where α_1 and α_2 are as defined before Theorem 7. The roots of $\partial f/\partial a_1 = 0$ are

$$a_1^* = \frac{\alpha_1 \pm \sqrt{\alpha_1^2 - 3a_5\alpha_2}}{3a_5}. \tag{A2}$$

The minimum of f is obtained with the plus sign for $a_5 > 0$ and negative sign for $a_5 < 0$, i.e.,

$$a_1^* = \frac{\alpha_1 + \text{sgn}(a_5)\sqrt{D}}{3a_5}.$$

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Computable Minimum Lattice-Like ARMA Synthesis

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***Abstract*—A cascade synthesis for ARMA digital filters is given which uses the minimum number of delay elements, this number being the degree of the transfer function in z. The resulting structure results in the very convenient lattice form when the zeros of the transmission are not on the unit circle. The theory rests upon conversion to scattering parameters and the use of a Richards' function valid for complex zeros of rational functions with complex coefficients. Results for zeros of transmission on the unit circle are obtained via a parallel combination of lattices. By a suitable transformation the sections are made computable while preserving the cascade form.**

Manuscript received October 21, 1986; revised June 18, 1987 and December 21, 1987. This work was supported in part by the National Science Foundation under Grant ECS and MIP-85-06924, and by the US–Spain Joint Commission under Grant CCB 84-002-002. This paper was recommended by Associate Editor P. P. Vaidyanathan.

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IEEE Log Number 8819870.

I. Introduction

There has been considerable interest in lattice structures in the realization of AR (AutoRegressive) filters because of the nice properties possessed [1]–[5]. For example, lattice filters result directly from the reflection coefficients found in the estimation process of the maximum entropy method and they allow for updating by simply adding sections without recalculating previous sections.

However, AR filters are somewhat limited in that they have all zeros of transmission at $z = 0$; that is, they are all pole in $1/z$. When one adds MA (Moving Average) portions, that is, zeros at other than $z = 0$, one runs into a new situation since the zeros must be incorporated in the structure. To do this, one can 1) divert from the lattice structure [6], 2) obtain a cascade realization of lattices by cascading separate numerator and denominator lattices [7, p. 1004], or 3) revert to general factorization methods [8], or 4) use higher dimensional cascades [9]. In the first instance one loses the nice properties of the lattice; in the second instance the structure is not minimal in the number of delay elements; in the third and four instances one may use overly powerful techniques with attendant numerical difficulties.

Here we extend to complex zeros the method of Vaidyanathan and Mitra [10] whereby non-unit circle zeros of transmission are realized by cascade lattices via use of Richards' functions, analogous to the analog filter technique of extracting Hazony sections [11, p. 158]. Zeros of transmission on the unit circle must be treated separately and in such cases we obtain a cascade section that is the parallel connection of lattices, hence, our title of "lattice-like." Since the resulting sections are not computable (except when the zeros are at zero or infinity), we transform them to computable sections following the ideas of Deprettere and Dewilde [12]. This allows us to obtain in a very simple way the cascade of lattice-like sections for stable ARMA filters. Since Hazony sections are lossless, this technique to be discussed has the advantages of digital lattice filters as well as those of the LBR approach [13]. With the recent interest in complex signal processors and networks [14], we point out that complex coefficients are an inherent result of the theory, as we show below. Thus the results are most convenient for programmed realizations where this method should offer computational advantages. However, when one starts with real-rational functions one can obtain a cascade of minimal degree real sections by suitable transformation and combination of sections.

Notations

$1/z$ is used to represent unit delay and we consider when relevant that $z = \exp(-p)$ with $p = \sigma + j\omega$ being the complex (Laplace transform) frequency. Superscript ^{T} denotes matrix transposition; superscript * means complex conjugation; subscript $_*$ means para-Hermitian conjugation, that is, for a matrix $A(z)$

$$A_*(z) = A^{*T}(1/z^*). \tag{1}$$

1_m denotes the $m \times m$ identity matrix, $\dot{+}$ is the matrix direct sum and

$$J = 1_m \dot{+} (-1_m) \tag{2}$$

We write $\delta[A]$ for the McMillan degree in z of a matrix $A(z)$

and call a structure realizing A with $\delta[A]$ delay elements a minimum degree structure.

II. Background Review

Our technique is closely related to one of 2-port synthesis through the use of scattering matrices; thus we rely heavily upon the concepts developed in [15] and recall that the n-port scattering matrix $S(z)$ maps incident n-vector variables, v^i, to reflected variables, v^r, $v^r = Sv^i$. In the case where $n = 2m$ we partition the incident and reflected variables into two m-vectors, subscripted by 1 for input and 2 for output, as shown in Fig. 1, where the leads represent signal flow lines, and introduce the transfer scattering matrix $\theta(z)$. We then have

$$\begin{bmatrix} v_1^r \\ v_2^r \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} v_1^i \\ v_2^i \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} v_1^i \\ v_1^r \end{bmatrix} = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} v_2^r \\ v_2^i \end{bmatrix}. \tag{4}$$

