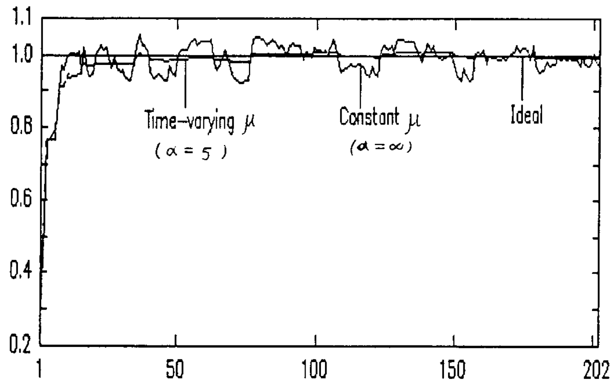

Fig. 1. Adaptive weight versus time. Stationary model.

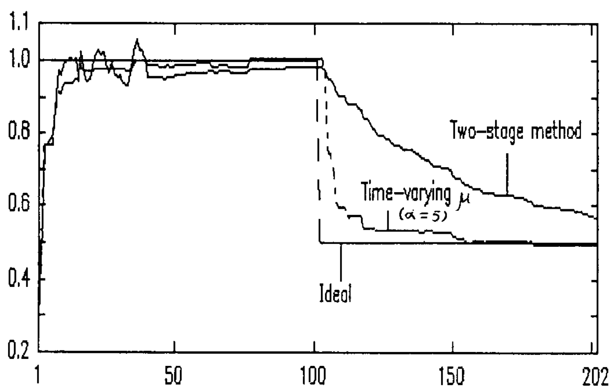

Fig. 2. Adaptive weight versus time. Time-varying model.

misadjustment of w than the new algorithm with $\alpha = 5$; both have the same convergence rate.

The second example compares the new algorithm with the two-stage algorithm. Here, the model is not stationary:

$$d(n) = x(n) + 0.1x(n-1), \qquad \text{if } n < 100 \tag{10}$$

$$d(n) = \frac{1}{2}x(n) + 0.1x(n-1), \qquad \text{if } n > 100. \tag{11}$$

The two-stage algorithm assumes that the algorithm converges when $n = 40$ and sets $\mu = \hat{\mu}/10$ as $n > 40$. Fig. 2 illustrates that the two-stage algorithm does not follow the time-varying model as well as the new algorithm.

In this paper, the constant convergence factor, μ, of the LMS algorithm is modified to $\mu(n)$ (see (9)), which depends not only on the input, but on the error output as well. In (9), $\hat{\mu}$ is a constant which is smaller than or equal to the upper bound of μ for the conventional LMS algorithm. Actually, this can be further extended by replacing $\hat{\mu}$ in (9) by the μ's suggested by MLMS, HA, or IA. In these cases, $\hat{\mu}$ is time-varying. It incorporates the merits of MLMS, HA, or IA, namely, stability and fast convergence; it also overcomes their chief drawback, i.e., large misadjustment. Thus the conventional constant μ, the MLMS, HA, and IA are all its special cases by letting the damping parameter α be infinity.

Computer simulations show that by picking a suitable damping parameter α, the new algorithm possesses a fast convergence rate and yields very small misadjustment. The new algorithm works well under nonstationary environments, in sharp contrast with the two-stage method.

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Realization of the Connected Sum of Two Identical Torus Knots

RASSA RASSAI AND ROBERT W. NEWCOMB

***Abstract*—In this paper we realize the connected sum of two identical torus knots by giving the details of a previously described technique for realizing the connected sum of two arbitrary knots. The theory is generalized from linear to Van der Pol oscillators and an example carried out for the connected sum of two (3,4)-torus Van der Pol knots.**

I. Introduction

To tie a knot seems to be part of every day life of a person. Therefore, it is of interest to design equipment that can tie knots. For example, one might wish to have a robot tie a knot in a hostile environment or one might desire to guide the arm of a paraplegic in tying a knot. In these cases the existence of electronic circuits with outputs that simulate motion that is a trajectory on a knot could be of use in controlling or designing the system.

