CIRCUITS SYSTEMS SIGNAL PROCESS
VOL. 9, NO. 2, 1990

SOLID HOLED TORUS KNOT OSCILLATORS*

Rassa Rassai,[1] *George Syrmos,*[1] *and Robert W. Newcomb*[1]

Abstract. In this paper we develop oscillators whose outputs are solid holed torus knots in real three-dimensional space. Here solid holed knots are closed trajectories on the solid holed torus where the solid holed torus is formed by three perpendicular circles revolved around one another. An electronic circuit is presented that can generate any desired solid holed torus knot.

1. Introduction

We recall that a torus can be considered as a meridian circle revolved around an axial circle to form the doughnut-shaped torus. By revolving the torus around another circle, called the longitudinal circle, we obtain the solid holed torus [1, p. 123]. This is illustrated in Figure 1 where c_1 is the longitudinal circle, c_2 is the axial circle, and c_3 is the meridian circle. In visualizing this solid holed torus we see that the hollowness of the c_2–c_3 torus is filled in as this torus is revolved around c_1 but that the doughnut hole in the c_2–c_3 torus leaves a hollow core in the final solid holed torus, thus, giving the name.

Given the solid holed torus we place a knot on it by using the trajectory of a point traveling on the three circles as the trace of the knot; we call this knot a solid holed torus knot or, alternatively, since three circles are involved, an (m_3, m_2, m_1)-torus knot where m_i is the radian frequency of traveling around the ith circle. Such a trajectory can be obtained by using three oscillators, one for each circle, and combining their outputs into a single trajectory in the underlying real-world three-dimensional space. Since knots are closed we need a closed trajectory which will be obtained if the frequencies of the oscillators are rationally related.

As background we recall that an (m_3, m_2)-torus knot is a trajectory on a torus which goes m_3 times around the meridian circle and m_2 times around

* Received September 16, 1988. Research supported by NSF Grant MIP 87-19886.

[1]Microsystems Laboratory, Electrical Engineering Department, University of Maryland, College Park, Maryland 20742, USA.

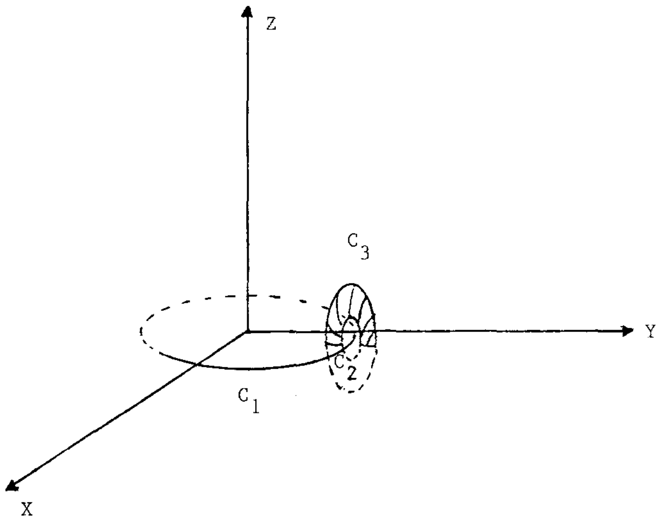


Figure 1. Formation of a solid holed torus.

the axial circle of a torus [2]–[5]. The set of semistate equations resulting in an (m_3, m_2)-torus knot are given in [2]. In Section 2 we give a brief review of the results found in [2] for obtaining the semistate equations resulting in an (m_3, m_2)-torus knot oscillator, reformulating these so that they can be generalized to the solid holed case. In Section 3 we give the semistate equations for solid holed torus (m_3, m_2, m_1)-knots and illustrate these by means of an example. In Section 4 we give the proposed electronic circuit oscillator design and in Section 5 we give the conclusions.

