SEMISTATE EQUATIONS FOR SOLID-HOLED TORUS KNOTS*

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Abstract

In this paper we present a set of semistate equations for solid-holed torus knots in real 3-dimensional space. Here solid-holed torus knots are closed trajectories in the solid-holed torus, where the solid-holed torus is formed by rotating a punctured disk around a longitudinal circle.

I. Introduction

We recall that the standard torus, that is, the hollow torus, is formed by rotating a circle, called the meridian circle, around another circle, the axial circle [1, p.5][2, p.3]. Now, if a disk is rotated around a third circle, the longitudinal circle, then a solid-holed torus results. By taking trajectories that wind around in the disk and the longitudinal circle, a knot is formed which we call a solid-holed torus knot [3]. This knot can be generated from a set of semistate equations [4], these latter which generalize those of torus knots are given in this paper. In particular, in part II, we present in detail background work for creating an equation describing hollow torii in real 3-dimensional Euclidean space. In part III, we extend our previous work in a natural, intuitive, way to forming solid-holed torus knot trajectories. Also, a few graphical representations of these knots are included.

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II. Hollow Torus Equation for Hollow Torus Knots

As background material upon which the extension to creating a solid-holed torus is based, in this part of the paper we present the semistate equations that produce a hollow torus. Trajectories travelling on this torus, result in hollow torus knots.

The hollow torus can be considered as a topological product of two circles [5, p.15] reduced into real 3-dimensional Euclidean space. In the four dimensional circle space the dynamical portion of the semistate equations for a hollow torus are given as (here ′ denotes d/dt)

$$\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix}' = \begin{bmatrix} 0 & -m_1 & 0 & 0 \\ m_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -m_2 \\ 0 & 0 & m_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} \qquad \begin{matrix} (1a) \\ (1b) \\ (1c) \\ (1d) \end{matrix}$$

These generate the two circles

$$c_1 : X^2 + Y^2 = R_1^2 \qquad (2a)$$

$$c_2 : Z^2 + W^2 = R_2^2 \qquad (2b)$$

with radii R_1, R_2 fixed by initial conditions, which can be chosen to insure that $Z=R_2\cdot\cos(m_2t)$ and $W=R_2\cdot\sin(m_2t)$. The (X,Y)-plane circle, c_1, is traversed at a radian frequency m_1, and the (Z,W)-plane circle is traversed at a radian frequency m_2. These circles, c_1 & c_2, form a hollow torus, T, in the real four dimensional Euclidean space (X,Y,Z,W), [6, p. 161]. Hence, Eqs. (1a-d) can be considered to describe (m_1,m_2)-torus knots [2, p.185]. However, we are interested in viewing the torus in real 3-dimensional Euclidean space, R^3.

To achieve this end, we reduce the torus, T, to R^3. This reduction is obtained by forming the quotient system [7, p. 21] for which, following Parris [8], we introduce the algebraic constraints

$$D=W+\Delta \quad ; \quad \Delta=\text{constant} \qquad (3)$$

$$\underline{X}=X/D,\ \underline{Y}=Y/D,\ \underline{Z}=Z/D \qquad (4a,b,c)$$

where Δ is chosen so that $D>0$.

To see that the quotient system yields trajectories on a torus in three dimensional space, we perform the following algebraic steps. It should be stressed here that this step by step development of the equation that satisfies the hollow torus will aid significantly in understanding the natural extension of our work from the hollow torus to solid-holed torus.

First, we divide Eq. (2b) by D^2, this resulting in the following equation

$$(Z/D)^2+(W/D)^2=(R_2/D)^2 \qquad (5)$$

By using Eq. (5) along with Eqs. (3) and (4c) we get

$$\underline{Z}^2 = -((D-\Delta)/D)^2+(R_2/D)^2 \qquad (6a)$$

$$= -(1-(\Delta/D))^2+(R_2/D)^2 \qquad (6b)$$

$$= -(1-2(\Delta/D)+(\Delta^2/D^2))+R_2{}^2/D_2{}^2 \qquad (6c)$$

$$= -\{((\Delta^2/D^2)-(R_2{}^2/D^2))-2(\Delta/D)\}-1 \quad (6d)$$

$$= -\{[(\Delta^2-R_2{}^2)^{1/2}/D]^2-[(\Delta^2-R_2{}^2)^{1/2}/D]\xi$$

$$+ \xi^2\}-(1-\xi^2) \qquad (6e)$$

where ξ is found by making the $\{\bullet\}$ term a perfect square. Identifying the 1/D terms in Eqs. (6d) and (6e) to achieve this perfect square, we have

$$\Delta=(\Delta^2-R_2{}^2)^{1/2}\bullet\xi \iff \xi=\Delta/(\Delta^2-R_2{}^2)^{1/2}$$

$$(6f,g)$$

Using Eqs. (6a) and (6e), results in

$$\underline{Z}^2 = -[((\Delta^2-R_2{}^2)^{1/2}/D)-\xi]^2-(1-\xi^2) \qquad (7a)$$