As was mentioned in [1] and [2], it is possible to make certain basic knots and then splice these knots together to form more complicated knots, for example, the granny knot can be considered as two trefoil knots spliced together. Splicing two knots K_1 and K_2 mathematically can be looked upon as the connected sum of the two knots, denoted by $K_1 \# K_2$. Basic knot theory is nicely written up in [3] where the idea for the realization of the connected sum of two knots is described. This realization was developed for general knots in [1] while for the particular case of torus knots a different technique was discussed in [2].

The idea as developed in [1], is based on keeping one knot fixed while moving and stretching the other knot to give a certain kind of intersection thus yielding a connected sum via a step function that chops out the unnecessary overlap.

Here we use the method discussed in [1] to realize the connected sum of two identical (m_1, m_2)-torus knots giving the details and comparing the results with the ones obtained in [2]. In this paper we show that the method described in [1], which can be applied for finding the connected sum of any two knots, can be used for finding the connected sum of two torus knots and

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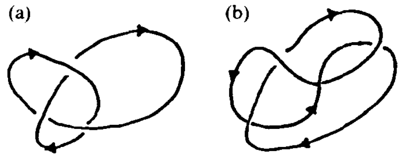

Fig. 1. (a) Trefoil knot. (b) Figure eight knot.

that the results are identical to the ones found in [2]. For this we should mention that in [2] we realized the equations for the connected sum of two torus knots using hysteresis planes. We should also mention that in [2], for the generation of torus knots, we used linear oscillators in order to bring out the philosophy even though the linear oscillators are structurally unstable. However, the method described in [1] actually gives the connected sum of the two torus knots no matter how the knots are generated. Thus to illustrate the basic equations in the simplest case, in Section II we use linear oscillators. In Section III we use Van der Pol torus knots, that is, we replace the linear oscillators by Van der Pol nonlinear oscillators that are structurally stable and thus more practical for actual constructions. We should also mention that in both sections we use identical torus knots; that is, torus knots which have the same frequencies of oscillations and are wrapped around geometrically identical tori.

In Section II we give a review of the results obtained in [1] for the connected sum of any two knots and in Section III we give an example and use the technique of [1] in order to realize the connected sum of two identical (m,n)-torus knots. Section III contains discussion and conclusion.

II. Realization of the Connected Sum of Two Knots

As a subfield of topology, knot theory forms the core of a wide range of problems dealing with the position of one manifold imbedded within another. In such a topological setting a knot in real three dimensional space is usually specified by a projection onto a plane. Fig. 1 shows, for example, the projected images of a trefoil knot and a figure eight knot. A knot K is in regular position (with respect to a specified projection plane) if the only multiple points of the projection of K are double points and no double point is the image of any vertex of the knot [4, p. 6]. The second condition insures that every double point represents a genuine crossing of strands in the knot. Therefore, each double point of the projected image of a polygonal knot in regular position is the image of two and only two points of the knot. In (X,Y,Z)-space the one with the larger Z coordinate, is called an overcrossing, and the other is the corresponding undercrossing.

As one traces along these knots, one follows in time t a trajectory $K(t)$ in real three-dimensional Euclidean space, [$^{\top}$ denotes the transpose]

$$K(t)=[X(t),Y(t),Z(t)]^{\top}. \tag{1}$$

In the case of (m_1,m_2)-torus knots these trajectories can be described by a set of four differential equations which are reduced by algebraic constraints to three dimensions. Thus these knots are described by differential-algebraic equations, which are sometimes called semistate equations [5], [6]. Such semistate equations can be put into the canonical form [7];

$$ES'=A(S,t)+Bu \tag{2a}$$

$$y=CS \tag{2b}$$

where E, B, and C are constant matrices, with E generally singular, $A(\cdot,t)$ is a nonlinear vector valued function, $'=d/dt$, u and y are the system input and output, respectively, and S is the semistate vector.

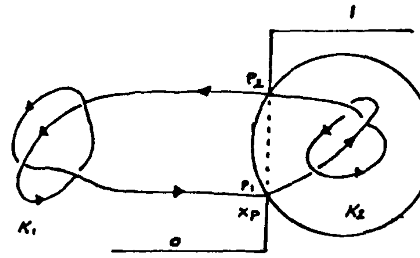

Fig. 2. Connected sum of two (2,3)-torus knots.

