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SOME CIRCUITS AND SYSTEMS APPLICATIONS OF SEMISTATE THEORY*

R. W. Newcomb[1] *and B. Dziurla*[1]

Abstract. The semistate theory of circuits and systems is reviewed with regard to its applications. Among these latter discussed are electronic circuit design and analysis, hysteresis, knot generation, neural networks, and constrained robot characterizations.

1. Introduction

An importance of systems theory is that it allows us to formulate a full system's operation in terms of the descriptions of subsystems that are connected to make the full system. When broken into their most primitive parts these subsystems can usually be described in terms of either differential equations or algebraic equations. The full system description is obtained by subjecting the primitive subsystems to the connection laws from which we more often than not obtain a mixed differential-algebraic description of the full system. In many cases the algebraic portions of the full system description can be eliminated, leading to the state-variable description. Indeed, the state-variable description has been the prevailing tool in systems theory as developed following World War II. But in some cases the elimination of algebraic constraints is not possible or if possible it may lead to loss of accuracy or to loss of necessary information about the system. What is even more important is that when state-variable-described systems are interconnected the resultant system may not have a state-variable description. Consequently, the semistate-variable description of systems has been under development recently with the practical result that the shortcomings of state-variable theory are often overcome. In this paper we discuss some applications to circuit and system design, including robotics, where the semistate equations offer an advantageous description.

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[1] Microsystems Laboratory, Electrical Engineering Department, University of Maryland, College Park, MD 20742, USA.

The class of systems that have a semistate description has different nomenclature in different fields. For example, control theorists and mathematicians call them "singular systems" [31], [39], [3], [4], since the matrix on the derivative of the semistate is generally singular, or sometimes they call them "generalized state-space systems" [1], [28], [57], or at times "extended state-space systems" [29]. On the other hand, the engineering economic systems community calls them "descriptor" [36], [37] since they give the natural description of the system, while numerical analysts call their descriptions "differential-algebraic equations" (DAEs) [26], [10], [38], [48] or "differential equations with algebraic constraints" [35]. In the circuits area the original name was "pseudostate" [30] but more recently these systems have been called "semistate systems" [18] since they are almost, though not quite, state described.

But all of these fields have a commonality in that the describing equations are essentially the same, in which case results in one field naturally carry over to the others. For example, placing the equations which describe an electronic circuit in semistate form allows us to perform a computer-aided analysis of the circuit using the powerful methods under development by numerical analysts.

As a starting point for the mathematics we realize that many physical systems can be described by the nonlinear fully implicit form

$$F(x', x, t, u, y) = 0, \tag{1.1}$$

where $F(\cdot, \cdot, \cdot, \cdot, \cdot)$ is a vector-valued function of its arguments. These arguments are the input vector u, the output vector y, the semistate vector x, and time t which may be either continuous or discrete. In the continuous-time case, which is what we consider here unless comment is made otherwise, $'$ denotes time differentiation (that is, $' = d/dt$). In the discrete-time case $'$ could mean replace t by $t+1$ or, alternatively, it could mean replace t by $t+1$ and then subtract the original vector (without this t replacement). In any event, (1.1) is rather too general for most applications and usually results when internal variables are eliminated via some sort of feedback. Consequently, we work with the specific canonical semistate equations, presented as (2.3) below, which can also be derived under appropriate conditions from the fully implicit form (1.1) [26, p. 7], [6].

The semistate theory of systems has many and varied applications in engineering and science, indeed, too many for us to discuss here. What we concentrate on are design aspects, again in a somewhat limited framework. However, before proceeding to the technical details we comment that semistate theory is driven by very practical applications. This is especially true in the circuit design area where, for example, the main circuit analysis program, ASTAP, used by IBM is based upon the semistate formulation of circuits [58] and the need for industrial concerns to handle operational amplifier circuits [53].

2. The canonical semistate equations

Here we present the canonical semistate equations (2.3) in a general framework, showing how they come about and their relationship to (1.1).

Usually a system is built up of basic components which are either dynamic, having a differential equation model, or static, having an algebraic (or functional) model. The variables of the basic components are interrelated by the connections used to form the basic components into the full system. Before we make the connections to form the full system we can think of the basic components as independent items, somewhat like components in a stockroom. As stockroom items, for the basic differential components, let x_1 and $x_2 = x_1'$ be vectors of "dependent" variables with x_3 the vector of variables on which they depend and let x_4 be the vector of variables used to describe the algebraic components. For example, in electronic circuits, x_1 could be capacitor charges, with x_2 and x_3 the capacitor currents and voltages, respectively; x_4 could be voltages and currents describing resistors and low-frequency behavior of op-amps and transistors. Then as unconnected components we have

$$x_1' = x_2, \tag{2.1a}$$

$$x_1 = f(x_3, t), \tag{2.1b}$$

$$0 = g(x_4, t), \tag{2.1c}$$

where $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ are vector-valued functions and 0 is a zero vector. On top of these equations there is some law of connection (and variable identification) which interrelates the variables of these basic components, along with the input u; for example, Kirchhoff's laws in the case of electronic circuits. Thus,

$$K \cdot \begin{bmatrix} x \\ u \end{bmatrix} = 0 \qquad \text{with} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \tag{2.1d, e}$$

in which K is a constant matrix if the connections are fixed and invariable; x is the *semistate* vector. Often there is also a relation $0 = h(x, t)$ that further constrains the semistate, as, for example, when a robot arm must write on a plane. Finally, we can read off the output y as a combination of the internal variables $y = C \cdot x$ where again C is a constant matrix if the outputs are fixed in terms of the semistate.

Putting all of these equations together we explicitly get (with the dots and I's denoting, respectively, zero and identity matrices of appropriate sizes; the K_i are relevant submatrices of K)

$$\begin{bmatrix} I & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} x' = \left\{ \begin{bmatrix} \cdot & I & \cdot & \cdot \\ -I & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ K_1 & K_2 & K_3 & K_4 \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} x + \begin{bmatrix} \cdot \\ f(x_3, t) \\ g(x_4, t) \\ \cdot \\ h(x, t) \end{bmatrix} \right\} + \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ -K_5 \\ \cdot \end{bmatrix} u \tag{2.2}$$

along with the output equation, $y = Cx$. We note that the coefficient matrix of x' is generally singular and possibly not even square, so that (2.2) is usually not in state-variable form.

