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Nonregular Semistate Systems: Examples and Input-Output Pairing*

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ABSTRACT

By the presentation of several examples it is shown that nonregular semistate described systems can be met in practice. A means of pairing inputs and outputs for such systems is discussed.

I. INTRODUCTION

Semistate described systems, which are often referred to by other names (such as singular systems [1]) are of interest since the semistate equations explicitely contain both dynamic and non-dynamic portions. These semistate described systems can be broken down into regular and nonregular systems [2]. Since nonregular systems may have many or no solutions, the question arises as to whether we would ever meet them in practice [3]. Here we show that indeed we do meet them, at least when we idealize certain systems, possibly as design, simulation, or analysis tools.

II. SEMISTATE SYSTEMS & REGULARITY

We will work with the canonical linear time-invariant semistate equations

$$\mathfrak{A}\dot{x} + \mathfrak{B}\cdot x = \mathfrak{D}u \qquad (1a)$$

$$y = \mathfrak{F}x \qquad (1b)$$

where x is the semistate k-vector, u is the input n-vector, y is the output m-vector and $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{D}$, and $\mathfrak{F}$ are constant matrices. These equations are called <u>regular</u> if $(\mathfrak{A}s+\mathfrak{B})$ is nonsingular and nonregular if it is singular. The significance of the semistate equations being regular is that a transfer function T(s) exists since

$$T(s) = \mathfrak{F}\cdot(\mathfrak{A}s+\mathfrak{B})^{-1}\cdot\mathfrak{D} \qquad (2)$$

Therefore, a nonregular (linear time-invariant) system is one which may have no transfer function. This latter can occur by virtue of $\mathfrak{A}s+\mathfrak{B}$ being square but of zero determinant or if $\mathfrak{A}s+\mathfrak{B}$ is nonsquare. Here we only consider the former case, though we do allow $m \neq n$.

II. SOME NONREGULAR EXAMPLES

<u>Example</u> <u>a</u>. Consider the gyrator circuit of Fig. 1 which, when the capacitor is absent, will result in a norator (having v=arbitrary and i=arbitrary when $G_2=-G_1=G\neq 0$) and a nullator (having v=0 and i=0 when $G_1=-G_2=G\neq 0$) [3, p. 13]. We take the input as the current of the current source and the output as the voltage across this source. If we choose the semistate as $x \triangleq [v_1,v_2]^T$ where the superscript T denotes the transpose, we find canonical semistate equations as

$$\begin{bmatrix} C & -C \\ 0 & 0 \end{bmatrix}\dot{x} + \begin{bmatrix} -G_1 & -G \\ G_1-G & G_2+G \end{bmatrix}x = \begin{bmatrix} -1 \\ 0 \end{bmatrix}u \qquad (3a)$$

$$y = [-1,0]x \qquad (3b)$$

which has $\det(\mathfrak{A}s+\mathfrak{B})=\{sc(G_1+G_2)-(G_1G_2+G^2)\}$. The choices $G_1=-G_2=\pm G$ render these equations nonregular.

<u>Example</u> <u>b</u>. In this example we consider the single-input multi-output op-amp integrator amplifier of Fig. 2. We assume that the op-amps are ideal and, thus, characterized by the equations

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}v - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}i = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (4)$$

where $v=[v_1,v_2]^T$, $i=[i_1,i_2]^T$ for the top op-amp and $v=[v_3,v_4]^T$, $i=[i_3,i_4]^T$ for the second one. If we choose

$$x = [v_1,v_2,v_3,v_4,i_1,i_2,i_3,i_4]^T \qquad (5a)$$

for the semistate, Kirchhoff's laws give

$$\mathfrak{A} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ C&-C&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} \qquad (5b)$$

$$\mathfrak{B} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&R \\ 0&0&1&0&0&0&0&0&0&0&R \\ 0&0&0&0&1&0&1&0&1&1&-1 \\ 0&0&1&-1&0&0&0&0&0&-R_2&0 \\ 0&0&0&0&0&0&0&0&-1&0&0 \\ 0&0&0&0&0&1&0&0&-1&0&0 \\ 0&0&0&0&0&0&0&1&0&-1&0 \\ 1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&-1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&-1&0&0&0&0 \end{bmatrix} \qquad (5c)$$

$$\mathfrak{D} = [1,1,0,0,0,0,0,0,0,0,0]^T \qquad (5d)$$

$$\mathfrak{F} = \begin{bmatrix} 0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0 \end{bmatrix} \qquad (5e)$$

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If we subtract rows 8 & 10 from rows 1 & 2, respectively, the new rows 1 & 2 are identical showing that $\det(\mathcal{A}s+\mathcal{B})=0$; these semistate equations are nonregular! To see why this might come about we note that

$$u = \frac{-R}{sCR_2}(R_2y_1 + sCy_2) \qquad (6)$$

in which case one can not solve for unique outputs given the input (even though one can in the individual cases of an integrator and an amplifier).