In the following we assume that a knot can be described as an output trajectory in time t of a nonforced time-invariant semistate system of equations. In the canonical form of (2) such a system is described by

$$ES'=A(S) \tag{3a}$$

$$g=CS \tag{3b}$$

where the output is denoted as g given by

$$g(t)=K(t). \tag{3c}$$

In (3) E and C are constant matrices defined in the next section, $A(\cdot)$ is a nonlinear but time-invariant operator, and the output g consists of the coordinates of points on the knot trajectory in three-dimensional space. Both E and C are generally singular but C has rank three for a knot with true crossings.

Now, by review of [1] we discuss how the two subknots are pieced together and how this is represented in terms of the semistate equations. Consider any two knots, K_1 & K_2, described by (3) with semistates S_1 and S_2 and outputs g_1 and g_2, respectively. We assume that these two knots have already been twisted and stretched in order to guarantee that there are two strands, one on each knot, the portion on K_1 being for larger $X=X_1$ value and the portion on K_2 being for smaller $X=X_2$ values are shown in Fig. 2. With this assumption on hand we will keep the origin for K_1 fixed and move the origin for K_2. First we locate the maximum X_1 value, $X_{\max 1}$ on K_1 and the minimum X_2 value, $X_{\min 2}$, on K_2. Then we move linearly the X_2 origin so that there is a slight overlap of X values on the two knots, that is, such that $X_{\max 1}=X_{\min 2}+e$ for e which is a small positive number. There will be at least two points on K_1 with $X_1=X_{\max 1}-(e/2)$; we locate the one with the smallest Y_1 value and call it P_1, and linearly shift the Y_2 and Z_2 origins such that the two knots intersect at P_1. We then rotate K_2 around the $X_2=X_1=X$ axis such that the knots passing through P_1 have another intersection and we call the intersection point P_2. By our assumption on the two strands being joined we know there is a small enough e such that there are no other intersections of the two knots and we choose such a small enough e. At P_1 and P_2 we have achieved $g_1=g_2$ [1]. By linear motions of the axes we can also assume that these two intersections are in the $Z=0$ plane. Now we patch the two set of semistate equations together using step functions. With reference to Fig. 2 and the above, we let $1(\cdot)$ denote the unit step function, X_p denote the common value of X at P_1 and P_2, and subscripts 1 and 2 on system quantities refer to knots K_1 and K_2, respectively. Then for $K=K_1\#K_2$ the full semistate system has the following form:

$$S=\begin{bmatrix} S_1 1(-(X-X_p)) \\ S_2 1(X-X_p) \end{bmatrix}. \tag{4a}$$

With this we have the first semistate vector component of S being zero when the trajectory is on the subknot K_2 and the second component zero when on K_1. Equation (4b) gives the output

$$g = [C_1 \quad C_2] S \tag{4b}$$

where S is given by (4a) and S_1 and S_2 are semistates for the canonical form of the two knots forming the connected sum (shown in detail in the next section for torus knots). From (4a),

$$S' = \begin{bmatrix} S_1' 1(-(X - X_p)) - S_1 \delta(X - X_p) X' \\ S_2' 1(X - X_p) + S_2 \delta(X - X_p) X' \end{bmatrix} \tag{5}$$

where $\delta(\cdot)$ is the unit impulse distribution. Using (3), we obtain the canonical semistate equations for the connected sum as

$$ES' = \left\{ \begin{bmatrix} 1(-(X - X_p)) & 0 \\ 0 & 1(X - X_p) \end{bmatrix} A_0(S) - E \begin{bmatrix} S_1 X' \\ -S_2 X' \end{bmatrix} \delta(X - X_p) \right\} \tag{6a}$$

$$g = [C_1 \qquad C_2] S \tag{6b}$$

where

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \tag{6c}$$

$$A_0(S) = \begin{bmatrix} A_1(S_1) \\ A_2(S_2) \end{bmatrix} \tag{6d}$$

The expression in the braces of (6a) is $A(S)$ for which we have to evaluate $S_1 X'$ and $-S_2 X'$ at X_p for the $\delta(\cdot)$ terms which set initial conditions when we transfer from one subknot to the other, [1]. Next, we specialize this method of describing the connected sum of two arbitrary knots to the realization of the connected sum of two identical torus knots.