In an effort to obtain a smaller set of equations we may at times prefer to make various eliminations in (2.2) while still maintaining the nature of the equations. With this in mind we call any set of equations of the form of (2.3a, b) *canonical semistate equations*:

$$Ex' = A(x, t) + Bu, \tag{2.3a}$$

$$y = Cx. \tag{2.3b}$$

As above, x is a semistate k-vector, u is the input n-vector, and y is the output m-vector; E, B, and C are constant matrices all generally being singular and $A(\cdot, \cdot)$ is a vector-valued function of the semistate and the independent variable (time) t. Most often the matrix E is square but it need not be; but usually we assume that it is square without further comment. We also take $A(\cdot, \cdot)$ to be single-valued. Any system described by (2.3) we call a *semistate described system* and note that for physical systems all of the quantities involved would normally be real-valued, though we do not so restrict the theory. In distributed parameter systems, such as RC lines in integrated circuits [52], it is important to consider infinite-dimensional vectors or Hilbert-space-valued vectors [16], and, thus, if so necessary, we consider the canonical semistate equations as being defined on Hilbert spaces. If originally the B and/or C matrices do vary in t, then a redefinition of semistate components, such as $x_5 = u$, $x_6 = Bx_5$, can yield constant B and C; an alternate is to define $x_5 = t$ which then gives nonlinear dependence which can be completely absorbed in A by the use of equivalences [44].

Of special interest is the *linear* case which we define by

$$A(x, t) = A(t) \cdot x. \tag{2.4}$$

If $A(t)$ is independent of t, then the linear semistate-described system is *time-invariant*, called *LTI*. For a linear semistate-described system we can replace $'$ by an operator, $' = p$, and use operator techniques to eliminate the internal (semistate) variables to obtain a *transfer function* description

$$y = T(p, t) \cdot u \qquad \text{with} \qquad T(p, t) = C \cdot [pE - A(t)]^{-1} \cdot B \tag{2.5a, b}$$

in which we have assumed that the indicated inverse exists and care should be taken since p and the t-dependent quantities do not commute. The situation where the indicated inverse does not exist is usually bypassed since in practical cases the full system most often has a transfer function. However, in describing subsystems and components from which the system is constructed the case where no transfer function exists is practically very important, as, for example, where devices such as op-amps are present; when $pE\text{-}A(t)$ is identically singular an input–output pairing, $F(p, t) \cdot y = G(p, t) \cdot u$, can be given [21] and is used to replace the transfer function. In the LTI case $T(p, t) = T(p)$ and we can interpret p as the Laplace transform variable.

3. Feedback

In the design of electronic circuits and control systems, feedback is an important concept. Here we show how the incorporation of feedback limits the state-variable theory of systems but places no restrictions on the semistate theory.

Feedback is defined through the signal flow graph of Figure 1 where system S_2 provides feedback around the forward system S_1; the intersection points in a signal flow graph mean that signals leaving the point are all identical to the sum of the signals coming into the point. Mathematically,

$$y = y_1 = u_2, \qquad u_1 = u + y_2. \tag{3.1a–c}$$

3.1. *Limitations of state-variable theory*

A critical limitation of state-variable theory is that it may not be able to describe the feedback connection of two state-described systems. This is best seen by example, the following being typical. We first note that the standard equations of linear state-variable theory are

$$x' = A(t)x + B(t)u, \tag{3.2a}$$

$$y = C(t)x + D(t)u, \tag{3.2b}$$

where A, B, C, and D are matrix functions of t.

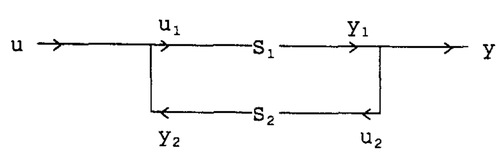

Figure 1. Feedback signal flow graph.

Example 1. For this example we take the forward system, indicated by subscript 1, and the system providing the feedback, indicated by subscript 2, to be described by

$$x_1' = -x_1 - u_1, \qquad x_2' = -x_2 - u_2, \tag{3.3a, c}$$

$$y_1 = x_1 + u_1, \qquad y_2 = 0 \cdot x_2 + u_2. \tag{3.3b, d}$$

Applying (3.1) we find that $y = x_1 + u + y$ which means that y cannot be expressed within a set of equations of the form of (3.2). We note that $T_1(p) = p/[p+1]$, $T_2(p) = 1$, and $T(p) = T_1/[1 - T_1 \cdot T_2] = p$ for the feedback system; that is, the feedback has created a differentiator and state-variable equations cannot directly describe a differentiator.

3.2. *Semistate feedback formulation*

In contrast we consider two semistate-described systems in the canonical form

$$E_1 x_1' = A_1(x_1, t) + B_1 u_1, \qquad E_2 x_2' = A_2(x_2, t) + B_2 u_2, \tag{3.4a, c}$$

$$y_1 = C_1 x_1, \qquad y_2 = C_2 x_2. \tag{3.4b, d}$$

Applying feedback, as per (3.1), gives the full system description

$$\begin{bmatrix} E_1 & \cdot \\ \cdot & E_2 \end{bmatrix} x' = \begin{bmatrix} A_1(x_1, t) + B_1 C_2 x_2 \\ A_2(x_2, t) + B_2 C_1 x_1 \end{bmatrix} + \begin{bmatrix} B_1 \\ \cdot \end{bmatrix} u, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{3.5b, a}$$

$$y = [C_1, \cdot\,] x. \tag{3.5c}$$

In the linear case we may calculate the transfer function using (3.1) from $y = y_1 = T_1 u_1 = T_1[u + y_2] = T_1[u + T_2 y]$ as

$$T(p, t) = [I - T_1(p, t) \cdot T_2(p, t)]^{-1} \cdot T_1(p, t) \tag{3.6}$$

which can also be found from the semistate equations (using (2.5) with (3.5)).

Example 2. In the case of Example 1 we first eliminate the D's by introducing new semistate components equal to the u_i with the resulting feedback system having a semistate description given, as per (3.5), as (in which we use $x_2 = u_1$, $x_4 = u_2$, calling the x_2 of (3.3)x_3)

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} x' = \begin{bmatrix} -1 & \cdot & \cdot & -1 \\ \cdot & -1 & \cdot & 1 \\ -1 & -1 & -1 & \cdot \\ 1 & 1 & \cdot & -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \\ \cdot \\ \cdot \end{bmatrix} u, \tag{3.7a}$$

$$y = [1 \quad 1 \quad \cdot \quad \cdot\,] x. \tag{3.7b}$$

Comparing with Example 1, this shows that semistate theory can succeed where state-variable theory may fail.

Next we illustrate the use of the canonical semistate equation for various design situations.

4. Design

In engineering the primary goal is usually that of design of a system from given specifications. For example, we may be given a transfer function and wish to construct an electronic circuit that has the given transfer function. In many such cases we can proceed via the semistate equations. But in so doing it is convenient to have specific canonical forms [7], [59] that are germane to the problem at hand. In the design of linear systems one such form is the following normal form of the semistate equations. We concentrate upon the time-invariant case where the most significant design results are available.