2. Realization of an (m_3, m_2)-torus knot oscillator

Here we rephrase the results of [2] in a form that leads to a convenient generalization to the solid holed torus. In [2] we found the semistate equations for the realization of an (m_3, m_2)-torus knot oscillator. In four-dimensional space the dynamical portions of these equations are as follows:

$$dx_3/dt = -m_3 y_3, \tag{1a}$$

$$dy_3/dt = m_3 x_3, \tag{1b}$$

$$dx_2/dt = -m_2 y_2, \tag{1c}$$

$$dy_2/dt = m_2 x_2, \tag{1d}$$

where initial conditions are chosen as

$$x_3(0) = 0, \qquad y_3(0) = R_3, \qquad x_2(0) = 0, \qquad y_2(0) = R_2, \qquad \text{(1e,f,g,h)}$$

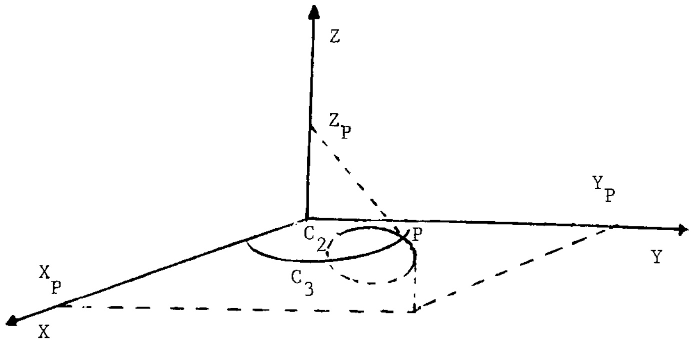


Figure 2. Torus coordinate frame.

in which R_3 and R_2 are the radii of the two circles used to construct the torus. With these initial conditions the solutions to (1) are the following:

$$x_3 = R_3 \cos(m_3 t), \tag{2a}$$

$$y_3 = R_3 \sin(m_3 t), \tag{2b}$$

$$x_2 = R_2 \cos(m_2 t), \tag{2c}$$

$$y_2 = R_2 \sin(m_2 t). \tag{2d}$$

The subset of (2a) and (2b) result in a circle c_3 and the subset of (2c) and (2d) result in a circle c_2 where we take the meridian circle to be c_3 and the axial circle to be c_2. When m_3 and m_2 are rationally related the trajectories obtained from (1) form an (m_3, m_2)-torus knot and by observing Figure 2 we can write the coordinates of any point on these trajectories in real three-dimensional XYZ-space as

$$X(t) = (R_3 + R_2 \cos(m_2 t)) \cos(m_3 t), \tag{3a}$$

$$Y(t) = (R_3 + R_2 \cos(m_2 t)) \sin(m_3 t), \tag{3b}$$

$$Z(t) = R_2 \sin(m_2 t). \tag{3c}$$

By direct substitution we see that (3) satisfy the equation of a torus, (4), in XYZ-space:

$$Z^2 + [R - [X^2 + Y^2]^{1/2}]^2 = r^2, \tag{4}$$

where the radii are set by initial conditions as

$$R = R_3 \quad \text{and} \quad r = R_2. \tag{5a,b}$$

Now we proceed to find the relations between the method described above and the one given in [2] and [3]. First, we notice that the X, Y, Z coordinates given by (3) are the coordinates of a point on the torus as time varies. The coordinates of a point on the torus using the method described in [2] are given by the following equations in which we use a prime, ′, to designate the use of a possibly different set of initial conditions at (1e,f,g,h), that is the replacement of R_2 and R_3 by R'_2 and R'_3:

$$X' = x'_3/D, \qquad Y' = y'_3/D, \qquad Z' = R'_3 x'_2/[D(\delta^2 - R'^2_2)^{1/2}], \quad \text{(6a,b,c)}$$

$$R = \delta R'_3/[\delta^2 - R'^2_2], \qquad r = R'_3 R'_2/[\delta^2 - R'^2_2], \quad \text{(6d,e)}$$