Fig. 2(a) and (b) shows two particular instances of lattice structures of interest for which we, respectively, calculate the transfer scattering matrices as

$$\Theta = \begin{bmatrix} 1_m & -K_1 \\ K_2 & 1_m \end{bmatrix} \tag{5a}$$

$$\Theta = \begin{bmatrix} 1/f(z) & 0 \\ 0 & g(z) \end{bmatrix}. \tag{5b}$$

For lossless structures we have the two equivalent results

$$S_* S = 1_n \tag{6a}$$

$$\Theta_* J \Theta = J \tag{6b}$$

in which case S is called para-unitary, with $S^{-1} = S_*$, and Θ is said to be J-para-unitary, with $\Theta^{-1} = J\Theta_* J$. As an example of interest, if k is any complex number and we treat a 2-port, that is, take $m = 1$, then

$$\Theta = \frac{1}{(1-k^*k)^{1/2}} \begin{bmatrix} 1 & -k \\ -k^* & 1 \end{bmatrix} \tag{7}$$

is J-para-unitary and results by scaling the transfer scattering matrix of (5a) with $K_1 = k$ and $K_2 = -k^*$ by $l = (1-k^*k)^{1/2}$. If, as in the case of interest to us, the n-port is passive then S is analytic in $|z| > 1$ and

$$1_n - S^{*T} S \geqslant 0 \tag{8a}$$

and, equivalently,

$$\Theta^{*T} J \Theta - J \geqslant 0 \text{ in } |z| > 1. \tag{8b}$$

In the case that we treat here, $S(z)$ is rational in z for which the passivity constraints mean that it is analytic on $|z| = 1$.

We are interested in cascades in which, with references to Fig. 1, we desire to calculate the input S_I (or load S_L) scattering coefficient in terms of the load (or input) scattering coefficient and the transfer scattering matrix entries. Thus

$$S_I = (\Theta_{21} + \Theta_{22} S_L)(\Theta_{11} + \Theta_{12} S_L)^{-1} \tag{9a}$$

$$S_L = (\Theta_{22} - S_I \Theta_{12})^{-1}(S_I \Theta_{11} - \Theta_{21}) \tag{9b}$$

which are found by using $v_1^r = S_I v_1^i$ and $v_2^i = S_L v_2^r$ in (4) (see fig. 2).

When considering cascades on the scattering basis it is the transfer scattering matrix which is of interest, since the cascade

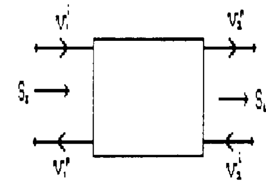

Fig. 1. Signal flowgraph like representation of 2 m-port.

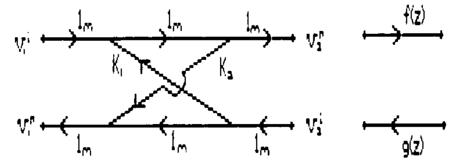

Fig. 2. Some lattices. (a) Lattice with cross-arm gains. (b) Lattice for diagonal transfer scattering matrix.

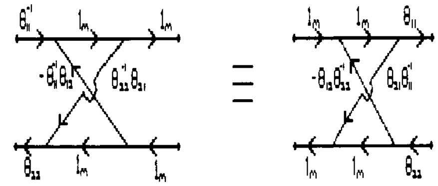

Fig. 3. General lattices from factorization of transfer scattering matrix.

using the port variable designations of Fig. 1 yields the transfer scattering matrix as the product of the individual transfer scattering matrices in the same order, that is, $\Theta = \Theta_1 \cdot \Theta_2$. If the diagonal entries of a given transfer scattering matrix are nonsingular, its factorizations

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \begin{bmatrix} \Theta_{11} & 0 \\ 0 & \Theta_{22} \end{bmatrix} \cdot \begin{bmatrix} 1_m & \Theta_{11}^{-1}\Theta_{12} \\ \Theta_{22}^{-1}\Theta_{21} & 1_m \end{bmatrix} \tag{10a}$$

$$= \begin{bmatrix} 1_m & \Theta_{12}\Theta_{22}^{-1} \\ \Theta_{21}\Theta_{11}^{-1} & 1_m \end{bmatrix} \cdot \begin{bmatrix} \Theta_{11} & 0 \\ 0 & \Theta_{22} \end{bmatrix} \tag{10b}$$

yield the two equivalent lattice structures of Fig. 3. However, these may be far from minimal degree structures if Θ_{11} and/or Θ_{22} are not constant in z.

Given a scattering matrix S we will have occasion to convert between it and the transfer scattering matrix Θ and the admittance matrix Y, the formulas being

$$S = (1_n + Y)^{-1}(1_n - Y) \tag{11}$$

$$= \begin{bmatrix} \Theta_{21}\Theta_{11}^{-1} & \Theta_{22} - \Theta_{21}\Theta_{11}^{-1}\Theta_{12} \\ \Theta_{11}^{-1} & -\Theta_{11}^{-1}\Theta_{12} \end{bmatrix} \tag{12}$$

$$Y = (1_n + S)^{-1}(1_n - S) \tag{13}$$

$$\Theta = \begin{bmatrix} S_{21}^{-1} & -S_{21}^{-1}S_{22} \\ S_{11}S_{21}^{-1} & S_{12} - S_{11}S_{21}^{-1}S_{22} \end{bmatrix}. \tag{14}$$

III. Computable Cascades

In order that the structure be useful for performing calculations it is necessary that there be no delay free loops [16, p. 272], since otherwise an iteration could not proceed due to a variable's new value depending upon that value itself. Even though a given

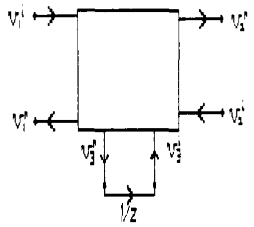

Fig. 4. Delay extraction.