III. INPUT-OUTPUT PAIRINGS

If a system is nonregular, it need not have a transfer function. However, it may still have a general description pairing [4, p. 47], that is a description of the form

$$A(s)Y(s) = B(s)U(s) \qquad (7)$$

where Y(s) and U(s) are Laplace transforms of the input and output. To obtain this pairing we proceed to obtain matrices $\mathcal{U}(s)$ and $\mathcal{V}(s)$ such that

$$\mathcal{U}(s)Y(s) = \mathcal{U}(s)\mathcal{F}X(s) = \mathcal{V}(s)(\mathcal{A}s+\mathcal{B})X(s) = \mathcal{V}(s)\mathcal{D}U(s) \qquad (8)$$

The method is as follows. First we assume that m=n, for if it does not we can force it by adding zeros appropriately and then cancelling them at the end. With this assumption we apply Theorem 23.1 of MacDuffee [5, p. 35] to obtain the greatest common right divisor, gcrd, of two square matrices (over a principal ideal ring, which we have when the system is time-invariant). By (8) we see that the Laplace transform of the semistate, X(s), is a common right divisor of $\mathcal{U}(s)\mathcal{F}$ and $\mathcal{V}(s)(\mathcal{A}s+\mathcal{B})$, so we look for a gcrd of $\mathcal{F}$ and $(\mathcal{A}s+\mathcal{B})$. This is done by forming, with submatrices X_{ij}'s nonsingular,

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} \mathcal{F} & 0 \\ \mathcal{A}s+\mathcal{B} & 0 \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \qquad (9a)$$

from which the gcrd is $D(s)=X_{11}(s)\mathcal{F}+X_{21}(s)(\mathcal{A}s+\mathcal{B})$ and $X_{21}(s)\mathcal{F}=-X_{22}(s)(\mathcal{A}s+\mathcal{B})$. Again on observing (8) we finally have

$$\mathcal{U}(s) = X_{21}(s), \qquad \mathcal{V}(s) = -X_{22}(s) \qquad (9b)$$

Consequently, the general description relating input and output quantities among themselves is

$$[X_{21}(s)]Y(s) = [-X_{22}(s)\mathcal{D}]U(s) \qquad (10)$$

If any zeros were inserted as final rows of Y(s) or U(s) to make m=n they may now be cancelled by removing the corresponding columns of the appropriate matrices in square brackets. A close look at the steps undertaken to obtain (10) shows that any linear time-invariant semistate described system has this general description input-output pairing.

IV. CONCLUSIONS

We have shown that common circuits using standard circuit elements can have nonregular semistate descriptions. Since nonregular semistate described systems may have many or no solutions, something strange has happened. This can be looked upon as coming through the idealization of the elements. For example, the op-amps of Example b are idealized, being of infinite gain with a true virtual ground at the input while the negative resistor of Example a are assumed linear over their full range when in practice they are quite nonlinear. When a linear time-invariant system is described by a nonregular set of semistate equations, we have shown above that the system has a general description relating input-output quantities among themselves, this being in line with the most general circuit-theoretic system descriptions [4, Chap. 3].

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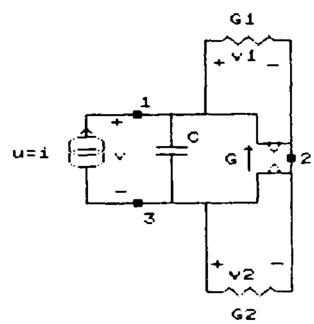

Figure 1
Nonregular Gyrator Circuit
$C_1=-G_2=+C$ or $-G$

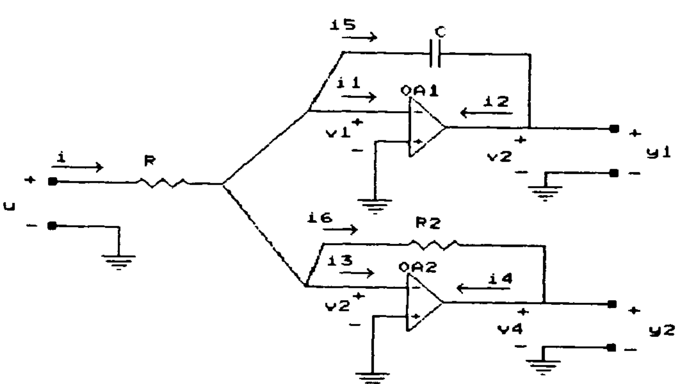

Figure 2
Nonregular Ideal Op-Amp Circuit