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Notes on Cochlear Models for Kemp Echos

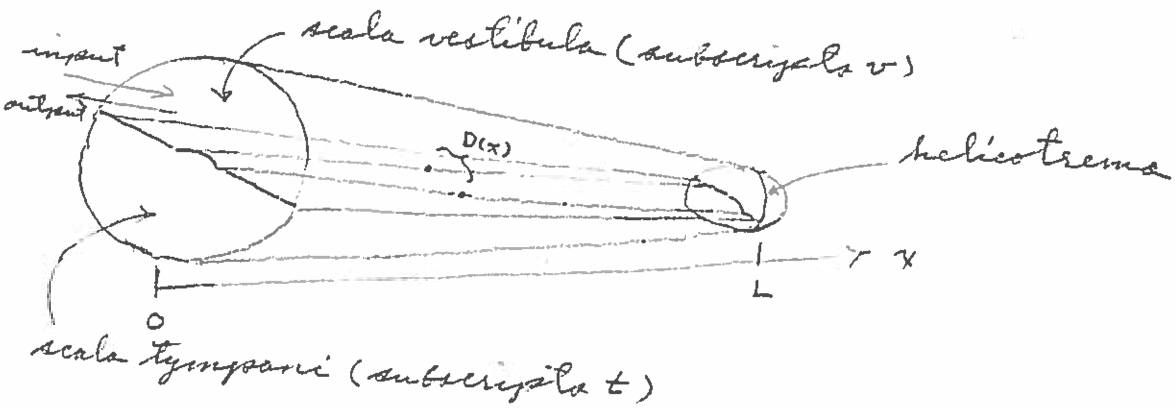
The equations describing motion of the fluid and the basilar membrane will be derived from three considerations:

I. Conservation of mass

II. Newton's law for force on a fluid

III. Basilar membrane motion equations

We idealize the cochlea as an exponentially tapered tube divided in two by a membrane that is stiff except for a portion that vibrates transverse to the fluid flow. The linear membrane taper is actually opposite to the exponential taper, the upper chamber is called the scala vestibuli and the lower one the scala tympani, the two are connected at the narrow end by an opening (the helicotrema), and at the wider end both chambers are sealed by a flexible membrane (the scala tympani by the round window and the scala vestibuli by the oval window). Sound inputs and outputs are at the oval window via the middle ear which acts as a transformer for signals transmitted from the outer ear at the ear drum.



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We use subscripts v & t for the vestibules and tympanic scala. Also t will denote time while x will be distance along the basilar membrane starting at the window end, a one-dimensional model being assumed. Thus

$$v_v(t,x),\ v_t(t,x) = \text{velocities of fluid in } x \text{ direction}$$
$$p_v(t,x),\ p_t(t,x) = \text{pressure in the fluid at } x \text{ \& } t.$$

1) $$S_v(x) = S_t(x) = \text{areas at } x$$

By virtue of the taper and the closed nature of the cochlea a velocity v_v in the scala vestibuli induces a velocity

2) $$v_t(t,x) = -v_v(t,x)$$

in the scala tympani. We take 2) as a basic assumption of the theory which allows us to work primarily with v_v. Also it is the pressure difference

3) $$p = p_v - p_t$$

which causes the basilar membrane to vibrate so we will desire our equations in terms of p.

By way of electrical analogies, since p is measured at v with respect to t we will take p analogous to voltage and v_v analogous to current.

I. Conservation of mass

We have for any fluid

I.1) $$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v}\cdot\nabla\rho$$

ρ = mass density
$\vec{v}$ = velocity of fluid
∇ = space partial derivative operator ($\nabla = \frac{\partial}{\partial x}$ here)

But

I.2a) Increase in mass in volume ΔV in time Δt

I.2b) $= \left(\frac{\partial}{\partial t}\iiint_{\Delta V} \rho\, dV\right)\Delta t$

I.2c) = fluid inflow in time Δt through surface S bounding ΔV

I.2d) $= \left(-\iint_S \rho\vec{v}\cdot d\vec{S}\right)\Delta t$ which by Gauss' law