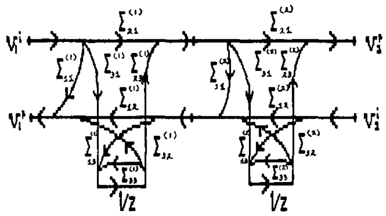

Fig. 5. Absence of delay free loops for $\Sigma_{22}=0$.

section may have no delay free loops, when it is cascaded with another such loops may show up. To see how such delay free loops can be avoided we undertake state-variable type of considerations. Although most of what we do holds for $2m$-port sections, since we are interested in degree one lossless 2-port sections, we now limit ourselves to them and pull out the delay element, as shown in Fig. 4. Assuming, as we may for our purposes, that we are left with a 3-port for which the scattering matrix Σ exists, it and its delay load are described by

$$\begin{bmatrix} v_1^r \\ v_2^r \\ v_3^r \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix} \begin{bmatrix} v_1^i \\ v_2^i \\ v_3^i \end{bmatrix} \tag{15a}$$

$$v_3^i = (1/z)\, v_3^r. \tag{15b}$$

From these we find that the 2-port scattering matrix is

$$S = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} + \begin{bmatrix} \Sigma_{13} \\ \Sigma_{23} \end{bmatrix} \frac{1}{(z-\Sigma_{33})} [\Sigma_{31} \quad \Sigma_{32}]. \tag{16}$$

It should be noted that this decomposition for $S(z)$ is the standard state-variable one in terms of the variable z [11, pp. 80–84]. Therefore, in order to obtain Σ we can obtain a state-variable realization for S and then identify the various submatrices, as in [17], after possibly transforming to obtain passivity and lossless conditions on Σ. Once that is done, loading of port three by $1/z$, rather than z (as one might expect from state-variable theory), yields S (according to (15) and Fig. 4).

Next consider Fig. 5 which shows in signal flowgraph form the cascade of two degree one sections having $\Sigma_{22}=0$ with the delay extracted as above. On inspecting Fig. 5 but with a possibly nonzero Σ_{22} included we see that a sufficient condition for delay free loops is

$$\Sigma_{22} = 0 \Rightarrow \text{no delay free loops}. \tag{17}$$

That is, any signal incident from the right must go through a delay before it gets fed back to being reflected on the right when $\Sigma_{22}=0$ [while if $\Sigma_{22}\neq 0$ then there can be delay free loops (one possibility would be the loop of $\Sigma_{22}^{(1)}$ with $\Sigma_{11}^{(2)}$)]. Since we wish to perform calculations on the transfer scattering matrix, we see from (12) and (16) that (17) is equivalent to

$$\Theta_{12} = 0 \text{ at } z=\infty \Rightarrow \text{no delay free loops} \tag{18}$$

where we have assumed that Θ_{11} is not zero at infinity, which will be the case for the functions we consider.

IV. The Lattice Synthesis Technique

We take as given a voltage transfer function $H(z)$ assumed to be rational in z which is absolutely stable, that is with no singularities outside of $|z|<1$. To obtain a lossless scattering matrix we calculate

$$M \geq \max_{|z|=1} |H(z)| \tag{19a}$$

and form

$$S_{21}(z) = H(z)/M. \tag{19b}$$

Consequently, we have on hand a rational $S_{21}(z)$ which is absolutely stable with magnitude bounded by unity on the unit z circle. S_{21} is to be interpreted as the transmission coefficient of a network described on the scattering basis. With this in mind we create the reflection coefficient $S_{11}=S_I$ by forming

$$1 - S_{21*}S_{21} = S_{I*}S_I \tag{20}$$

which we factor to obtain S_I. Knowing S_{21}, the factorization of (20) for S_I is straightforward though S_I is nonunique. Indeed one can introduce all-pass factors, that is factors $A(z)$ such that $A^*A=1$. Also we note that all-pass factors in S_{21} cancel out in this factorization and will have to be reintroduced to complete a realization of $H(z)$. We choose $S_I(z)$ to have the same poles as S_{21} and to be of the same degree as $S_{21}(z)$; if there are all-pass factors in S_{21} we put the same all-pass factors into S_I. Since, from (19b) and (20), $1-S_{I*}S_I = 1-S_I^*S_I \geq 0$ for $|z|=1$, S_I is bounded in magnitude by 1 in $|z|\geq 1$, being analytic there, and, hence, is like a passive scattering coefficient. From S_{21} and $S_I=S_{11}$ we can create a full 2×2 lossless scattering matrix but for our purposes this will be implicit as we proceed directly from S_I.