4.1. *Normal canonical form*

Here we treat the LTI case and use the following *normal canonical form*:

$$\begin{bmatrix} I & \cdot \\ \cdot & N \end{bmatrix} x' = \begin{bmatrix} A_1 & \cdot \\ \cdot & I \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \tag{4.1a}$$

$$y = [C_1, C_2]x, \tag{4.1b}$$

where N is a nilpotent matrix; that is, $N^i = 0$ for some integer power i, the smallest such integer being called the *index* of N and of the system. The transfer function can be readily calculated from (4.1) as

$$T(p) = C_1[pI - A_1]^{-1}B_1 + C_2[pN - I]^{-1}B_2 \tag{4.2a}$$

$$= \sum_{j=1}^{\infty} C_1(i/p)^{j+1}(-A_1)^j B_1 - \sum_{j=0}^{i-1} C_2(-pN)^j B_2, \tag{4.2b}$$

where we have made a Taylor series expansion about $p = \infty$, by using the power series expansion of $[I - X]^{-1}$, along with the fact that N has index i. The rightmost term of (4.2b) shows that the index can be physically interpreted as one more than the highest power of p that occurs at $p = 0$ in the transfer function. The index, which is of considerable importance to numerical computation [7], [6], therefore differs from the degree of $T(p)$ at infinity, the latter being the smallest number of differentiators needed, as we discuss below.

Before discussing how to get the normal canonical form of (4.1) we observe that Figure 2 shows a signal flow graph realization of the normal canonical form equations using integrators (for the upper subsystem) and differentiators (for the lower subsystem) (in Figure 2 the vectors x_1 and x_2 are the subvectors of x corresponding to the subdivision indicated for E and A in (4.1a)). In the continuous-time case this uses as many integrators as the size of the identity submatrix in E of (4.1a) and as many differentiators

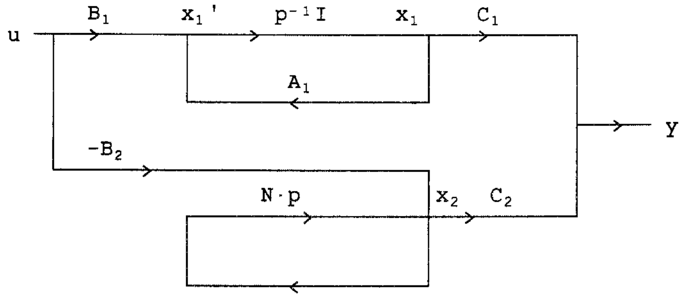

Figure 2. Signal flow graph for normal canonical form.

as needed to construct the nilpotent matrix N, the minimum number of which is the rank of N.

Because of this importance of the normal canonical form in constructing physical systems via integrators and differentiators, we show how to obtain it, the method also bringing out the assumptions under which the normal canonical form is used. Thus, assume the linear time-invariant case with square E and $pE-A$ nonsingular. Although there may be particular numbers p_n for whgich p_nE-A is singular this will not be the case for almost all p_n. Thus, we choose one of the p_n for which p_nE-A is nonsingular and add and subtract p_nE to A in the canonical semistate equations of (2.3a) followed by premultiplication of the resulting equation by the inverse of $A-p_nE$ to get $[A-p_nE]^{-1}Ex'=(I+p_n[A-p_nE]^{-1}E)x+[A-p_nE]^{-1}Bu$. Following this we perform a similarity transformation via a Q_1 to obtain $Q_1[A-p_nE]^{-1}EQ_1^{-1}$ in Jordan form, that is as the direct sum of a nonsingular matrix $E_{\rm ns}$ and a nilpotent matrix $E_{\rm np}$. Multiplying the full equation by Q_1 and letting $\hat{x}=Q_1x$, at this point we have

$$[E_{\rm ns}\oplus E_{\rm np}]\hat{x}'=[(I+p_nE_{\rm ns})\oplus(I+p_nE_{\rm np})]\hat{x}+Q_1(A-p_nE)^{-1}Bu$$

where we use $\oplus$ to denote the direct sum of two matrices. Next premultiply by $Q_2=[E_{\rm ns}\oplus(I+p_nE_{\rm np})]^{-1}$ which finally gives the normal canonical form with $A_1=E_{\rm ns}^{-1}+p_nI$, C replaced with CQ_1^{-1}, B replaced by $Q_2Q_1(A-p_nE)^{-1}B$ and $N=[I+p_nE_{\rm np}]^{-1}E_{\rm np}$; N is nilpotent since $N^2=[I+p_nE_{\rm np}]^{-1}E_{\rm np}^2[I+p_nE_{\rm np}]^{-1}$ with the latter following since the two factors in N commute as can be seen by pre- and post-multiplying $E_{\rm np}$ by $I+p_nE_{\rm np}$. We comment that in the t-dependent case the substitution for $x=Q_1^{-1}\hat{x}$ leads to an added term $-Q_1'Q_1^{-1}\hat{x}$ which in general destroys the normal canonical form. Although this does not completely negate the possibility of obtaining the normal canonical form with t-dependent matrices [25, p. 721] there is another canonical form that can more often be obtained, this being [5, p. 1103]

$$\begin{bmatrix} I & E_0 \\ \cdot & N \end{bmatrix}x'=\begin{bmatrix} \cdot & \cdot \\ \cdot & I \end{bmatrix}x+Bu, \tag{4.3a}$$

$$y=Cx. \tag{4.3b}$$

As with equation (4.1) and Figure 2, a signal flow graph can be given for this canonical form which conveniently leads to the design of time-variable systems.

4.2. *Realization of transfer functions*

A typical design problem is: Given a transfer function $T(p)$ find a system that has $y = T(p) \cdot u$. In the case that $T(p)$ is rational in p we can proceed by finding a set of constant matrices $\{A, B, C, E\}$, called a *realization* of $T(p)$, for which $T(p) = C[pE - A]^{-1}B$, that is, in view of (2.3), (2.5), a canonical semistate description holds to give the transfer function.

The *minimal realizations* are of considerable interest since they in some sense minimize the space used for dynamics (such as chip area due to capacitors in integrated circuits), these being realizations in which E has minimum rank. This minimum size also coincides with the McMillan degree in p of $T(p)$, $\delta[T(p)]$, which has been extensively developed from its network synthesis roots [42, pp. 176–185]. There are now a number of ways to obtain minimal realizations [8], [9], one being to augment $T(p)$ to be square, interpret u and y as voltages and currents, do a minimum capacitor RC-active synthesis, and then analyze the resulting circuit to get the canonical semistate equations. Probably of more interest mathematically is the following. Treating p as a complex variable, expand $T(p)$ using a matrix partial fraction expansion into $T(p) = T_o(p) + T_\infty(p)$ where T_o contains all the finite poles and is zero at $p = \infty$ and T_∞ is the principal part at infinity, as per (4.2). A classical minimal state-space realization [43, pp. 80–84], [63] can be obtained for both $T_o(p)$ and $T_\infty(1/s)$ (where we use $s = 1/p$ to transfer the poles at infinity to $s = 0$). Thus, $T_o(p) = C_o[pI - A_o]^{-1}B_o$ and $T_\infty(1/s) = D_\infty + C_\infty[sI - E_\infty]^{-1}B_\infty$ which is $T_\infty(p) = D_\infty + C_\infty[(1/p)I - E_\infty]^{-1}B_\infty$. Introducing x_2 as the semistate-like vector for $T_\infty(p)$, the last gives a decomposition $Ix_2 = p[E_\infty x_2 + B_\infty u]$ with $y_\infty = C_\infty x_2 + D_\infty u$; further introducing $x_3 = B_\infty u$ and $x_4 = D_\infty u$ these become $p[E_\infty x_2 + x_3] = Ix_2$, $0 = x_3 - B_\infty u$, $0 = x_4 - D_\infty u$, $y_\infty = C_\infty x_2 + x_4$ which are semistate equations for $T_\infty(p)$. We observe that the degree of $T_\infty(p)$ is the same as that of $T(p)$ at infinity, δ_∞; as such δ_∞ is equal to the minimum number of differentiators in the system and x_2 is a δ_∞ vector.