I.2e) $= \left(-\iiint_{\Delta V} \nabla\cdot(\rho\vec{v})\,dV\right)\Delta t$

The minus sign coming from $d\vec{S}$ pointing out & $\vec{v}$ in. Equating terms under $\iiint$:

I.3) $$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\vec{v}) = -\vec{v}\cdot\nabla\rho - \rho\nabla\cdot\vec{v}$$

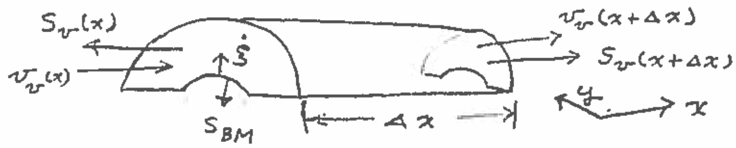
Substituting I.3) into I.1) gives $d\rho/dt = -\rho\nabla\cdot\vec{v}$ or

I.4) $$\nabla\cdot\vec{v} = -\frac{1}{\rho}\frac{d\rho}{dt}$$

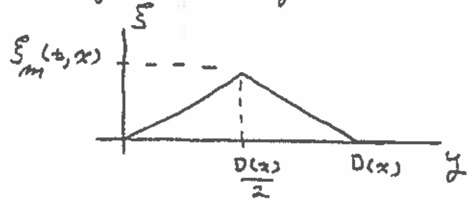
But $\nabla\cdot\vec{v}$ is the divergence and divergence is calculated from

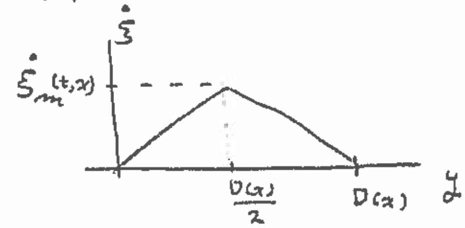
I.5) $$\nabla\cdot\vec{v} = \lim_{\Delta V\to 0}\left\{\frac{\oiint_S \vec{v}\cdot d\vec{S}}{\Delta V}\right\}$$

Now consider a section of length Δx of the scala vestibuli:



Here the velocity perpendicular to the external surface is zero since this is a stiff boundary as is it over the stiff portion of the dividing membrane. For the basilar membrane, BM, we approximate the displacement $\xi(t,x,y)$ and velocity $\partial\xi(t,x,y)/\partial t = \dot{\xi}$ by triangular curves:





Then

I,6a,b) $$\xi = \begin{cases} \dfrac{2\xi_m}{D} y \\ -\dfrac{2\xi_m}{D} y + 2\xi_m \end{cases} \quad , \quad \dot{\xi} = \begin{cases} \dfrac{2\dot{\xi}_m}{D} y & 0 \le y \le \frac{D}{2} \\ -\dfrac{2\dot{\xi}_m}{D} y + 2\xi_m & \frac{D}{2} \le y \le D \end{cases}$$

Here $\xi_m(t,x)$ is the maximum displacement at time t and position x of the BM.

For the right side of I,5) for the section

I,7) $$\oint\!\!\!\oint_S \vec{v}\cdot d\vec{s} = \iint_{at\ x} \vec{v}\cdot d\vec{s} + \iint_{at\ x+\Delta x} \vec{v}\cdot d\vec{s} + \iint_{BM} \vec{v}\cdot d\vec{s}$$

For which

I,8a,b) $$\iint_{BM} \vec{v}\cdot d\vec{s} = \left(-\int_0^{D(x)} \dot{\xi}(t,x,y)\, ds\right)\Delta x = -2\int_0^{D(x)/2} \dot{\xi}(t,x,y)\, ds\ \Delta x$$

where ds is the differential arc length of BM:

I,8c) $$ds = \sqrt{dy^2 + \left(\frac{\xi_m}{D/2} dy\right)^2}$$

or

I,8d) $$\iint_{BM} \vec{v}\cdot d\vec{s} = -2\int_0^{D(x)/2} \frac{2\dot{\xi}_m(t,x)}{D(x)}\, y \sqrt{1 + \left(\frac{2\xi_m(t,x)}{D(x)}\right)^2}\, dy \cdot \Delta x$$