Given S_I of the type just mentioned, that is, rational, absolutely stable, and of passive type, we proceed to extract lossless cascade sections. The nature of the sections depends upon the zeros of transmission, that is the zeros of $H(z)$, or equivalently by (19b), of $S_{21}(z)$, or by (20) where $1=S_{I*}S_I$. There are essentially two different types of cases, one where zeros of transmission are not on the unit z-plane circle and one where they are on $|z|=1$. In the former case there are two further subcases depending upon $|z|>1$ or $|z|<1$. When the zeros of transmission are not on the unit circle we do a Richards' type of transformation and factor out an all-pass factor, with this factor corresponding to a cascade lattice section. In the case of a unit circle zero of transmission the theory is more difficult; for it we transform to an admittance type of description and extract a lossless cascade section which is the parallel connection of two lattices. Finally, there is the termination section where a constant reflection coefficient is met.

Correspondingly, we let α be a zero of transmission, that is a zero of $H(z)$, and consider the three nontermination cases just mentioned.

Case 1: $|\alpha|>1$

Let

$$k = -S_I(\alpha)^* \tag{21a}$$

and form the Richards' function

$$S_L(z) = \frac{1-\alpha^* z}{z-\alpha} \cdot \frac{S_I(z)+k^*}{kS_I(z)+1}. \tag{21b}$$

Then if we let

$$f(z) = (1-\alpha^* z)/(z-\alpha). \tag{21c}$$

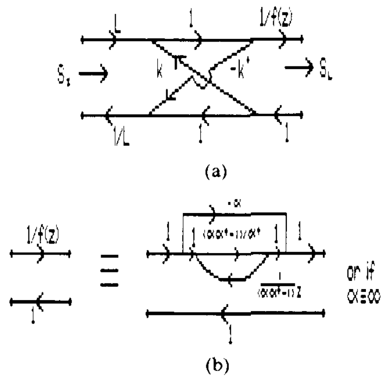

Fig. 6. (a) Lattice for a zero of transmission in $|z|>1$. (b) Degree one realization for $1/f$.

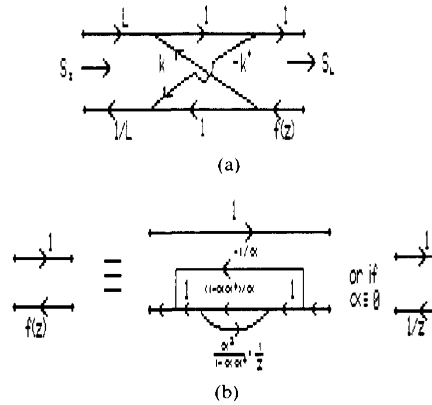

Fig. 7. (a) Lattice for a zero of transmission in $|z|<1$. (b) Degree one realization for f.

S_L results from S_I, as is seen using (9b), via the transfer scattering matrix

$$\Theta=\frac{1}{l}\begin{bmatrix}1 & -k\\ -k^* & 1\end{bmatrix}\begin{bmatrix}f(z) & 0\\ 0 & 1\end{bmatrix},\qquad l=(1-k^*k)^{1/2}. \tag{21d}$$

This transfer scattering matrix is realized by the signal flowgraph lattice (with scaling by l for losslessness) of Fig. 6(a) with $S_I(z)$ resulting at the input by loading in $S_L(z)$ at the output. As can be seen directly from Fig. 6(a), where the upper right arm opens up at a pole of f, there is a zero of transmission at the pole of $f(z)$, that is, at $z=\alpha$, as desired. The realization of the $1/f$ transmittance is shown in Fig. 6(b) which, when $\alpha\neq\infty$ results from

$$1/f(z)=[1-(\alpha/z)]/[(1/z)-\alpha^*] \tag{22a}$$

$$=-\alpha+[1-\alpha^*\alpha]/[(1/z)-\alpha^*] \tag{22b}$$

$$=-\alpha+\frac{(\alpha^*\alpha-1)/\alpha^*}{1-\dfrac{(\alpha^*\alpha-1)}{\alpha^*}\cdot\dfrac{1}{(\alpha^*\alpha-1)\cdot z}}. \tag{22c}$$

This expansion is interpreted in terms of a degree one structure through a feedback loop, as shown in Fig. 6(b) when α is finite. Should the zero of transmission be at infinity, then

$$1/f(z)=1/z\ \{\text{if } \alpha=\infty\} \tag{22d}$$

which is a simple delay. We note that in this case of a zero of transmission at infinity, $\Sigma_{22}=0$, and there are no delay free loops; in contrast (22c) gives a nonzero Σ_{22} (except in the case of an all-pass zero where $k=0$). These facts are verified by calculating Σ directly from Fig. 6; thus

$$\Sigma=\frac{1}{\alpha^*}\cdot\begin{bmatrix}-\alpha^*k^* & \alpha^*l & 0\\ -l & -k & 1\\ l(\alpha\alpha^*-1) & k(\alpha\alpha^*-1) & 1\end{bmatrix},\qquad \text{for } \alpha\neq\infty \tag{23a}$$

or

$$\Sigma=\begin{bmatrix}-k^* & l & 0\\ 0 & 0 & 1\\ l & k & 0\end{bmatrix},\qquad \text{for } \alpha=\infty. \tag{23b}$$

Before going further, we note that $f(z)$ is all-pass and that if α is an all-pass zero of transmission then $k=0$ and the technique still applies.