Observing (4.2) we see that the decomposition just discussed gives a semistate realizations for $T(p)$ by piecing together the ones for $T_o(p)$ and $T_\infty(p)$. We set up the full semistate minimal realization as the following:

$$\begin{bmatrix} I & \cdot & \cdot & \cdot \\ \cdot & E_\infty & I & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} x' = \begin{bmatrix} A_o & \cdot & \cdot & \cdot \\ \cdot & I & \cdot & \cdot \\ \cdot & \cdot & I & \cdot \\ \cdot & \cdot & \cdot & I \end{bmatrix} x + \begin{bmatrix} B_o \\ \cdot \\ -B_\infty \\ -D_\infty \end{bmatrix} u, \tag{4.4a}$$

$$y = [C_o, C_\infty, \cdot, I]x. \tag{4.4b}$$

Here the sizes of the identities in the second and third rows are the same, being equal to the degree of $T(p)$ at infinity, δ_∞, as are the ones in the fourth row and output equation. Comparing with the normal canonical form, (4.1), we see that

$$N=\begin{bmatrix} E_\infty & I & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}, \qquad N^i=\begin{bmatrix} E_\infty^i & E_\infty^{i-1} & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \tag{4.4c}$$

for which, in carrying out the power series expansion of (4.2b), we see that if N has index i, then E_∞ has index $i-1$, since $E_\infty^{i-1}=0$ and $E_\infty^{i-2}\neq 0$, while E_∞ has size $\delta_\infty\times\delta_\infty$; thus the minimum number of differentiators, $\delta_\infty=$ size E_∞, and the highest power, $i-1=$ index E_∞, of p present in the principal part of $T(P)$ at infinity are both nicely expressed in terms of E_∞. In terms of physical constructions we use the fact that equations (4.4) are in the normal canonical form of (4.1) for which Figure 2 holds.

In summary, given a rational transfer function, we can give a construction for it using the minimum number of integrators and differentiators by creating the minimal realization of (4.4) from which the system is constructed via Figure 2.

Example 3. Here we give a minimal semistate realization of the 3×2 transfer function matrix

$$T(p)=\left[\frac{1}{p+3}\right]\begin{bmatrix} 2p^3+9p^2+10p & -p^3-4p^2-p+3 \\ 2p^2+5p & 3 \\ -p & p^2+3p+3 \end{bmatrix} \tag{4.5a}$$

$$=p^2\begin{bmatrix} 2 & -1 \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}+p\begin{bmatrix} 3 & -1 \\ 2 & \cdot \\ \cdot & 1 \end{bmatrix}+\begin{bmatrix} 1 & 2 \\ -1 & \cdot \\ -1 & \cdot \end{bmatrix}+\left[\frac{1}{p+3}\right]\begin{bmatrix} -3 & -3 \\ 3 & 3 \\ 3 & 3 \end{bmatrix} \tag{4.5b}$$

$$=\begin{bmatrix} 1 & 2 \\ -1 & \cdot \\ -1 & \cdot \end{bmatrix}+\begin{bmatrix} 1 & 2 & \cdot \\ \cdot & 1 & 1 \\ \cdot & \cdot & 1 \end{bmatrix}\left\{\begin{bmatrix} s & \cdot & \cdot \\ \cdot & s & \cdot \\ \cdot & \cdot & s \end{bmatrix}-\begin{bmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}\right\}^{-1}\begin{bmatrix} -1 & 1 \\ 2 & -1 \\ \cdot & 1 \end{bmatrix}$$

$$+\begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix}[p-(-3)]^{-1}[1 \quad 1] \tag{4.5c}$$

in which we have set $s=1/p$ (that is, we obtained a minimal realization in s of $T_\infty(1/s)$ which has a finite pole at $s=0$). Equation (4.5c) checks that $\delta_\infty=3$ which is also directly found by counting, with their order, poles at

$s=0$ in the middle term on the right in the following Smith-MacMillan form [40, pp. 580-595]:

$$T_\infty(1/s)-D_\infty=\begin{bmatrix} 1+1.5s & -0.5 & 1.5 \\ s & 1 & 1 \\ \cdot & 1 & \cdot \end{bmatrix}\begin{bmatrix} 1/s^2 & \cdot \\ \cdot & 1/s \\ \cdot & \cdot \end{bmatrix}\begin{bmatrix} 2 & -1 \\ \cdot & 1 \end{bmatrix}. \tag{4.6}$$

Equations (4.5) give a minimal realization from which we see that the index of E_∞ is 2, which is the highest power of p present in $T_\infty(p)$, with the index of the system being 3, while the degree is 4 being the sum of the degree at $p=-3$, which is 1, and the degree at $p=\infty$ which is $\delta_\infty=3$. The minimal realization semistate equations are directly read off for (4.4) as

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} x' = \begin{bmatrix} -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} x$$

$$+\begin{bmatrix} 1 & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & -1 \\ -2 & 1 \\ \cdot & -1 \\ -1 & -2 \\ 1 & \cdot \\ 1 & \cdot \end{bmatrix} u, \tag{4.7a}$$

$$y=\begin{bmatrix} -3 & 1 & 2 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 3 & \cdot & 1 & -1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 3 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} x. \tag{4.7b}$$

In summary, the transfer function matrix of (4.5a) results from the semistate equations (4.7) and, since these are for a minimal realization, a direct construction of the system via these semistate equations uses the minimal number of integrators and differentiators.

4.3. *Hysteresis*

While the design formulations given above have been for linear systems the real power of semistate theory seems to be associated with nonlinear systems. Of particular interest is hysteresis where by *hysteresis* we mean multivalued input-output functions in the steady state ($x'=0$). Hysteresis has a number of practical uses, one being to minimize the effect of noise in systems by maintaining a desired state in a system that would otherwise switch states.