I,8e,f) $$= -2\Delta x \cdot \frac{2\dot{\xi}_m}{D} \cdot \sqrt{1 + \left(\frac{2\xi_m}{D}\right)^2} \cdot \frac{1}{2} y^2 \Big|_0^{D/2} = -\frac{D}{2} \cdot \dot{\xi}_m \cdot \sqrt{1 + \left(\frac{2\xi_m}{D}\right)^2}\ \Delta x$$

Since

I.8g) $$D(x) >> 2\xi_m(t,x)$$

using $\sqrt{1+x} \approx 1 + \frac{1}{2}x$, $|x|$ small

I.8h) $$\iint_{BM} \vec{v}\cdot d\vec{S} \approx -\frac{D(x)}{2}\cdot \dot{\xi}_m(t,x)\cdot\left[1 + 2\left(\frac{\xi_m(t,x)}{D(x)}\right)^2\right]\Delta x$$

For $\nabla\cdot\vec{v}$ we then have for I.5)

I.9a) $$\nabla\cdot\vec{v} = \lim_{\Delta x\to 0}\left\{\frac{-S_v(x)\,v_v(t,x) + S_v(x+\Delta x)\,v_v(t,x+\Delta x) - \frac{D}{2}\dot{\xi}_m\left[1+2\left(\frac{\xi_m}{D}\right)^2\right]\Delta x}{S_v(x)\,\Delta x}\right\}$$

I.9b) $$= \frac{1}{S_v(x)}\left\{\frac{\partial\left(S_v(x)\,v_v(t,x)\right)}{\partial x} - \frac{D(x)}{2}\dot{\xi}_m(t,x)\left[1 + 2\left(\frac{\xi_m(t,x)}{D(x)}\right)^2\right]\right\}$$

I.9c) $$= -\frac{1}{\rho}\frac{d\rho}{dt} \qquad (\text{by I.4}))$$

We next assume that the fluid in the cochlea is incompressible, i.e.

I.10) $$\frac{d\rho}{dt} \equiv 0$$

in which case we find

I.11) $$\nabla(S_v v_v) = \frac{\partial(S_v v_v)}{\partial x} = \frac{D}{2}\dot{\xi}_m\left[1 + 2\left(\frac{\xi_m}{D}\right)^2\right]$$

This is one of our main equations. By symmetry and the fact that $S_t + S_v$ is constant in time;

I.12) $$\frac{\partial S_t v_t}{\partial x} = -\frac{D}{2}\dot{\xi}_m\left[1 + 2\left(\frac{\xi_m}{D}\right)^2\right]$$

II. Newton's Law

By $d[\cdot]/dt = \partial[\cdot]/\partial t + \vec{v}\cdot\nabla[\cdot]$ and using subscript $i = v$ or t

II.1a) $$\frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + (v_i \cdot \nabla) v_i = -\frac{1}{\rho} f_i.$$

where

II.1b) $-f_i$ = force on S_i/volume exerted by the medium on a particle of fluid

II.1c) $$= -(\nabla p_i + R_i v_i)$$

and R_i is the coefficient of frictional force. We have

II.2) $$R_v = R_t$$

Therefore, using $v_t = -v_v$

II.3a) $$\frac{\partial v_v}{\partial t} + v_v \nabla v_v = -\frac{1}{\rho}(\nabla p_v + R_v v_v)$$

II.3b) $$\frac{\partial v_t}{\partial t} + v_t \nabla v_t = -\frac{\partial v_v}{\partial t} + v_v \nabla v_v = -\frac{1}{\rho}(\nabla p_t - R_v v_v)$$

Subtracting II.3b) from II.3a) & using $p = v_v - p_t$

II.4) $$2\frac{\partial v_v}{\partial t} = -\frac{1}{\rho}(\nabla p + 2R_v v_v)$$

or

II.5) $$\nabla p = -\frac{2}{S_v}\left[R_v(S_v v_v) + \rho\frac{\partial(S_v v_v)}{\partial t}\right]$$

which is our second main equation.