where

$$D = \delta + y'_2 \quad \text{(6f)}$$

Substitution in (4) written with primes shows that the trajectories of (6) lie on the same torus as the trajectories of (5). Equating R and r from (6d,e) with those found at (5a,b) gives the relation between the two sets of initial conditions and the free parameter $\delta > R'_2$ to give these identical tori, Thus,

$$R'_3 = \delta[1 - (r/R)^2]R, \qquad R'_2 = \delta r/R. \quad \text{(7a,b)}$$

3. (m_3, m_2, m_1)-Torus knots

In Section 2 we presented two means to find X, Y, and Z for a torus knot. Now we proceed to generalize the equations to the solid holed torus knots. For this we add the differential equation for the third circle:

$$dx_1/dt = -m_1 y_1, \quad \text{(8a)}$$

$$dy_1/dt = m_1 x_1, \quad \text{(8b)}$$

with initial conditions

$$x_1(0) = 0, \qquad y_1(0) = R_1. \quad \text{(8c,d)}$$

To obtain the XYZ coordinates of a point on the solid holed torus knot we observe that each circle in an x_i-y_i-z_i coordinate system has coordinates $[R_i \cos(m_i t), R_i \sin(m_i t), 0]$. Then we do coordinate transformations to express the coordinates of a point traveling on the knot in terms of the base frame coordinates, that is in terms of the x_1-y_1-z_1 coordinate system. We do this for transformation of the second and third frames into the first.

We start by considering rotations and transformations of axes of the circles involved. Consider, as shown in Figure 3, that the $(i + 1)$st circle c_{i+1} is in a plane p_{i+1} perpendicular to the plane p_i of c_i and with p_i containing the radius vector R_i of c_i. In the $(i + 1)$st plane p_{i+1}, define a coordinate system with X_{i+1} along R_i, and having $Y_{i+1} = Z_i$ with Z_{i+1} yielding a right-hand coordinate system (as unit vectors we would have $Z_{i+1} = X_{i+1} \times Y_{i+1}$ where $\times$ denotes the cross product) and centered at the center of c_{i+1}. To relate the $(i + 1)$st to the ith coordinate system, we can perform in order the following steps:

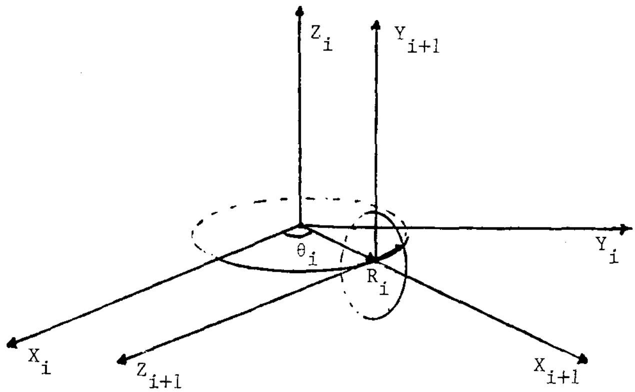


Figure 3. The ith and $(i+1)$st coordinate frames.

(a) We rotate the ith frame around the radius vector of the ith circle to line up the Z_i and Y_{i+1} axes; this is a 90° rotation

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} P', \tag{9a}$$

where the three-vectors P and P' give coordinates of a point in the coordinate frame used before and the one used after the rotation, respectively.

(b) We rotate around the Y_{i+1} axis by an angle of $-\theta_i$ going from Z to X, where $\theta_i = m_i t$,

$$P'' = \begin{bmatrix} C_i & 0 & -S_i \\ 0 & 1 & 0 \\ S_i & 0 & S_i \end{bmatrix} P' \quad \text{or} \quad P' = \begin{bmatrix} C_i & 0 & S_i \\ 0 & 1 & 0 \\ -S_i & 0 & C_i \end{bmatrix} P'' \tag{9b}$$

in which we use the designations $C_i = \cos\theta_i$ and $S_i = \sin\theta_i$.