III. The Connected Sum of Two (m_1, m_2)-Torus Knots

A. Using Linear Oscillators

An (m_1, m_2)-torus knot is a trajectory on a torus which goes m_1 times around the meridian circle and m_2 times around the axial circle of the torus [3]. From [2] the following set of equations, (7), describe a (m_1, m_2)-torus knot in the real four-dimensional Euclidean space R^4. The two circles of radii R_1 and R_2, making up the torus are defined by the following differential part of the semistate equations:

$$dx/dt = y \tag{7a}$$

$$dy/dt = -m_1^2 x \tag{7b}$$

$$dz/dt = w \tag{7c}$$

$$dw/dt = -m_2^2 z \tag{7d}$$

with initial conditions at time t_0 set to obtain the maximum value of the following $X(0)$, with the development of the connected sum in mind as

$$x(t_0) = R_1, \quad y(t_0) = 0, \quad z(t_0) = 0, \quad w(t_0) = -R_2. \tag{7e}$$

To get the reduction to three dimensions we use the algebraic portion of the semistate equations and set [8]

$$D = w + \delta, \qquad \delta = \text{constant} \tag{7f}$$

with

$$X = m_1 x / D \tag{7g}$$

$$Y = y / D \tag{7h}$$

$$Z = (R_1 m_2 z) / \left(D [\delta^2 - R_2^2]^{1/2} \right). \tag{7i}$$

Here δ is any constant larger than R_2 but it is often convenient to choose it as $\delta = (R_1^2 + R_2^2)^{1/2}$ since Z then becomes $m_2 z / D$ (we will, however, use the general δ in our development). These equations result in a trajectory that is constrained to be on the three-dimensional torus described in (X, Y, Z)-space by

$$Z^2 + \left[R - [X^2 + Y^2]^{1/2} \right]^2 = r^2 \tag{8a}$$

where

$$R = \delta R_1 / [\delta^2 - R_2^2] \tag{8b}$$

$$r = R_1 R_2 / [\delta_2 - R_2^2]. \tag{8c}$$

Therefore, (7a)–(7e) with connection of (7f)–(7h) yield trajectories that lie on the torus described by (8); any such trajectory describes a (m_1, m_2)-torus knot in the real three-dimensional Euclidean space. From the above expressions, one takes the semistate S for a torus knot as

$$S = [x, y, z, w, D, X, Y, Z]^T. \tag{9}$$

In (9), X, Y, Z are defined by (7) in terms of x, y, z, and D.

Now, with reference to Fig. 2 and using (3), one can realize the connected sum of the two identical (m_1, m_2)-torus knots in the three-dimensional space. We set, as the individual semistates,

$$S_1 = \begin{bmatrix} x_1(t) \\ y_1(t) \\ z_1(t) \\ w_1(t) \\ D_1(t) \\ X_1(t) \\ Y_1(t) \\ Z_1(t) \end{bmatrix} \tag{10a}$$

$$S_2 = \begin{bmatrix} x_2(t) \\ y_2(t) \\ z_2(t) \\ w_2(t) \\ D_2(t) \\ X_2(t) \\ Y_2(t) \\ Z_2(t) \end{bmatrix} \tag{10b}$$

along with E of (6c) where E_1 and E_2 are in this case direct sums of 4×4 identity matrices with 4×4 zero matrices,

$$E_1 = E_2 = \begin{bmatrix} 1_4 & 0 \\ 0 & 0_4 \end{bmatrix} \tag{11}$$

$$A_1(S_1) = \begin{bmatrix} +y_1 \\ -m_1^2 x_1 \\ +w_1 \\ -m_2^2 z_1 \\ D_1 - w_1 - \delta \\ X_1 - m_1 x_1/D_1 \\ Y_1 - y_1/D_1 \\ Z_1 - \left(R_1 m_2 z_1/\left[D_1(\delta^2 - R_1^2)^{1/2}\right]\right) \end{bmatrix}. \tag{12}$$