Now we check the properties of S_L and show that they are the same as those of S_I, except that the degree has decreased, $\delta[S_L]=\delta[S_I]-1$, in which case we can continue on S_L as if it were S_I. First we observe that S_L is rational in z with S_I. Second we see that $z-\alpha$ and $1-\alpha^*z$ both cancel in numerator and denominator of S_L by the choice of k. Thus S_L is analytic where S is analytic, in particular in $|z|>1$. And because of this cancellation, $\delta[S_L]\leqslant\delta[S_I]-1$. However, since the lattice section uses only one delay element, the degree can not decrease by more than one, in which case equality holds to show that the degree decreases by exactly 1. Finally $1-S_L^*S_L\geqslant 0$ on $|z|=1$, and, hence, in $|z|>1$, as we see by noting that $|(1-\alpha^*z)/(z-\alpha)|=1$ and $|(S_I+k^*)/(kS_I+1)|<1$ on $|z|=1$ {this latter being seen by noting that $|k|<1$ in $|z|>1$, as $1-S_x^*S>0$ there, and letting $S_I=a+jb$, $k=c+jd$}.

Case 2: $|\alpha|<1$

This case of zeros of transmission inside the unit circle is almost identical to that for zeros outside of the unit circle. Here we let

$$k=-1/S_I(\alpha) \tag{24a}$$

and form

$$S_L(z)=\frac{z-\alpha}{1-\alpha^*z}\cdot\frac{S_I(z)+k^*}{kS_I(z)+1}. \tag{24b}$$

Here S_L results from S_I via

$$\Theta=\frac{1}{l}\begin{bmatrix}1 & -k\\ -k^* & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & f(z)\end{bmatrix},\qquad l=(1-k^*k)^{1/2}. \tag{24c}$$

Fig. 7(a) shows the corresponding lattice where $f(z)$ is given by (21c) as in Case 1. To realize the $f(z)$ transmittance, as is done in Fig. 7(b), we make the following expansion, first assuming that the zero of transmission is not at zero:

$$f(z)=[1-\alpha^*z]/[z-\alpha] \tag{25a}$$

$$=[(\alpha^*/\alpha)-1/(z\alpha)]/[(1/z)-(1/\alpha)] \tag{25b}$$

$$=-\frac{1}{\alpha}+\frac{(1-\alpha\alpha^*)/\alpha}{1-\dfrac{1-\alpha\alpha^*}{\alpha}\cdot\dfrac{\alpha^2}{(1-\alpha\alpha^*)z}}. \tag{25c}$$

If the zero of transmission is at zero then

$$f(z)=1/z\quad\{\text{if } \alpha=0\}. \tag{25d}$$

This situation of a zero of transmission at zero corresponds to the case of AR filters most frequently treated via lattice structures. Again, $\Sigma_{22}=0$ when $\alpha=0$ but is nonzero otherwise, since

$$\Sigma=\begin{bmatrix}-k^* & -\alpha^*l & (1-\alpha\alpha^*)l/\alpha\\ l & -\alpha^*k & (1-\alpha\alpha^*)k/\alpha\\ 0 & \alpha & \alpha\end{bmatrix},\qquad \text{for } \alpha\neq 0 \tag{26a}$$

or

$$\Sigma = \begin{bmatrix} -k^* & 0 & l \\ l & 0 & k \\ 0 & 1 & 0 \end{bmatrix}, \qquad \text{for } \alpha = 0., \tag{26b}$$

Case 3: $|\alpha| = 1$

In this case the extractions of the previous two cases do not work directly since $f(z) = -1/\alpha$ and the Richards' function factorizations of (21b) and (24b) lose their z dependence. Consequently, we proceed by forcing a pole of the input admittance at α. To do the latter we force an input reflection coefficient to be -1 at the zero of transmission which is accomplished by extracting a scaling complex constant section having transfer scattering matrix:

$$\Theta = \begin{bmatrix} -S_I(\alpha)^* & 0 \\ 0 & 1 \end{bmatrix}. \tag{27a}$$

To see the effect of this transformation we note from (9b) that the load scattering coefficient is $S_{L1}(z) = -S_I(\alpha)^* S_I(z)$ which at $z = \alpha$ becomes $S_{L1}(\alpha) = -|S_I(\alpha)|^2$ while (20) is $|H(\alpha)/M|^2 = 0 = 1 - |S_I(\alpha)|^2$ which shows that $|S_I(\alpha)| = 1$ giving $S_{L1}(\alpha) = -1$. Defining this S_L as S_{I1} the scaling through (27a) yields a complex constant diagonal lattice as in Fig. 2(b) and with it we assume that $S_{I1}(\alpha) = -1$. Next form

$$y_I(z) = [1 - S_{I1}(z)]/[1 + S_{I1}(z)] \tag{27b}$$

which has a pole at $z = \alpha$ which must be simple and with a residue of $2r\alpha$, with $r > 0$ (by the passive nature of $y_I(z)$). This gives