In many instances semistate theory can be used to create multivalued input-output characteristics from single-valued ones [19]. To see how this can occur it is probably best to proceed with a simple example for which we let $1(\cdot)$ denote the unit step function and consider the system defined by (all x_i, u, and y are scalar here)

$$Ex' = \begin{bmatrix} x_1 - x_2 \\ a1(x_2) - x_3 \\ bx_1 - x_4 \\ x_3 - x_4 \end{bmatrix} + \begin{bmatrix} -1 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} u, \tag{4.8a}$$

$$y = [\cdot, \cdot, 1, \cdot]x. \tag{4.8b}$$

The situation is as shown in Figure 3 where the two functions of the second and third row of (4.8a) are plotted versus x_1 (found for $x'=0$) in Figure 3(a) and 3(b), respectively. The first row of (4.8a) introduces the input as a running parameter in the plot of the function of the second row, and the fourth row constrains the y axis of both plots to be the same. Thus, as u

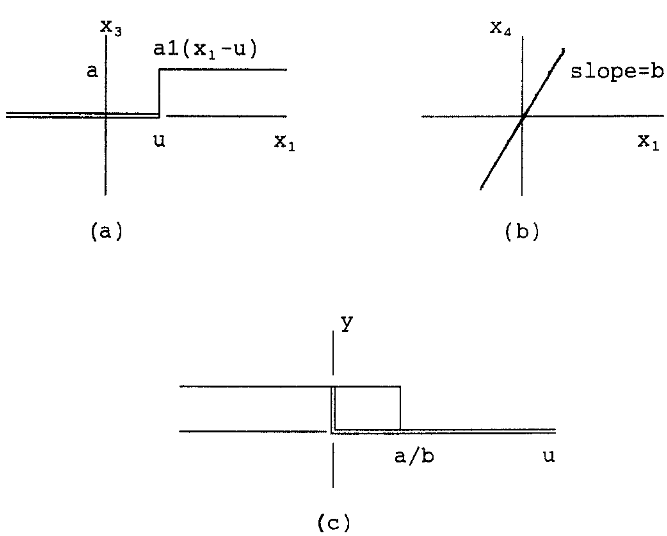

Figure 3. Binary hysteresis generation.

changes the plot for the function of the second row slides by the function of the third row with their intersections being solutions at dc (i.e., when $x'=0$) for the semistate equations. The resulting hysteresis, called *binary* hysteresis, is plotted, as in Figure 3(c), by sliding the function of Figure 3(a) by the function of Figure 3(b) and observing the intersections, which give the output, of the two functions as u is varied. By proper choice of the design constants a and b, multiple solutions for the same u can be obtained, giving the hysteresis. If we desire a different hysteresis, we would insert different (nonlinear) functions in the second and third row of (4.8a) as well as possibly a different means of inserting the input u as a parameter. When we practically build the hysteresis system, E will not be zero and, hence, the parasitic dynamics will ensure that solutions of the full semistate equations are not multiple valued but lie somewhere on a branch of the hysteresis curve.

Of course, only certain types of hysteresis can be generated in this manner. For example, it does not appear possible to obtain those formed by integrating via Stieltjes integrals in which the differentials are differential binary hysteresis loops of the form of Figure 3(c). But very intriguing curves with many branches can readily be obtained, though the general theory still remains to be developed.

4.4. *Torus knot oscillators*

Another nonlinear system for which the design via semistate equations is useful is that used to create torus knots' trajectories, these being trajectories which form torus knot curves in real three-dimensional space. Practically these could arise by the desire to have a robot tie a knot, say in an environment hostile to humans.

We recall that a torus is the direct product of two circles. Thus, we create two uncoupled circles via oscillators with these, however, being in four-dimensional space, and then follow up by a reduction of the result into three-dimensional space via an algebraic constraint. For the oscillators we choose the following with t continuous time and initially $f(\cdot)=0$:

$$x_1' = m_1[x_2 - \mu f(x_1)], \qquad x_3' = m_2[x_4 - \mu f(x_3)], \tag{4.9a, c}$$

$$x_2' = -m_1 x_1, \qquad x_4' = -m_2 x_3. \tag{4.9b, d}$$

These have solutions (trajectories) when $f(\cdot)=0$ that satisfy the circle equations $x_1^2+x_2^2=R_1^2$ and $x_3^2+x_4^2=R_2^2$ where the radii R_1 and R_2 are set by initial conditions. We see that the x_1-x_2 plane and x_3-x_4 plane circles are traversed at the radian frequencies m_1 and m_2, respectively. Thus, if m_1 and m_2 are irrationally related the full torus is covered by the solutions of (4.9) but if they are rationally related the trajectories are closed and form (m_1, m_2)-torus knots. To reduce the torus to three-dimensional space we let, following [47],

$$x_5 = x_4 + \delta \tag{4.10}$$

with δ a constant greater than R_2. The final $[X, Y, Z] = [x_6, x_7, x_8]$ coordinates are found by setting

$$X = x_1/x_5, \qquad Y = x_2/x_5, \qquad Z = \alpha x_3/x_5 \qquad \text{with} \quad \alpha = R_2/[\delta^2 - R_2^2]^{1/2}. \tag{4.11}$$

Direct calculation with these latter quantities gives

$$Z^2 + \{R - [X^2 + Y^2]^{1/2}\}^2 = r^2 \tag{4.12a}$$

with

$$R = \delta R_1/[\delta^2 - R_2^2] \quad \text{and} \quad r = R_1 R_2/[\delta^2 - R_2^2] \tag{4.12b, c}$$

showing that indeed the trajectory is on a three-dimensional torus of axial circle with radius R and meridian circle of radius r (revoled around the Z axis).

The describing equations for the torus knot are readily put in the canonical semistate variable form:

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} x' = \begin{bmatrix} \cdot & m_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -m_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & m_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -m_2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} x + \begin{bmatrix} -m_1 \mu f(x_1) \\ \cdot \\ -m_2 \mu f(x_3) \\ \cdot \\ \delta \\ -x_1/x_5 \\ -x_2/x_5 \\ -\alpha x_3/x_5 \end{bmatrix}, \tag{4.13a}$$

$$y = [\cdot, \cdot, \cdot, \cdot, \cdot, 1, 1, 1]x \tag{4.13b}$$

Also an improvement can be made by using Van der Pol oscillators [50], rather than linear ones, since Van der Pol oscillators are structurally stable while linear ones are not. In the case of Van der Pol oscillators we can choose $f(\cdot)$ to be a cubic polynomial for the oscillators of (4.9), (4.13); however, it is practically much better to use a piecewise linear function, such as $f(v) = a_0 v - b_0 - a_1|v - b_1| + a_2|v - b_2|$ with appropriate choice of the slope and breakpoint constants a_i and b_1 for $i=0,1,2$. With a desired negative resistance between the two breakpoints, $f(\cdot)$ can be physically built using convenient electronic devices. If the Van der Pol oscillator is used to construct the knot, then the limit cycle is independent of initial conditions and is no longer a circle, in which case the surface on which the knot is "drawn" is no longer given by the torus equation of (4.12). That is, rather than using the direct product of two circles we form a torus as the direct product of two Van der Pol limit cycles with the actual knot being the same as obtained via the linear oscillators.