(c) We translate the origin a distance R_i along the resulting x_{i+1} axis. We have

$$P''' = P'' - R_i \quad \text{which is} \quad P'' = P''' + R_i. \tag{9c}$$

The $'''$ frame is seen to be tracking a point on c_{i+1} as measured in the $'$ frame.

(d) We set

$$P = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_i, \qquad P''' = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{i+1}. \tag{9d,e}$$

The composition of these steps gives the coordinates of a point in the ith coordinate frame in terms of the coordinates of the same point in the $(i + 1)$st coordinate frame, thus

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} C_i & 0 & S_i \\ 0 & 1 & 0 \\ -S_i & 0 & C_i \end{bmatrix} \left\{ \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{i+1} + \begin{bmatrix} R_i \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} C_i & 0 & S_i \\ S_i & 0 & -C_i \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{i+1} + \begin{bmatrix} C_i R_i \\ S_i R_i \\ 0 \end{bmatrix}. \tag{10}$$

Written in component form (10) is

$$X_i = (X_{i+1} + R_i) \cos \theta_i + Z_{i+1} \sin \theta_i, \tag{11a}$$

$$Y_i = (X_{i+1} + R_i) \sin \theta_i - Z_{i+1} \cos \theta_i, \tag{11b}$$

$$Z_i = Y_{i+1}. \tag{11c}$$

The above equations are subject to the absence of an $(n + 1)$st coordinate set in the case of n circles, where $n = 3$ in our case. That is, we set the boundary condition

$$X_{n+1} = Y_{n+1} = Z_{n+1} = 0, \tag{12}$$

which gives

$$X_n = R_n \cos \theta_n, \tag{13a}$$

$$Y_n = R_n \sin \theta_n, \tag{13b}$$

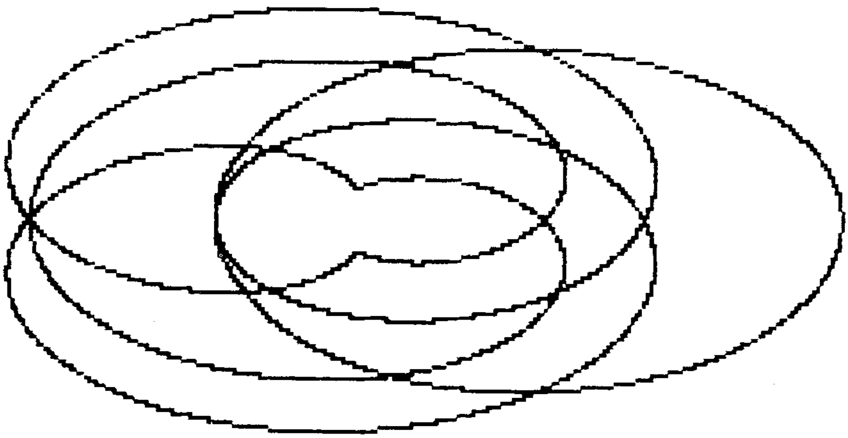
$$Z_n = 0. \tag{13c}$$

These last equations show that we are tracking a point on the last numbered circle, viewing it in the frames attached to the other lower numbered circles.

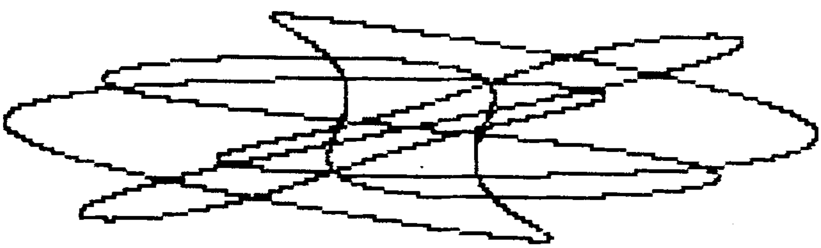
In the case of interest to us we want to express the coordinates of a point in the $(i + 2)$nd frame in terms of the ith for $i = 1$. Thus we perform another iteration on (10) to get