$$y_I(z) = r \cdot \frac{z+\alpha}{z-\alpha} + y_L(z). \tag{28a}$$

Here the degree goes down when going from Y_I to Y_L since we have $\delta[y_L] = \delta[y_I] - 1 = \delta[S] - 1$ while passivity is retained since on $|z| = 1$, $y_{I*} + y_I = y_{L*} + y_L$. To continue, form

$$S_L(z) = [1 - y_L(z)]/[1 + y_L(z)] \tag{28b}$$

which in turn has degree one less than S_I, $\delta[S_L] = \delta[y_L] = \delta[S_I] - 1$. We have then extracted a degree one section to obtain S_L from the original S_I and this results from the degree one coupling transfer scattering matrix:

$$\Theta = [-S_I(\alpha) \dotplus 1] \cdot [1_2 + g(z) J u u^T] = \Theta_1 \cdot \Theta_2 \tag{29a}$$

where

$$g(z) = \frac{r}{2} \cdot \frac{z+\alpha}{z-\alpha} \tag{29b}$$

$$u = [1, 1]^T. \tag{29c}$$

As is checked by calculating S_I from S_L using (9a) with (29a) on noting that $g_* = -g$, Θ is J-paraunitary and, thus comes from a lossless 2-port. The resulting 2-port can be realized by obtaining the scattering matrix and obtaining the state-variable realization for it [as discussed below (16)] and is shown in Fig. 8. Thus considering just Θ_2 since $\Theta_1 = [-S_I(\alpha) \dotplus 1]$ is already in a form for lattice realization, we find its scattering matrix using (12) and then make a partial fraction expansion to get the state-variable realization of (16):

$$S_2(z) = \frac{1}{1+g(z)} \begin{bmatrix} -g(z) & 1 \\ 1 & -g(z) \end{bmatrix} \tag{30a}$$

$$= \frac{1}{2+r} \begin{bmatrix} -r & 2 \\ 2 & -r \end{bmatrix} + \frac{-4\alpha r}{(2+r)^2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \left[z - \alpha \left(\frac{2-r}{2+r} \right) \right]^{-1} [1, \ 1]. \tag{30b}$$

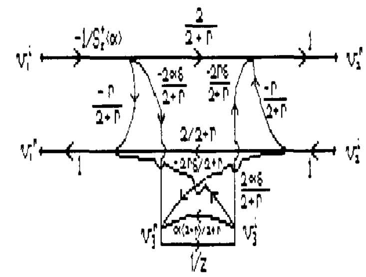

Fig. 8. Realization of unit circle zeros of transmission.

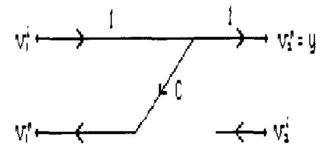

Fig. 9. Termination section.

Performing a transformation on the state by inserting a design parameter δ into the numerator and denominator of the $-4\alpha r$ term we identify $2\alpha/\delta$ with the (3,1) and (3,2) entries of Σ and $-2r\delta$ with the (1,3) and (2,3) entries while observing the formation of a scattering matrix $S(z)$ from the constant coupling scattering matrix Σ shown in (16) to set

$$\Sigma = \frac{1}{2+r} \begin{bmatrix} -r & 2 & -2r\delta \\ 2 & -r & -2r\delta \\ 2\alpha/\delta & 2\alpha/\delta & \alpha(2-r) \end{bmatrix}. \tag{30c}$$

Following the structure of Fig. 5 the case 3 section is then realized as shown in Fig. 8. We note that $\Sigma_{22} \neq 0$ in which case the section described by (30c) is not computable. However, as can be checked by direct calculation, by a proper choice of the design parameter $\delta 1_3 - \Sigma^{*T}\Sigma$ can be chosen to be positive semidefinite. Indeed the choice of

$$\delta = \left[1 - \alpha\alpha^* (\{2-r\}/\{2+r\})^2\right]^{1/2} = 2(2r)^{1/2}/(2r) \tag{31a}$$

yields

$$1_3 - \Sigma^{*T}\Sigma = (2+r)^2 \begin{bmatrix} 4 & 4 & 2(2-r)\delta \\ 4 & 4 & 2(2-r)\delta \\ 2(2-r)\delta & 2(2-r)\delta & [(2-r)\delta]^2 \end{bmatrix} \tag{31b}$$

which is positive semidefinite and, hence, yields a passive coupling section. It should be noted that this is not in itself lossless since the delay load is not in itself lossless.

Termination

After all the dynamics is extracted by repeatedly carrying out the lossless extractions of the above cases a constant reflection coefficient is met. Thus we eventually achieve

$$S_I(z) = C = \text{constant}, \qquad |C| \leq 1. \tag{32}$$

Fig. 9 shows the 2-port realization for this S_I to be used as a terminating section.