III. Basilar Membrane Motion

We consider the basilar membrane to be a second order system with an equivalent mass, $\mu(x)$, friction, $\sigma(x)$, and spring constant $\phi(x)$, all per unit area. Then, using the average displacement over $D(x)$ as $\xi_m/2$

$$\text{III.1)} \quad -p = \mu \frac{\partial^2 \xi_m/2}{\partial t^2} + \sigma \frac{\partial \xi_m/2}{\partial t} + \phi\, \xi_m/2$$

where the minus sign results since p is the force down on the membrane. We rewrite this as

$$\text{III.2)} \quad -p = \frac{1}{D}\left[\mu \frac{\partial^2 \left(\frac{D\xi_m}{2}\right)}{\partial t^2} + \sigma \frac{\partial \left(\frac{D\xi_m}{2}\right)}{\partial t} + \phi \left(\frac{D\xi_m}{2}\right)\right]$$

which is the final main equation desired.

To summarize, we let

IV.1) $$u_v = S_v v_v$$

IV.2) $$s = \frac{\partial[\cdot]}{\partial t}$$

Then I.11), II.5), III.2) are

IV.3a) $$\nabla u_v = \left[1 + \frac{8}{D^4}\left(\frac{D\xi_m}{2}\right)^2\right] \cdot s \left(\frac{D\xi_m}{2}\right)$$

IV.3b) $$\nabla p = -\frac{2}{S_v}\left[R_v + \rho s\right] u_v$$

IV.3c) $$-p = \frac{\left[\mu s^2 + \sigma s + \phi\right]}{D}\left(\frac{D\xi_m}{2}\right)$$

We can eliminate $D\xi_m/2$ by solving for it from IV.3c) and substituting in IV.3a). Thus

IV.4) $$\frac{D\xi_m}{2} = \frac{-D}{\mu s^2 + \sigma s + \phi} \cdot p = -\frac{1}{s\,Q(x,s)}\,p(t,x)$$

where

IV.5a) $$Q(x,s) = \frac{\mu(x)\,s^2 + \sigma(x)\,s + \phi(x)}{D(x)\,s}$$

Let

IV.5b) $$P(x,s) = \frac{2}{S_v(x)}\left[R_v + \rho s\right]$$

Then we have

IV.6a) $$\nabla p = -P(x,s)\,u_v$$

IV.6b) $$\nabla u_v = -\left[1 + \frac{8}{D^4}\left(\frac{1}{sQ}\cdot p\right)^2\right] \cdot \frac{1}{Q(x,s)}\,p$$

Linearizing by assuming

IV.7a) $$1 >> \frac{8}{D^4}\left(\frac{1}{sQ}\,p\right)^2$$

gives

IV.7b) $$\nabla p = -P\,u_v$$

IV.7c) $$\nabla u_v \approx -\frac{1}{Q}\,p$$

V. Scattering Treatment – Linear Case

Convert to force scattering-type variables by

V.1(a,b,c) $$\begin{bmatrix} p \\ u_v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ G(x,a) & -G(x,a) \end{bmatrix}\begin{bmatrix} p^i \\ p^r \end{bmatrix} ; \quad \begin{bmatrix} p^i \\ p^r \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & R(x,a) \\ 1 & -R(x,a) \end{bmatrix}\begin{bmatrix} p \\ u_v \end{bmatrix} ; \; R = 1/G$$