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_i = \begin{bmatrix} C_i & 0 & S_i \\ S_i & 0 & -C_i \\ 0 & 1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} C_{i+1} & 0 & S_{i+1} \\ S_{i+1} & 0 & -C_{i+1} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{i+2} + \begin{bmatrix} C_{i+1} R_{i+1} \\ S_{i+1} R_{i+1} \\ 0 \end{bmatrix} \right\} + \begin{bmatrix} C_i R_i \\ S_i R_i \\ 0 \end{bmatrix} \tag{14}$$

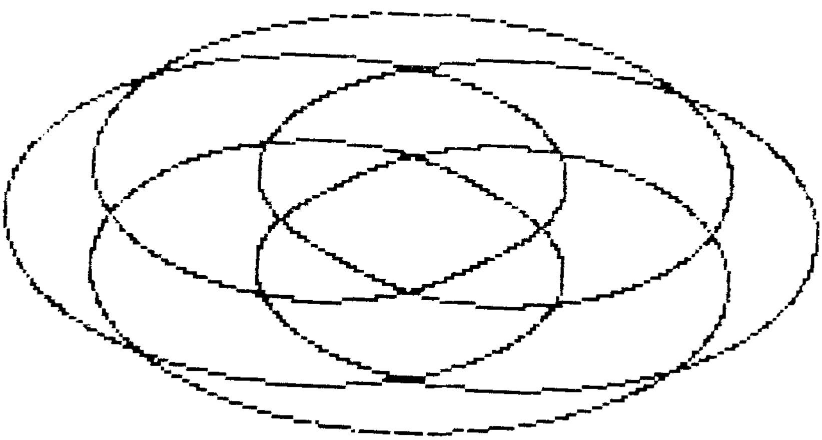
$$= \begin{bmatrix} C_i C_{i+1} & S_i & C_i S_{i+1} \\ S_i C_{i+1} & -C_i & S_i S_{i+1} \\ S_{i+1} & 0 & -C_{i+1} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{i+2} + \begin{bmatrix} C_i C_{i+1} R_{i+1} \\ S_i C_{i+1} R_{i+1} \\ S_{i+1} R_{i+1} \end{bmatrix} + \begin{bmatrix} C_i R_i \\ S_i R_i \\ 0 \end{bmatrix}.$$



(a)
TOP



(b)
SIDE



(c)
$R_3=0$

Figure 4. Views of a (4, 3, 5)-torus knot.

Finally, for our case of the solid holed torus, on using (13) for X_3, Y_3, Z_3, we have the solid holed torus knot described by

$$X(t) = [(R_3 \cos(m_3 t) + R_2) \cos(m_2 t) + R_1] \cos(m_1 t) + R_3 \sin(m_3 t) \sin(m_1 t), \tag{15a}$$

$$Y(t) = [(R_3 \cos(m_3 t) + R_2) \cos(m_2 t) + R_1] \sin(m_1 t) - R_3 \sin(m_3 t) \cos(m_1 t), \tag{15b}$$

$$Z(t) = [R_3 \cos(m_3 t) + R_2] \sin(m_2 t). \tag{15c}$$

Example. We consider a solid holed torus with $R_1 = 10$, $R_2 = 4$, $R_3 = 2$ and $m_1 = 5$, $m_2 = 3$, $m_3 = 4$. Using various viewing spins and tips [6] Figure 4 shows the top view (part (a) where spin = 0, tip = 90°) and the side view (part (b) where spin = 90°, tip = 0) of the resulting solid holed torus knot. In Figure 4(c) we set $R_3 = 0$ and look down upon the knot from the top (spin = 0, tip = 90°) to show the (m_2, m_1)-torus knot resulting by deleting the third circle from the solid holed torus.