V. Computable Sections

Except in the cases of zeros of transmission at zero or infinity the extractions discussed so far do not lead directly to computable sections. Consequently, we show here how to modify the degree one cascade sections found to this point so that they

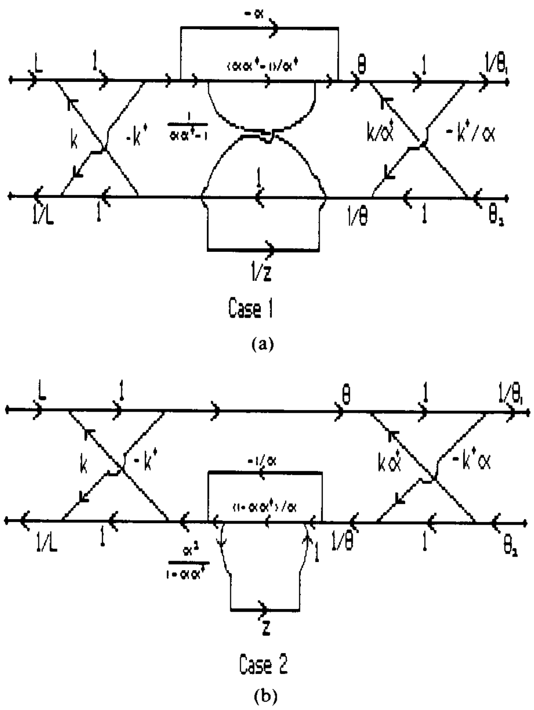

Fig. 10. Computable sections in cascade of lattices form.

become computable. This is accomplished by making for each section a further J-unitary extraction so that Σ_{22} of the combined extractions becomes zero. However, it is much easier to proceed from Θ, rather than Σ, so we perform the transformations on Θ and use (18), that is, we force Θ_{12} to be zero at $z=\infty$. This is done by factoring out on the right another J-unitary transfer scattering matrix, called here Θ_c for computability. Thus following [12, p. 252], note that any J-unitary 2×2 matrix has the following form:

$$\Theta_c = \frac{1}{\theta}\begin{bmatrix} 1 & R \\ R^* & 1 \end{bmatrix}\begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}, \qquad \theta = (1-RR^*)^{1/2}, \qquad |\theta_1| = |\theta_2| = 1. \tag{33}$$

We wish to multiply the transfer scattering matrices of (21d), (24c), (29a) on the right by matrices of the type of (33) and find the corresponding R's and θ_1 and θ_2 such that the (1,2) entries of the products become zero at infinity. Carrying out these calculations in the three cases we find (independently of the phase terms θ_1 and θ_2), on using the fact from (21c) that $f(\infty) = -\alpha^*$,

$$R = k/f(\infty) = -k/\alpha^*. \tag{34a}$$

Case 2:

$$R = kf(\infty) = -k\alpha^*. \tag{34b}$$

Case 3:

$$R = -r/(2+r). \tag{34c}$$

The result is that in all cases we can obtain degree one sections that are computable, these resulting as a cascade of three sections (a nondynamic J-unitary one to prepare the delay extraction, a dynamic J-paraunitary one, and a nondynamic J-unitary one to correct for computability]. In the Cases 1 and 2 situations these sections are all lattices, as shown in Fig. 10, but their computable equivalents, which can be drawn as per Fig. 5, are not. Indeed we calculate Σ in a straightforward manner from Fig. 10 and get, on recalling that $l = (1-kk^*)^{1/2}$ and $\theta = (1-RR^*)^{1/2}$:

For Case 1 (Fig. 10a):

$$\theta = \left(1 - \frac{kk^*}{\alpha\alpha^*}\right)^{1/2} \tag{35a}$$

$$v_1^r = \frac{-1}{\theta^2\alpha\alpha^*}\left[k^*(\alpha\alpha^*-1)v_1^i - (\alpha^* l\theta\theta_2)v_2^i + (k^* l)v_3^i\right] \tag{35b}$$

$$v_2^r = \frac{\theta_{1*}}{\alpha^*\theta^2}\left[(-l)v_1^i + 0\cdot v_2^i + v_3^i\right] \tag{35c}$$

$$v_3^r = \frac{1}{\alpha^*\theta^2}\left[l(\alpha\alpha^*-1)v_1^i - k(\alpha\alpha^*-1)\theta\theta_2 v_2^i + (l^2)v_3^i\right]. \tag{35d}$$

Note that, as expected and seen by the middle term of (35c), $\Sigma_{22} = 0$. Further, since Θ_c transforms an input scattering coefficient into a load one, we can apply (9b) to get the final load scattering coefficient after the computable extraction. Combining the original Richards' function extraction and the computability section extraction we can obtain a single expression for the final load in terms of the given input scattering coefficient, this being best found by multiplying all of the transfer scattering matrices involved and using (9b). Thus

$$S_{Lc}(z) = \frac{\theta_1\alpha^*}{\theta_2\alpha}\cdot\frac{S_l(z)[\alpha(1-\alpha^*z)+kk^*(z-\alpha)] + k^*[\alpha(1-\alpha^*z)+(z-\alpha)]}{[kk^*(1-\alpha^*z)+\alpha^*(z-\alpha)] + kS_l(z)[(1-\alpha^*z)+\alpha^*(z-\alpha)]}. \tag{35e}$$

In this $z-(1/\alpha^*)$ and $z-\alpha$ cancel in numerator and denominator such that $\delta[S_L] = \delta[S_l]-1$ as expected [as is seen by evaluating the numerator and denominators at $z=\alpha$ and $z=1/\alpha^*$ while using $S_l(\alpha) = (1/S_{l*}(\alpha)) = (1/S_l^*(1/\alpha^*)) = -k^*$].