$A_2(S_2)$ is given by the same expression as $A_1(S_1)$ except that x, y, z, w, D, X, Y, Z have subscripts two. Next, with reference to the last term of (6a), we need to evaluate

$$\begin{bmatrix} S_1 \\ -S_2 \end{bmatrix} X \quad \text{at} \quad X = X_p. \tag{13}$$

For the evaluations of $A_1(S_1)$, $A_2(S_2)$, and the initial conditions of (13) we proceed to find the knot trajectories and then carry out the splicing.

Following [9] and [2] we determine the parameters guaranteeing and specifying the splicing of the two knots by taking (14a)–(14h) for the solutions to the set of differential equations (7a)–(7e). Here we should mention that in (14) both the solutions and the time for variables with subscripts 1 and 2 are of generic type and in reality there are two different running times for the two knots in their own frames of references.

$$x_1(t) = R_1 \cos(m_1 t + \phi)/m_1 \tag{14a}$$

$$x_2(t) = R_1 \cos(m_1 t + \theta)/m_1 \tag{14b}$$

$$y_1(t) = -R_1 \sin(m_1 t + \phi) \tag{14c}$$

$$y_2(t) = -R_2 \sin(m_1 t + \theta) \tag{14d}$$

$$z_1(t) = -R_2 \sin(m_2 t + \alpha)/m_2 \tag{14e}$$

$$z_2(t) = -R_2 \sin(m_2 t + \beta)/m_2 \tag{14f}$$

$$w_1(t) = -R_2 \cos(m_2 t + \alpha) \tag{14g}$$

$$w_2(t) = -R_2 \cos(m_2 t + \beta) \tag{14h}$$

where θ, ϕ, α, and β are phase angles.

Now we consider a good means to connect the two knots. We note that we desire the knots to intersect near the maximum X value of the first one and the minimum X value of the second. We would also like to have the intersections near and symmetric about the $Z=0$ and $Y=0$ planes. If we start time at $t=0$ for both knots near the first intersect point, these mean that we desire all the above phase angles to be zero. And we can achieve complete symmetry in time for the two knots if we rotate the second knot 180° in the $X-Y$ plane and then rotate an angle Φ about the X axis to line up the Z intersect values, with such rotations making no change in the connected sum. This means replacing X_1, Y_2, Z_2 by $-X_2, -Y_2\cos\Phi - Z_2 \sin\Phi, -Y_2 \sin\Phi + Z_2 \cos\Phi$, respectively. Finally we choose the center of the first torus to be at (0,0,0) at the center of the second one at $(a,0,0)$. At the points where the two knots intersect we have

$$X_1 = -X_2 + a \tag{14i}$$

$$Y_1 = -Y_2 \cos\Phi - Z_2 \sin\Phi \tag{14j}$$

$$Z_1 = -Y_2 \sin\Phi + Z_2 \cos\Phi. \tag{14k}$$

To allow an analytic solution we choose

$$D_1 = D_2. \tag{14l}$$

In view of (14l) and letting t_1 and t_2 be the times that go with subscripts 1 and 2 in (14i–14l) we have for (14i)–(14k)

$$R_1 \cos(m_1 t_1) = -R_1 \cos(m_1, t_2) + a[\delta - R_2 \cos(m_2 t_2)] \tag{15a}$$

$$-R_1 \sin(m_1 t_1) = R_1 \sin(m_1 t_2)\cos\Phi + R_z R_2 \sin(m_2 t_2)\sin\Phi \tag{15b}$$

$$-R_z R_2 \sin(m_2 t_1) = R_1 \sin(m_1 t_2)\sin\Phi - R_z R_2 \sin(m_2 t_2)\cos\Phi \tag{15c}$$

where

$$R_z = R_1/(\delta^2 - R_2^2)^{1/2}. \tag{15d}$$

Noting that the period of traversing either subknot is 2π, the symmetry introduced by rotations of the second knot allows us to choose

