposition merely seperates the signal subspace from the noise subspace, but in the DOA matrix method it gives the original basis of the signal subspace directly. Without requiring any spectral peak searches, the 2-D DOA matrix method is highly efficient in computation. Thus it gives more generalized and more perfect results.

References

[1] V. Pisarenko, "The Retrival of Harmonics from Covariance Functions," Geophys. J. Roy. Astron. Soc., 1972, pp.347-366.

[2] R. Schmidt, "Multiple Emitter Location and Signal Parameter Estimation," Proc. of RADC Spectrum Estimation Workshop, Oct. 1979, pp.243-258.

[3] R. Kumaresan and D. Tufts, "Estimating the Angles of Arrival of Multiple plane Waves," IEEE Trans. on Aerosp. Electron. Syst. vol. AES-19, Jan. 1983, pp. 134-139.

[4] A. Paulraj, R. Roy, and T. Kailath, "A Subspace Rotation Approach to Signal Parameter Estimation," Proceedings of the IEEE, vol. 74, no. 7, July 1986, pp. 1044-1045.

[5] R. Roy, A. Paulraj, and T. Kailath, "ESPRIT -- A Subspace Rotation Approach to Estimation of Parameters of Cisoids in Noise," IEEE Trans. on ASSP, vol. ASSP-34, no. 5, Oct. 1986, pp. 1340-1342.

[6] R. Roy, A. Paulraj, and T. Kailath, "Comparative Performance of ESPRIT and MUSIC for Direction-of-Arrival Estimation," Proc. IEEE ICASSP-87, Dallas, Texas, April 1988, pp. 2344-2347.

[7] Q. Yin, R. Newcomb, and L. Zou, "Estimating 2-D Angles of Arrival via Two Parallel Linear Array," in Proc. IEEE ICASSP-89, Glasgow, Scotland, May 1989, pp. 2803-2806.

[8] Q. Yin, R. Newcomb, and L. Zou, "Estimation of 2-D Angles-of-Arrival for Coherent signals," Proc. of 1989 International Conference on Circuits and Systems, Nanjing, China, July 1989, pp. 541-544.

[9] Q. Yin,"Estimating Directions-of-Arrival of Multiple Sources," Ph. D. dissertation, Xi'an Jiaotong University, Xi'an, P. R. China, 1989.

[10] M. Zoltonski,"Solving the Semi-Definite Generalized Eigenvalue Problem with Application to ESPRIT Algorithm," Proc. of IEEE ICASSP-87, Dallas, Texas, 1987, pp. 2316-2319.

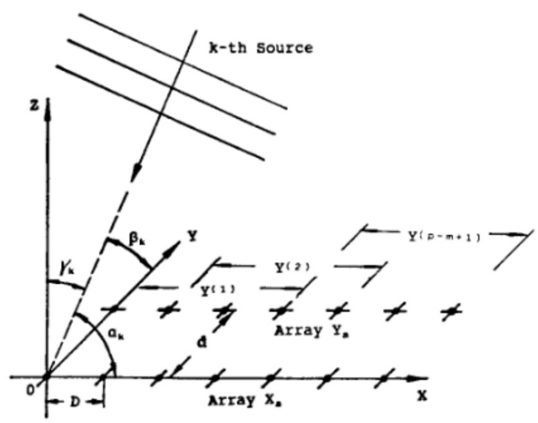

Fig. 1. The geometry of the array system