$$= -[((\Delta^2-R_2{}^2)^{1/2}/D)-\Delta/(\Delta^2-R_2{}^2)^{1/2}]$$

$$-[1-\Delta^2/(\Delta^2-R_2{}^2)] \qquad (7b)$$

$$= [\Delta/(\Delta^2-R_2{}^2)^{1/2}-(\Delta^2-R_2{}^2)^{1/2}\bullet(1/D)]^2$$

$$+R^2/(\Delta^2-R_2{}^2) \qquad (7c)$$

Next we substitute for 1/D. To find the term to use, look at Eq. (2a), in the following manner. Divide Eq. (2a) by D^2 to get

$$(X/D)^2+(Y/D)^2=(R_1/D)^2 \qquad (8a)$$

or, equivalently,

$$1/D = +(\underline{X}^2+\underline{Y}^2)^{1/2}/R_1 \qquad (8b)$$

It should be noted that in Eq. (8b) we have chosen the (+) sign on the square root as $D>0$ is assumed ($D\neq 0$ is required) as is $R_1>0$. Substituting Eq. (8b) in Eq. (7c) gives

$$\underline{Z}^2+[(\Delta/(\Delta^2-R_2{}^2)^{1/2})-((\Delta^2$$

$$-R_2{}^2)^{1/2}/R_1)\bullet(\underline{X}^2+\underline{Y}^2)^{1/2}]^2$$

$$= R_2{}^2/(\Delta^2-R_2{}^2) \qquad (9)$$

At this point there are several choices that can be made to define the final (x,y,z) coordinates of the torus in R^3. In all of these, we derive the equation describing the hollow torus in real 3-dimensional Euclidean space to be

$$z^2 + [R-(x^2+y^2)^{1/2}]^2 = r^2 \qquad (10)$$

The choice undertaken previously [9] was

$$x=\underline{X},\ y=\underline{Y},\ z=(R_1/(\Delta^2-R_2{}^2)^{1/2})\bullet\underline{Z}$$

$$(11a,b,c)$$

in which (after correcting the misprint of [9])

$$R=R_1\bullet\Delta/(\Delta^2-R_2{}^2),\ r=R_1\bullet R_2/(\Delta^2-R_2{}^2)$$

$$(11d,e)$$

However, another choice, and one more useful for rotating a disk, rather than a circle, around the longitudinal circle is the one given by Eqs. (20a,b,c) in part III of this paper. In this latter, we have obtained $R=R_1$, which is independent of R_2.

III. Solid-Holed Torus Equation for Solid-Holed Torus Knots

The solid-holed torus can be considered as the topological product of a disk and a circle. For the solid-holed torus knot we need trajectories in a disk to rotate around a longitudinal circle. For this we consider trajectories that are in the disk, defined in polar coordinates by $\rho_m\le\rho\le\rho_M$ and $0\le\theta\le2\pi$.

To relate to the solid-holed torus knot, we choose $\rho=R_2$ and let θ vary as a new parameter, $\theta_3=m_3t$. Thus, a trajectory of the type defined can be given by

$$\rho=R_2(\theta_3)=\underline{R}_2+R_3\bullet\sin(\theta_3) \qquad (12a)$$

where

$$\theta=\theta_3=m_3t; \quad 0\le t\le 2\pi \qquad (12b)$$

As a result of the above redefinition of R_2, which now varies in θ_3, we define solutions for Z, W analogous to those of Eqs. (1c,d), to be given in terms of polar coordinates as

$$Z=R_2(\vartheta_3)\cdot\cos(\vartheta_2) \quad (13a)$$

$$W=R_2(\vartheta_3)\cdot\sin(\vartheta_2) \quad (13b)$$

This requires (recall that $'=d/dt$)

$$Z'=R_2'(\vartheta_3)\cdot\cos(\vartheta_2)-R_2(\vartheta_3)\cdot\cos(\vartheta_2)\cdot\vartheta_2' \quad (14a)$$

$$W'=R_2'(\vartheta_3)\cdot\sin(\vartheta_2)-R_2(\vartheta_3)\cdot\cos(\vartheta_2)\cdot\vartheta_2' \quad (14b)$$

Converting the right side of Eqs. (14a,b) back to rectangular coordinates we have

$$Z' = -m_2\cdot W + m_3\cdot U\cdot[Z/(Z^2+W^2)^{1/2}] \quad (15a)$$

$$W' = m_2\cdot Z + m_3\cdot U\cdot[W/(Z^2+W^2)^{1/2}] \quad (15b)$$

Here we have substituted for $R_2'(\vartheta_3)$ by

$$R_2'(\vartheta_3) = m_3\cdot U \quad (16a)$$

Because R_3 is a constant, and $\vartheta_3=m_3t$, $R_2'(\vartheta_3)$ can be realized via a third oscillator as

$$U' = -m_3\cdot V \quad (17a)$$

$$V' = m_3\cdot U \quad (17b)$$

These have

$$V = \underline{R}_2 + R_3\cdot\sin(m_3t) \quad (18a)$$

$$U = R_3\cdot\cos(m_3t) \quad (18b)$$

as solutions showing that $\underline{R}_2=V(0)$ and $R_3=U(0)$. It should be pointed out that in converting the polar coordinates of the right side of Eqs. (14a,b) to the rectangular coordinates of the right side of Eqs. (15a,b), we have used the fact that $R_2{}^2(\vartheta_3)=Z^2+W^2$ i.e.

$$R_2(\vartheta_3) = (Z^2+W^2)^{1/2} \quad (19)$$

As before we generate the longitudinal circle via Eqs. (1a,b).