4. Electronic circuit realization

For an electronic oscillator which gives a solid holed torus knot as the output we want to obtain signals that give the coordinates of the solid holed torus knot, as (15). However, in order to create these signals we first create sinusoidal oscillators to the responses of the three differential equations from which we have proceeded, that is of

$$dx_i/dt = -m_i y_i, \tag{16a}$$

$$dy_i/dt = m_i x_i. \tag{16b}$$

Then, noting that these solutions are

$$x_i = R_i \cos \theta_i, \tag{17a}$$

$$y_i = R_i \sin \theta_i, \tag{17b}$$

we substitute into the torus knot, coordinates (15) to get

$$X = [(x_3 + R_2)(x_2/R_2) + R_1](x_1/R_1) + y_3(y_1/R_1), \tag{18a}$$

$$Y = [(x_3 + R_2)(x_2/R_2) + R_1](y_1/R_1) - y_3(x_1/R_1), \tag{18b}$$

$$Z = (x_3 + R_2)y_2/R_2. \tag{18c}$$

Therefore, using three linear oscillators we can create $x_1, y_1, x_2, y_2, x_3, y_3$ and then hook the oscillators up via multipliers and adders to create the combinations needed for (18).

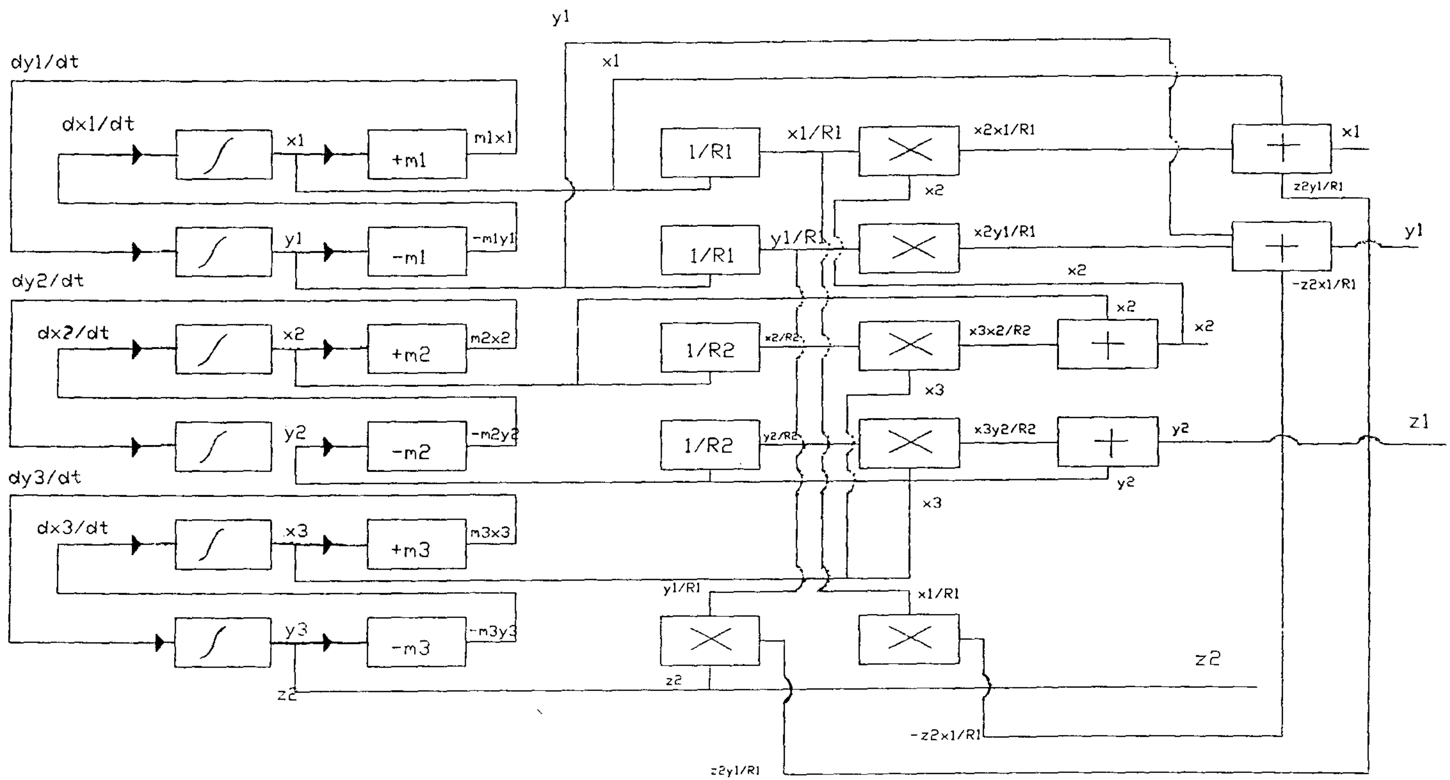


Figure 5. Electronic circuit design for a solid holed torus knot.

An attractive alternative is to return to the coordinate transformations of (10) and perform those transformations directly with hardware connections of the three linear oscillators. For this we use (10), for $i = 2$,

$$X_2 = (X_3 x_2/R_2) + x_2, \tag{19a}$$

$$Y_2 = (X_3 y_2/R_2) + y_2, \tag{19b}$$

$$Z_2 = Y_3, \tag{19c}$$

along with (10), for $i = 1$,

$$X = X_1 = [1 + (X_2/R_1)]x_1 + (Z_2/R_1)y_1, \tag{20a}$$

$$Y = Y_1 = [1 + (X_2/R_1)]y_1 - (Z_2/R_1)x_1, \tag{20b}$$

$$Z = Z_1 = Y_2, \tag{20c}$$

while, for $i = 3 = n$,

$$X_3 = x_3, \qquad Y_3 = y_3 \tag{21a,b}$$

and $x_i = R_i \cos(m_i t)$, $y_i = R_i \sin(m_i t)$. Figure 5 shows the block diagram for the solid holed torus knot oscillator circuit realization of (16) and (19)–(21). All the blocks in Figure 5 can be realized using operational amplifiers and multipliers.

5. Conclusions