4.5. *Neural networks*

Neural networks are another class of nonlinear systems that are ideally suited to the semistate framework. Generally these have processing elements, which are commonly called neurons, interconnections defined through weight matrices, and constraints. Dynamics is placed in the neurons and the weight matrices are adjusted for the network to perform a specified task. The key to the operation then is the specific neurons chosen and the means of adjusting the weights, which is an extensive story in itself that we can only touch upon here.

Let us consider just the circuit of Tank and Hopfield [56]. It consists of two types of neurons, signal ones and constraint ones. The signal neurons have the simplest possible dynamics and consist of a parallel conductance–capacitor circuit (of conductance g, capacitance c) fed by a current source input with the capacitor voltage v fed through a voltage-to-current converter of monotone nonlinearity $g(\cdot)$ to give its output current $g(v)$. The input is the sum of two types of currents, the first being external inputs and the second being the weighted sum of constraint neuron outputs where a constraint neuron is a possibly nonlinear current–current converter, of nonlinear current transfer characteristic $f(\cdot)$, with an input current that is the weighted sum of signal neuron outputs and constant bias currents i_B. Writing W_1 and W_2 for the weight matrices of the signal and constraint neurons, respectively, and assuming all neurons with the same parameters c, g, $f(\cdot)$, $g(\cdot)$ we can write the following semistate equations:

$$\begin{bmatrix} cI & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} x' = \begin{bmatrix} -gI & \cdot & \cdot & -W_1 \\ \cdot & I & \cdot & \cdot \\ \cdot & -W_2 & I & \cdot \\ \cdot & \cdot & \cdot & I \end{bmatrix} x + \begin{bmatrix} \cdot \\ -g(x_1) \\ -i_B \\ -f(x_3) \end{bmatrix} + \begin{bmatrix} B_1 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} u, \tag{4.14a}$$

$$y = [\cdot, I, \cdot, \cdot]x, \tag{4.14b}$$

where $g(\cdot)$, as well as $f(\cdot)$, now denote the vectors whose components are the above $g(\cdot)$ and $f(\cdot)$ that act on each of their vector argument's components in the same way. In [56] $f(i) = -i1(-i)$ is taken where $1(\cdot)$ is the unit step function and the constraints $W_2 y \geq i_B$ are imposed. The weights satisfy $W_1 = -W_2^T$, with superscript T the transpose, and are chosen such that an associated energy function will seek a minimum. We note that if constraints are not imposed, then we can choose the identity function for $f(\cdot)$ in which the nonconstrained case is seen to be a special case of the constrained. In actual constructions, and indeed in the circuits actually drawn in [56] and related references, it is often more convenient to use voltages out of the constraint amplifiers, via $f(\cdot)$, which are then applied to resistors. Doing so adds another term $W_1 x_1$ in (4.14a) which may, but need not, change the operation depending upon the relationship of W_1 to gI.

Many other philosophies of neural-type systems exist. One prior but very similar to the above is that of Morishita [41] which has been used to model the cerebellum for computed structures [15]. Most of the recent philosophies take the amplitude of the neuron signal as representing the firing rate of biological neurons. However, another philosophy is to code information in neural-type pulse occurrence times, this being more in line with actual biological systems and more noise immune in electronic realizations. Within this latter philosophy semistate theory is the key to our formulation [22], [23] since the circuits to accomplish the coding rely upon hysteresis. In this latter formulation there are neural-type cells, which contain the fundamental dynamics for signal processing, neural-type junctions, which combine the signals algebraically, and neural-type lines which serve to transfer and shape signals and which, when present, move the theory into Hilbert space. Still another theory of neural networks, in which semistate formulation is important, is more closely aligned to physiological properties using the fact that there are long-term and short-term processes ongoing [14].

4.6. *Constrained robots*

Since robots are both kinetic and kinematic in their operation they too are ideally suited to semistate formulation. Considering a fixed robot with the only moving parts being the arm links, the dynamic portion involves the (Newton's) equations of motion of the arm and the algebraic portion involves the translation of joint coordinates to the world reference frame of the robot. The algebraic portion is highly nonlinear since it uses trigonometric functions of joint angles while the dynamic portion can be quite nonlinear if friction is involved. Besides these, there is often another set of constraints placed on the robot arm, for example, that it must remain on a certain surface or in a certain region. Some of these constraints are in terms of inequalities and therefore require subterfuges for the use of semistate equations, but others can be conveniently expressed in terms of nonlinear function equations. For example, if we desire the end of the robot arm to trace a knot on a torus, then the equality of (4.12a) could be inserted in the semistate equations as a constraint.

Following [60, p. 297] we note that the dynamical behavior of an n link robot arm can be written in the form

$$N(\theta)\theta''+C(\theta,\theta')+G(\theta)=T_i+B_eT_e, \tag{4.15}$$

where θ is the vector of link variables, joint angles for rotary joints and radial displacement for prismatic joints [54, p. 148], $N(\theta)$ is nonsingular (actually positive definite) being the inertia matrix, $C(\theta,\theta')$ is a vector of centrifugal, coriolis, and frictional terms, while $G(\theta)$ is a vector of gravity terms; T_i is the generalized link force/torque vector generated by the robot (internally via its motors) with T_e being the same for externally applied quantities which, however, are assumed to act at the end effectors (of which there may be several, for example, on a multimanipulator robot). The vector of position world coordinates, that is, X's, Y's, and Z's, of the robot end effectors, called P_{ef}, which is usually what is desired, is given in terms of the link variables via a set of kinematic transformations, $K(\cdot)$, which map coordinate systems lined up with the links into the world coordinate frame in which the position is to be determined; thus $P_{ef}=K(\theta)$. Besides these there are often other constraints, due to the arm being constrained to move on a surface [61] or due to loads the robot may be asked to manipulate [39]. An example of the former could be that the robot is asked to wrap a chain around a cylinder to hold something in place while an example of the latter is that the robot is asked to lift an item of significant weight. In the former case there are added algebraic constraints defining the surface $0=S(P_{ef})$ while in the latter case there is dynamics incorporated in $T_e=mP''_{ef}+mg$ where g is the gravity vector in the world frame of reference.