$$t_1 = 2\pi - t_2 \tag{15e}$$

which in view of the sin's and cos's is equivalent to using $t_1 = -t_2$. In this case (15a)–(15c) give, on solving for $\cos\Phi$ and $\sin\Phi$ from (15b) and (15c),

$$2R_1 \cos(m_1 t_1) + aR_2 \cos(m_2 t_1) = a\delta \tag{15f}$$

$$A\cos\left\{\left[R_1^2 \sin(m_1 t_1)^2 - R_z^2 R_2^2 \sin(m_2 t_1)^2\right]/\left[R_1^2 \sin(m_1 t_1)^2 + R_z^2 R_2^2 \sin(m_2 t_1)^2\right]\right\}. \tag{15g}$$

Once the parameter a is known the first of these, (15f), can be solved for t_1 and the last, (15g), solved for Φ to obtain the intersection points (one could find Φ from $\sin\Phi$ or $\tan\Phi$ but (15g) numerically seems the most accurate).

Next we need to find suitable a. For this we desire a to be slightly less than twice the maximum X value, $X_{\max}$, of the left subknot, where, since the maximum is set to occur at $t=0$, we have

$$X_{\max} = R_1/[\delta - R_2]. \tag{16a}$$

Then we choose with γ small

$$a = 2X_{\max} - \gamma \tag{16b}$$

Having chosen a by a small enough choice of α we solve for t_1 to satisfy (15f), using the largest and the smallest values of t_1 within the range $0 < t_1 < 2\pi$. Since we desire a small t_1 we can expand (15f) in a Taylor series using the approximation $\cos(u) = 1 - u^2/2$ to get

$$t_1 = \left\{2(2R_1 + a[R_2 - \delta])\Big/\left(2R_1 m_1^2 + aR_2 m_2^2\right)\right\}^{1/2} \tag{17}$$

which can be slightly improved by using $\cos(u) = 1 - (u^2/2) + (u^4/24)$. The other solution of interest has its time value for the first knot as $2\pi - t_1$. Finally, by the symmetry used the two values of X_p will be identical.

In summary, we choose a small γ and find the center of the second knot via a shift of the x-axis to a found from (16b). The intersection points are given by evaluating X, Y, Z at the largest and the smallest t_1 satisfying (15f), with the latter probably found via (17). If other intersections occur between these two, as can be determined readily by plotting the result, then we choose a smaller γ and repeat the process. Normally $\gamma = 0.1$ is acceptable but for large m_1 and/or m_2 we may need a much smaller γ. For the connected sum itself we start time at the first intersect point and run time on the first knot until we hit the second intersect point. We then switch to running time on the second knot until we return to the first intersect point to complete the knot. For example, for $R_1=10$, $R_2=1$, $m_1=3$, $m_2=4$, $\delta=10$, $\gamma=0.1$, we find $a=2.12222$, $t_1=0.09205$, $X(t_1)=1.06111$, $Y(t_1)=-0.30071$, $Z(t_1)=-0.039897$.

To set up the semistate equations of (6) one can now evaluate numerically the entries for S_1 and S_2 at the intersection points so that S_1X' and $-S_2X'$ can be evaluated. For evaluating X' at the intersection points we note that $DX=m_1x$ so that $D'X+DX'=m_1x'=m_1y$ and $D'=w'=-m_2^2z$ so that

$$X'=\left[m_1y+m_2^2zX\right]/D=m_1Y+m_2XZ\left(\left[\delta^2-R_2^2\right]^{1/2}/R_1\right) \tag{18}$$

which is readily evaluated numerically at the intersect points. Having evaluated the various entries, the semistate equations are filled out following (6) with the output $g(t)$ being in the following form:

$$g(t)=\left[\begin{array}{cccccccc|cccccccc}0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1\end{array}\right]\begin{bmatrix}S_1 1(-(X-X_p))\\S_2 1(X-X_p)\end{bmatrix} \tag{19}$$

C_1 C_2

where S_1 and S_2 are given by (10).

Comparing these equations with the ones obtained in [2] we can make the conclusion that these equations are identical. Since $\delta(X-X_p)=0$ everywhere except at $X=X_p$, where X_p is the X value of the points of intersection of the two tori. Now we proceed to use the method described above using Van der Pol nonlinear oscillators.