For Case 2 (Fig. 10b),

$$\theta = (1-kk^*\alpha\alpha^*)^{1/2} \tag{36a}$$

$$v_1^r = \frac{1}{\alpha\theta^2}\left[-k^*\alpha(1-\alpha\alpha^*)v_1^i - (\alpha\alpha^*\theta l\theta_2)v_2^i + l(1-\alpha\alpha^*)v_3^i\right] \tag{36b}$$

$$v_2^r = \frac{\theta_{1*}}{\alpha\theta}\left[(\alpha l)v_1^i + 0\cdot v_2^i + k(1-\alpha\alpha^*)v_3^i\right] \tag{36c}$$

$$v_3^r = \frac{\alpha}{\theta^2}\left[(\alpha k^* l)v_1^i + \theta\theta_2 v_2^i + l^2 v_3^i\right]. \tag{36d}$$

As for Case 1 we can also express the load directly in terms of the input.

$$S_{Lc}(z) = \frac{\theta_1 S_l(z)[(z-\alpha)+kk^*\alpha(1-\alpha^*z)] + k^*[(z-\alpha)+\alpha(1-\alpha^*z)]}{\theta_2[kk^*\alpha^*(z-\alpha)+(1-\alpha^*z)] + kS_l(z)[\alpha^*(z-\alpha)+(1-\alpha^*z)]}. \tag{36e}$$

From the expressions of (35) and (36) we see that in both cases we have achieved $\Sigma_{22}=0$ and that all other Σ_{ij} are nonzero. The cascade structures then do take the form of Fig. 5 with all non Σ_{22} transmittances nonzero and read off from (35) or (36).

A similar treatment holds for transmission zeros on the unit circle since (33c) quarantees $\Sigma_{22}=0$ in that case also.

It should be noted that we may also find S_{Lc} by looking at S_L through the inverse of Θ_c, which is the same as Θ_c except that R is replaced by $-R$ and the θ_1's appear on the right; thus

$$S_{Lc}=[\theta_1(-R^*+S_L)]/[\theta_2(1-RS_L)]. \quad (37)$$

VI. Discussion

Here we have presented a means to realize digital filters in minimal degree cascade form. When the transmission zeros are off of the unit circle in the z plane, the resulting structure comes from the cascade of lattices via Richards' function extractions. It is in this use of the Richards' function that the advantages of this method lie for it gives a computationally simple method of obtaining minimal degree sections which can then be manipulated to obtain other desired properties, such as computability of the structure. The degree one sections so obtained will have complex valued transmittances when the transmission zeros are complex, and, hence, a Richards' function valid for complex zeros must be used, these being given in (21b) or (24b) depending upon whether the zero is inside or outside of the unit circle. Unfortunately, these Richards' functions can not be used for zeros of transmission on the unit circle so other techniques must be used there. In any of the cases, however, it is possible to make the sections computable by cascading with a constant lattice. After this cascading is made, a calculation of the resulting scattering matrix of the nondynamic portions leads to degree one computable sections, with these only being lattices in the case of zeros of transmission at zero or infinity (in which case the extra effort to cascade with the constant lattice is not necessary). By first normalizing the original transfer function to be less than unity on the unit circle it is possible to proceed solely with passive structures, indeed lossless nonterminating sections, for which the structure must necessarily be relatively insensitive to calculation errors and transmittance choices. Thus the technique presented should prove valuable for future investigations in the field. And, as with the passive network synthesis techniques from which the method stems, generalizations to multiports is possible while of probably equal value would be the incorporation of nonlinearities, the cascade structure being very suitable for introducing nondynamic nonlinearities in both the forward and the backward paths.

In the case of complex valued zeros of transmission for transfer functions having only real coefficients the method uses complex valued sections and originally results in complex loads. However, by the incorporation of J-unitary transformations for adjacent complex conjugate zero of transmission sections, these can be transformed and then may be combined to yield real-valued transmittances, though the resulting structure is no longer in the nice lattice form. Although the complex valued sections are convenient for complex signal processing, as the reviewers have pointed out, these real element structures are of interest, especially for constructions in terms of hardware. Consequently, this portion of the work could use further development.

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Inversion of Singular Systems

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***Abstract*—A novel method for inverting square singular systems is presented. The construction explores the flexibility of singular system representation, and provides a simple and closed form for the constructed inverse. One important property is that the minimality of the inverse (in the sense of Verghese) is equivalent to that of the original system.**

I. Introduction

The dynamical systems considered in this paper is represented by

$$E\dot{x}=Ax+Bu \quad (1a)$$

$$y=Cx+Du \quad (1b)$$

where x is an n-vector of internal variables, u is an m-vector of inputs and y is an m-vector of outputs. E, A, B, C, D are matrices of appropriate dimensions. In particular, E may be

Manuscript received February 27, 1987; revised July 2, 1987. This paper was recommended by Associate Editor R. W. Liu.

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IEEE Log Number 8819858.