Putting these equations together we get, on taking T_i as the input and writing $h(x)=-N(\theta)^{-1}\{C(\theta,\theta')+G(\theta)-T_i-B_eT_e\}$,

$$\begin{bmatrix} I & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & I & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & I & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & mI & \cdot \end{bmatrix} x' = \begin{bmatrix} \cdot & I & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & I & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & I & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & I & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & I \end{bmatrix} x + \begin{bmatrix} \cdot \\ h(x) \\ \cdot \\ -K(x_1) \\ S(x_4) \\ \cdot \\ -mg \end{bmatrix}$$

$$+\begin{bmatrix} \cdot \\ \cdot \\ -I \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} u \qquad \text{with } x=\begin{bmatrix} \theta \\ \theta' \\ T_i \\ P_{ef} \\ P'_{ef} \\ T_e \end{bmatrix}, \tag{4.16a}$$

$$y=[\cdot,\cdot,\cdot,I,\cdot,\cdot]x. \tag{4.16b}$$

Here if the load dynamics is absent we can set $m = 0$ which eliminates the last set of rows and T_e from consideration. In the customary case where motion is very slow $x' = 0$ is usually assumed. If there is no surface constraint, $S(\cdot)$, then the only term of real interest in (4.16) becomes the kinematics constraint which gives $y = K(\theta)$ in which case the customary robot problem is seen to be that of determination of the robot link variables from the desired arm path. Since there are usually many more link variables than coordinates of the arms' paths, there are usually many solutions to the inverse kinematics problem of finding θ for given y.

Although these semistate equations give a nice formulation for robotics' problems, they have as yet been little recognized, and, hence, to date have not really been used in practical robot design. But their generality and nice form should prove useful in the long run.

Example 4. Consider the robot arm depicted in Figure 4 where the arm is constrained to move in the $Z = 0$ plane and the end effector is constrained to move on the line $Y = Y_c$. This is a somewhat modified version of the robot treated in [49, p. 129]. We first discuss the coordinate frames which allow us to obtain the kinematics.

As shown in Figure 4 we take a fixed coordinate frame, called the world coordinate frame, as the reference and starting coordinate system, in which we measure $X = x_0$, $Y = y_0$, $Z = z_0$. All of the coordinate axes unit vectors will be chosen to form a right-hand system with positive Z out of the paper, and, since for the robot all $z_i = 0$, no further mention of the Z behavior is made. The center of this world coordinate frame is taken as the robot base,

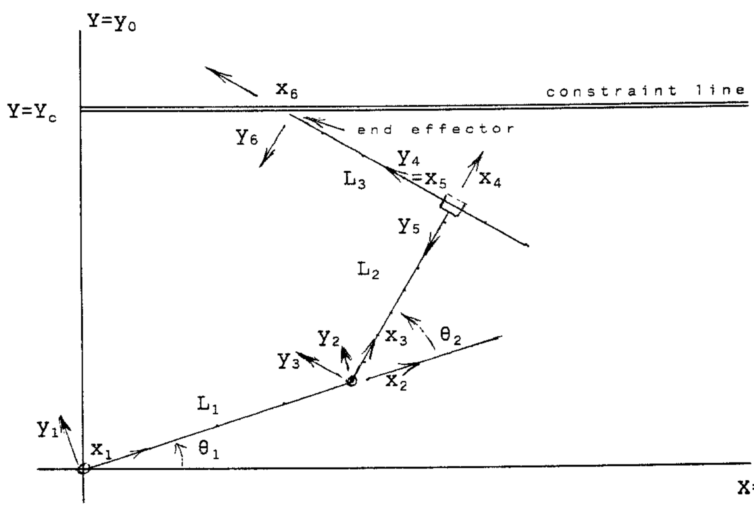

Figure 4. Two-dimensional constrained robot.

around which (at the first joint which is a rotary one) the first link rotates by a variable angle θ_1. The positive y_1 axis is fixed when we choose the x_1 axis along the first link with origin at the origin of the x_0 axis. Next the x_2 axis is taken to line up with the x_1 axis but shifted to the second (rotary) joint at the end of the first link, that is to $x_1 = L_1$. The second link makes a variable angle θ_2 with the x_2 axis with the x_3 origin at the x_2 origin but lined up along the second link of fixed length L_2. The x_4 origin is taken at the (prismatic) joint at the other end of the second link and, thus, is lined up with the x_3 axis but displaced by length L_2. The joint connecting the third link with the second one is a right-angle one, making $x_5 = y_4$ and $y_5 = -x_4$, that allows the third link to have variable length L_3. At the end of link three is the fixed end effector where we place the origin of the final coordinate system with $x_6 = x_5 - L_3$, $y_6 = y_5$. The transformations between the final and the original coordinate systems can be found via a series of one-parameter transformations tracing the above changes from x_i to x_{i+1} as follows (in which C_i and S_i respectively represent cos θ_i and sin θ_i, we write $\theta_4 = \theta_1 + \theta_2$, and the transformations are represented in homogeneous form):

$$\begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 & -S_1 & \cdot \\ S_1 & C_1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & L_1 \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} C_2 & -S_2 & \cdot \\ S_2 & C_2 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & L_2 \\ \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$\times \begin{bmatrix} \cdot & -1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & L_3 \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} x_6 \\ y_6 \\ 1 \end{bmatrix} \tag{4.17a}$$

$$= \begin{bmatrix} -S & -C_4 & C_1L_1 + C_4L_2 - S_4L_3 \\ C_4 & -S_4 & S_1L_1 + S_4L_2 + S_1L_1 \\ \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} x_6 \\ y_6 \\ 1 \end{bmatrix}. \tag{4.17b}$$

When evaluated at the end effector, where $x_{6ef} = y_{6ef} = 0$, these give the position, $P_{ef} = (X_{ef}, Y_{ef})^T$, of the end effector in the world coordinates, that is, $K(x_1)$ is determined for insertion in (4.16). This latter $x_1 = [\theta_1, \theta_2, L_3]^T$ as opposed to the scalar x_1 of Figure 4.

The end effector constraints are

$$y_{0ef} = Y_{ef} = Y_c. \tag{4.18}$$

For the dynamics we assume that the motors driving the three variables θ_1, θ_2, $\theta_3 = L_3$ are fixed to the first link attached to a joint and move only the next link via application of the respective torques τ_1 and τ_2 and force f_3. The fixed line $Y = Y_c$ also exerts a normal force, F_y, to counter the normal force applied to it by the robot arm, the latter of which we take as known. The force F_x exerted by the end effector is assumed to be taken up by the robot arm itself since, for simplicity, we ignore friction; likewise the force of gravity is so taken up by the robot arm since its effects cancel out (because everything takes place in the $Z=0$ plane). The dynamical (kinetics) equations are

$$I_1(\theta_2, L_3)\theta_1'' = \tau_1, \tag{4.19a}$$

$$I_2(L_3)\theta_2'' = \tau_2, \tag{4.19b}$$

$$M_3 L_3'' + F_y = f_3, \tag{4.19c}$$

in which M_3 is the mass of the rod for the third link and I_1 and I_2 are the moments of inertias driven by the torques τ_1 and τ_2, these latter depending upon the configuration of the subsequently numbered links (but not the previous ones by our assumption on the connection of the drive motors).