Equations (1a,b), (15a,b) and (17a,b) can be thought of as those for creating closed trajectories in a 6-dimensional space (of discs and circles). However, we are interested in trajectories in the solid-holed torus in real three dimensional Euclidean space R^3. To achieve this, we employ again the same reduction as in part II, namely the one described by the algebraic constraints given by Eqs. (3) and (4), to the X and Y of Eqs. (1a,b) and to the Z, W of Eqs. (15a,b).

Using, the D of Eq. (3) and variables of Eqs. (4), we let, for the real R^3 coordinates of the knot,

$$x = [(\Delta^2-R_2{}^2(\vartheta_3))/\Delta]\cdot\underline{X}, \quad (20a)$$

$$y = [(\Delta^2-R_2{}^2(\vartheta_3))/\Delta]\cdot\underline{Y}, \quad (20b)$$

$$z = [(\Delta^2-R_2{}^2(\vartheta_3))^{1/2}/\Delta]\cdot\underline{Z} \quad (20c)$$

Then the knot trajectories satisfy the following equation

$$z^2+[R_1-(x^2+y^2)^{1/2}]^2=(R_1\cdot R_2(\vartheta_3)/\Delta)^2 \quad (21)$$

where

$$R=R_1,\ r=R_1\cdot R_2(\vartheta_3)/\Delta \quad (22a,b)$$

Equations (21) & (22) describe a solid-holed torus with parameter ϑ_3 chosen in the range $0\le\vartheta_3\le2\pi$. In this ϑ_3 controls the radius of the disk that is rotated around the z axis, as described at the beginning of this section. For the knot itself ϑ_3 is given by m_3t while for the general solid-holed torus, it is a free parameter which when fixed yields a hollow torus. Thus the solid-holed torus can be considered as being constructed from overlapping layers of hollow torii.

In summary, Eqs. (1a,b), (15a,b), (17a,b) constitute the dynamical portion of the semistate equations for forming the solid-holed torus knots and Eqs. (3), (4) and (20) are the algebraic portion of the semistate equations.

V. Conclusions

The choice of coordinates (x,y,z) as they appear in Eqs. (20) is essential in order to get R independent of $R_2(\vartheta_3)$, as it appears in Eq. (22a), since $R_2(\vartheta_3)$ varies in the disk but we desire that R does not. Other choices for the coordinates (x,y,z) were also available, but not so convenient. Another point of importance is that an equivalent means of making the solid-holed torus is to rotate a (hollow) torus around a third longitudinal circle, though this appears to very complicated, a third way is presented in [10]. Finally, as far as the electronic realization of the solid-holed torus knot is concerned, it is apparent from Eqs. (15a,b) that this would be rather hard to implement, since the equations involve square-root and division nonlinearities.

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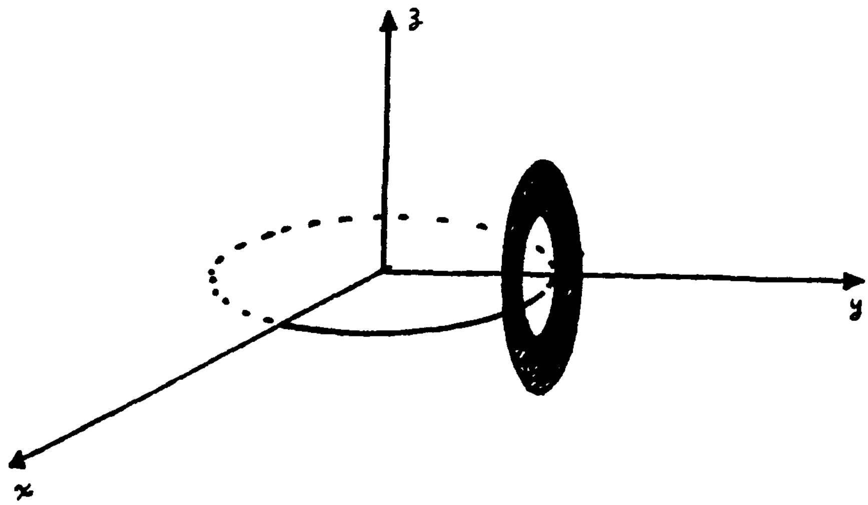

Figure 1
Formation of a Solid-Holed Torus

Figure 2
A Solid-Holed Torus