B. Connected Sum of Two Torus Knots using Van der Pol Nonlinear Oscillators

In [10] we gave the Van der Pol realization of torus knot oscillators. That is, in (7) we replace the linear oscillators by nonlinear oscillators of Van der Pol type. This replacement results in the equations given as follows:

$$dx/dt=y-\mu f(x) \tag{20a}$$

$$dy/dt=-m_1^2x \tag{20b}$$

$$dz/dt=w-\mu f(z) \tag{20c}$$

$$dw/dt=-m_2^2z. \tag{20d}$$

In (20) μ is the Van der Pol parameter and $f(\cdot)$ is the Van der Pol nonlinearity, which, because of its ease of realizability via electronic circuits, is taken to be of the piecewise-linear form given by

$$f(x)=-\begin{bmatrix}Gx-G', & x>b\\ -mx, & -b<x<b\\ Gx+G', & -b<x\end{bmatrix} \tag{21}$$

in which G, G', b, μ, and m are positive constants. In [10] we gave the exact solutions for each region of the nonlinearities. Now we assume we have two identical tori and by following the procedure described in Section II-A we give the semistate equations for the connected sum of the two torus knots.

The semistate S for the torus knot is the same as given by (9) except that δ in $D=w+\delta$ is not a constant any more and is given by the following equation [9]:

$$\delta=\left(x^2+y^2+z^2+w^2\right)^{1/2}. \tag{22}$$

The individual semistates are the same as given by (10). However, $A_0(\cdot)$ found via (6d) differs from that obtained in Section II-A. For this, $A_1(S_1)$ has the following form:

$$A_1(S_1)=\begin{bmatrix}+y_1-\mu f(x_1)\\-m_1^2x_1\\+w_1-\mu f(z_1)\\-m_2^2z_1\\D_1-w_1-\delta\\X_1-m_1x_1/D_1\\Y_1-y_1/D_1\\Z_1-(R_1m_2z_1)/\left[D_1\left(\delta^2-R_1^2\right)^{1/2}\right]\end{bmatrix}. \tag{23}$$

Now to evaluate

$$\begin{bmatrix}S_1\\-S_2\end{bmatrix}X' \quad \text{at } X=X_p$$

we need to know X_p. X_p can be found as discussed following (17) using the numerical solutions (for more details see [9]). In Section II-C we give a numerical example of this problem and also the computer generated graph of the connected sum of two torus knots using Van der Pol nonlinear oscillators.

C. Numerical Example

Fig. 3(a) shows the two identical (3,4)-torus knots which we will use to form the connected sum. The torus for K_1 is centered at (0,0,0) and K_2 is just K_1 shifted to have its torus' center at $(a,b,0)$ with $a=8$ chosen big enough to show the subknots separated. For the connected sum, shown in Fig. 3(b), the center of the torus for K_2 has been moved to $(a,b,0)$ with $a=6.738$ chosen to give two and only two points of intersection of K_1 and K_2. In fact Fig. 3(b) was obtained by applying the above theory using Van der Pol oscillators for each of the two knots (a was actually found from (16b) using $X_{max}=3.554$ and $\gamma=0.155$, with this γ chosen by observing numerical results giving only two intersections). Here we should mention there is a range of numbers for a, in order for the two tori to have an overlap; that is, if $a>a_{max}$ the two knots will not have an overlap, as shown in Fig. 3(a), and if $a<a_{min}$ the two knots will have more than two points in common which will not give the connected sum.