If we let semistate variables be θ_1, θ_2, L_3, θ_1', θ_2', L_3', τ_1, τ_2, f_3, X_{ef}, Y_{ef}, then our semistate equations can be written in the form of (4.16) as (here the last three matrix rows of (4.16a) are empty and, thus, dropped)

$$\begin{bmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix} x' =
\begin{bmatrix}
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{bmatrix} x$$

$$+ \begin{bmatrix}
\cdot \\ \cdot \\ \cdot \\ -\tau_1/I_1 \\ -\tau_2/I_2 \\ -(f_3+F_y)/M_3 \\ \cdot \\ \cdot \\ \cdot \\ -(C_1L_1+C_4L_2-S_4L_3) \\ -(S_1L_1+S_4L_2+C_4L_3) \\ -Y_c
\end{bmatrix} + \begin{bmatrix}
\cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot
\end{bmatrix} u, \tag{4.20a}$$

$$y=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}x. \tag{4.20b}$$

We see that the semistate description for a robot is relatively easily formulated and that in general there are a large number of zeros present meaning that numerical techniques for sparse systems should prove useful.

4.7. *Input for specified output*

Often it is necessary to determine an input, or class of inputs, that will yield a desired output. The problem has been extensively studied in terms of controllability and observability matrices for linear systems [34], [51], [55], [64] with a number of results nicely summarized on p. 17 of [33]. Here we mention the theory of [29] which we slightly extend in order to work with the output and semistates, rather than just the state, and for nonlinear t-varying systems.

Given the canonical semistate equations, (2.3), we find nonsingular (constant) matrices P_i, P_o and Q_i, Q_o to transform the input and output matrices (B and C) into identities and zeros to write

$$P_iEQ_o(Q_o^{-1}x)'=P_iA(Q_o(Q_o^{-1}x),t)+\begin{bmatrix} I \\ \cdot \end{bmatrix}Q_i^{-1}u, \tag{4.21a}$$

$$P_oy=[I,\cdot]Q_o^{-1}x. \tag{4.21b}$$

By writing $x_o=Q_o^{-1}x$, noting from (4.21b) that the first component of this is P_oy, and premultiplying the first equation by $Q_i[I,\cdot]$ we obtain

$$u=Q_i[I,\cdot]P_i\left\{EQ_o\begin{bmatrix} P_oy \\ x_{o2} \end{bmatrix}'-A\left(Q_o\begin{bmatrix} P_oy \\ x_{o2} \end{bmatrix},t\right)\right\}. \tag{4.22}$$

Thus, if we can observe enough of the state (i.e., rank $C=\dim y$) and control enough of the state (i.e., rank $B=\dim u$ and appropriate properties of E and $A(\cdot,\cdot)$) we can find a u to force y as desired. In so doing there are a number of free parameters, via the unobservable x_{o2}, such that some useful properties of x may be obtainable; but in any event these free parameters have no effect on the output (in theory—in practice it is another story, since errors in fixing C could cause x_{o2} to influence the output y).

Example 5. We consider the system defined by

$$\begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}x'=\begin{bmatrix} -2 & \cdot & -2 \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -1 \end{bmatrix}x+\begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}u, \tag{4.23a}$$

$$y=[1,2,-3]x. \tag{4.23b}$$

Transforming by

$$P_o = 1 = Q_i, \tag{4.24a, b}$$

$$Q_o = \begin{bmatrix} 1 & -2 & 3 \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}, \qquad P_i = \begin{bmatrix} 1 & \cdot & -\frac{1}{2} \\ \cdot & 1 & \frac{3}{2} \\ 1 & \cdot & 3 \end{bmatrix}, \tag{4.24c, d}$$

gives, with $x_o = Q_o^{-1}x$,

$$\begin{bmatrix} 1 & -2 & 3 \\ \cdot & \cdot & \cdot \\ 1 & -2 & 3 \end{bmatrix} x_o' = \begin{bmatrix} -2 & 4 & -\frac{15}{2} \\ -2 & 5 & -\frac{15}{2} \\ -2 & 4 & -4 \end{bmatrix} x_o + \begin{bmatrix} 1 \\ \cdot \\ \cdot \end{bmatrix} u, \tag{4.25a}$$

$$y = [1, \cdot, \cdot]x_o, \tag{4.25b}$$

which is readily solved for u in terms of $y = x_{o1}$ as

$$u = [1, -2, 3]\begin{bmatrix} y' \\ x_{o2}' \\ x_{o3}' \end{bmatrix} - [-2, 4, -\tfrac{15}{2}]\begin{bmatrix} y \\ x_{o2} \\ x_{o3} \end{bmatrix}. \tag{4.25c}$$

If $x_{o2} = x_{o3} = 0$, then $u = y' + 2y$ which checks the transfer function $T(p) = 1/(p+2)$; for example, under these conditions if $y = 3\exp(-t)1(t)$, then $u(t) = 3\exp(-t)1(t) + 3\delta(t)$ where $1(\cdot)$ and $\delta(\cdot)$ are the unit step and unit impulse functions, respectively.

5. Discussion

As we hope has been illustrated here, the applications of semistate theory are many, varied, and more practical than might be imagined at first glance. Unfortunately space and time constraints have limited us in the actual applications treated in the text so it is worth pointing out a number of other areas where applications have been, or are being, made. Probably the most useful of these is that of nonlinear electronic circuit analysis and design [58], [17], [62] where Kirchoff's laws are the algebraic connection constraints and the laws of the elements are either algebraic, such as for resistors, or dynamic, as for capacitors. Besides these, among other topics of interest are: forward-backward systems [32], [45], pole placement [11], [27], equivalent systems [2], [28], sensitivity [46], [20], stability of time-variable networks [30], optimal control [12], and surface acoustic wave (saw) system design [13]. Likewise there are now appearing a number of theses in the area. In the above we have used the independent variable as time but in some applications, such as picture processing, we may wish to consider spatial or other variables, sometimes even of higher dimension.

As can be seen from the treatment above, in applications it is convenient to work from standard forms since they allow general theories to be applied to specific cases. Among the standard forms of most use for semistate-described systems are the "canonical semistate equations" of (2.3) and the "normal canonical form" of (4.1). After some experience in working with these a feeling is obtained for what a system can do if various changes are made in the description. For example, by decreasing the size of the nilpotent matrix N in the normal canonical form the maximum number of differentiators needed in a physical construction can be decreased.

It is our feeling that the field of applications of semistate theory is just in its infancy and that many important uses lie ahead. For example, we feel that biometrical-engineering applications could be very significant and that among future applications to be explored is that of hurricane prediction [24] where the equations describing hurricane formation undoubtedly take semistate form since there are dynamics due to the motion of air and water, as well as algebraic constraints due to gas and thermodynamical laws and the connection conditions for adjoining media. An exciting future seems to lie ahead.

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