In this paper we have given the equations for a solid holed torus knot, formulating these from the geometric configuration of circles generating the solid holed torus. Since three circles are involved and each circle is traversed at the radian frequency m_i, $i = 1, 2, 3$, we have called these (m_3, m_2, m_1)-torus knots, the reversal of the numbering coming about as a matter of convenience for the geometrical set-up where we chose the circle c_1 to be the longitudinal one which gets treated last when we consider first forming a normal torus from the axial and meridian circles c_2 and c_3. It should be noted that various permutations of the m_i lead to various knots. The equations are such that they allow us to generate the knot with standard electronic circuits, there being three oscillators and a number of four-quadrant multipliers. Because four-quadrant multipliers are inconvenient, it is worthwhile looking for alternatives, one being to form the solid holed torus knot trajectories via the second method, basically due to Parris [3], described above for formation of a torus knot. This probably requires some new theory since the topological product of a torus and a circle needs to be reduced into the three-dimensional real-world space; that is the six-dimensional set of oscillations needs to be reduced into three-dimensional space.

It should be clear that we can revolve other closed path curves than circles and get equivalent knots; for example, we could use squares. Thus, we can use other than linear oscillators to make the solid holed torus knots physically. Again, the Van der Pol oscillators appear to be ideal since they are structurally stable [7]. In such a case the above theory needs some modifications but it appears that equations (18) are the proper ones to use.

As a possible application, we note that the end effector of a three-jointed robot arm will follow trajectories similar to those of a solid holed torus knot if the three links are all of the perpendicular rotational kind, depicted in Figure 2 [8], [9]. However, except for the robot base, the angles allowed for a robot arm are constrained while those for our knots are not.

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