We start at $t_1=0.0946$ at a point on K_1, which we call P_1 and which is chosen on the limit cycles of the Van der Pol oscillators by fixing $X(t_1)=3.3694$; this yields numerically the Y and Z values of P_1 as $Y(t_1)=-2.1258$, and $Z(t_1)=-0.3961$. With these values we use the set of semistate equations describing K_1, (21)–(23) subscripted by 1, in which we increment t to plot the points on K_1, [10] and [11], until we reach P_2 which is the next point of intersection of the two knots. At this point we switch to the set of semistate equations (21)–(23), which describe K_2,

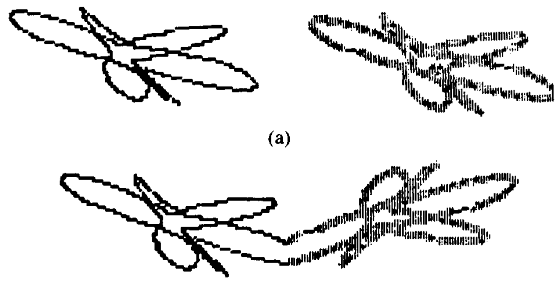

Fig. 3. Trajectories obtained for the connected sum of two (3,4)-torus knots using a computer program. (a) Separate knots before moving for connection. (b) Connected sum.

subscripted by 2, and follow K_2 by again solving the semistate equations until we again reach P_1.

For a (3,4)-torus knot since $m_1 = 3$ and $m_2 = 4$, $X_p = X(t_1)$ is found numerically using the method described in part a) and δ is given by (22), which is a function of time; we choose $\mu = 3$ for both Van der Pol oscillators. We point out that the connected sum of two torus knots may or may not be a torus knot [3].

IV. Discussion

Here we have given the semistate equations for the realization of the connected sum of two identical torus knots using the technique discussed in [1]; this assumes that the semistate equations of the individual knots are available. A numerical example using two (3,4)-torus knots illustrates the theory. We comment that we used identical knots so that the concepts would not be confused by notational difficulties; the ideas however readily apply to nonidentical subknots. Using the technique in [1], we notice that *in the semistate equations defining the connected* sum, we evaluate S_1X' and S_2X' at the connection points P_2 and P_1. This necessitates the use of impulses which serve to reset the pertinent portion of S from 0 to the proper initial value at the connection points to continue smoothly on the new connected sum knot. Of course this involves the assumption that there are trajectories for which we can choose our semistate variables and develop $A_0(S)$. But this existence assumption is implicit in our main assumption that two knots K_1 and K_2 and their semistate realizations are already on hand at the start. In short by applying the method discussed in [1] we find the semistate equations for the connected sum of two identical torus knots with the final equations agreeing with the ones obtained in [2].

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A Pole-Sensitivity Based Method for the Design of Digital Filters for Error-Spectrum Shaping

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***Abstract*—A method for realizing second-order digital filter structures which are amenable to error-spectrum shaping (ESS) was proposed recently by Diniz and Antoniou. This method minimizes the non-integer multiplier values for identifying the low-sensitivity structures. An alternative approach based on the concept of pole-sensitivity is presented in this paper for identifying the optimum structures. The proposed method is applicable for any pole location and it does not require a separate sensitivity analysis.**

I. Introduction

A pole-sensitivity based procedure is presented in this paper for obtaining low-sensitivity second-order digital filter structures which are amenable to error-spectrum shaping (ESS). Two types of second-order structures which are amenable to ESS were proposed recently by Diniz and Antoniou [1]. Low-sensitivity is achieved in these structures by introducing integer multiplier constants and choosing the optimum values of these constants such that they result in low values of the non-integer multiplier constants. In effect, the procedure of [1] makes use of the following steps for selecting the optimum structure.

1) Generation of low-sensitivity structures by forcing the non-integer multiplier constants to be low.

2) Selection of the optimum structure by a separate sensitivity analysis.

The proposed pole-sensitivity based procedure yields the optimum structure directly and hence the need for a separate sensitivity analysis is eliminated. Further, the proposed method is applicable for transfer functions with arbitrary pole locations. In general, low-sensitivity is desired for any pole location, though it is particularly important for structures with poles close to the unit circle as the sensitivity for such structures is high. We present in this paper a z-plane map containing several regions depending on the optimum set of integer multiplier constants. This map enables one to realize low-sensitivity second-order digital filter structures which are amenable to ESS with poles anywhere inside the unit circle.

A sensitivity comparison of the proposed method and the one given in [1] is also carried out. For a specific transfer function, the optimum structures are obtained by both the methods. The sensitivity performance of both these structures is compared by computing the error in the amplitude characteristics for different wordlengths of the multiplier constants.

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