Then

V.2) $$\nabla \begin{bmatrix} p \\ u_v \end{bmatrix} = -\begin{bmatrix} 0 & P \\ \frac{1}{Q} & 0 \end{bmatrix}\begin{bmatrix} p \\ u_v \end{bmatrix}$$

becomes, using $' = \partial[\,]/\partial x = \nabla$,

V.3a) $$\begin{bmatrix} 1 & 1 \\ G & -G \end{bmatrix}\nabla\begin{bmatrix} p^i \\ p^r \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ G' & -G' \end{bmatrix}\begin{bmatrix} p^i \\ p^r \end{bmatrix} = -\begin{bmatrix} 0 & P \\ \frac{1}{Q} & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ G & -G \end{bmatrix}\begin{bmatrix} p^i \\ p^r \end{bmatrix}$$

or

V.3b) $$\nabla\begin{bmatrix} p^i \\ p^r \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1 & R \\ 1 & -R \end{bmatrix}\left\{\begin{bmatrix} 0 & 0 \\ G' & -G' \end{bmatrix} + \begin{bmatrix} 0 & P \\ \frac{1}{Q} & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ G & -G \end{bmatrix}\right\}\begin{bmatrix} p^i \\ p^r \end{bmatrix}$$

V.3c) $$= -\frac{1}{2}\left\{RG'\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} R/Q & P \\ -R/Q & P \end{bmatrix}\begin{bmatrix} 1 & 1 \\ G & -G \end{bmatrix}\right\}\begin{bmatrix} p^i \\ p^r \end{bmatrix}$$

V.3d) $$= -\frac{1}{2}\begin{bmatrix} RG' + \frac{R}{Q} + GP & -RG' + \frac{R}{Q} - GP \\ -RG' - \frac{R}{Q} + GP & RG' - \frac{R}{Q} - GP \end{bmatrix}\begin{bmatrix} p^i \\ p^r \end{bmatrix}$$

V.3e) $$= -\frac{R}{2}\begin{bmatrix} G' + P\left(\frac{1}{PQ} + G^2\right) & -G' + P\left(\frac{1}{PQ} - G^2\right) \\ -G' - P\left(\frac{1}{PQ} - G^2\right) & G' - P\left(\frac{1}{PQ} + G^2\right) \end{bmatrix}\begin{bmatrix} p^i \\ p^r \end{bmatrix}$$

on observing the (1,2) and (2,1) terms, choose

V.4a) $$G^2 = \frac{1}{PQ}$$

which has

V.4b) $$2GG' = \nabla\left(\frac{1}{PQ}\right) \implies G' = \frac{1}{2G}\nabla(1/PQ)$$

Thus,

V.4c) $$\frac{R}{2}G' = \frac{1}{4G^2}\nabla\left(\frac{1}{PQ}\right) = \frac{PQ}{4}\nabla\left(\frac{1}{PQ}\right) = -\frac{1}{4PQ}\nabla(PQ)$$

and V.3e) gives

$$\text{V.5)}\qquad \nabla\begin{bmatrix} p^{l} \\ p^{h} \end{bmatrix} = \begin{bmatrix} -\frac{RG'}{2} - \frac{R}{Q} & \frac{RG'}{2} \\ \frac{RG'}{2} & -\frac{RG'}{2} + \frac{R}{Q} \end{bmatrix}\begin{bmatrix} p^{l} \\ p^{h} \end{bmatrix}$$

for which we define

$$\text{V.6a)}\qquad \rho(x,s) = -\frac{RG'}{2} = \frac{1}{4PQ}\nabla(PQ)$$

$$\text{V.6b)}\qquad \gamma(x,s) = \frac{R}{Q} = RG^2P = GP = \sqrt{\frac{P}{Q}}$$

This allows V.5) to be written as

$$\text{V.7)}\qquad \nabla\begin{bmatrix} p^{l} \\ p^{h} \end{bmatrix} = \begin{bmatrix} \rho - \gamma & -\rho \\ -\rho & \rho + \gamma \end{bmatrix}\begin{bmatrix} p^{l} \\ p^{h} \end{bmatrix}$$

In V.7) γ is like a propagation function, yielding delay through the section of length dx.

VI. Space Discretization - Linear Case

We consider a section of length Δx where for a total of N sections we have

VI.1) $$\Delta x = \frac{L}{N}, \quad x_k = \frac{kL}{N} = x_{k-1} + \Delta x$$

For the kth section we evaluate γ and ρ at $x = x_k$. We break the section into two portions, one in terms of γ with $\rho = 0$ and the other in cascade in terms of ρ with $\gamma = 0$.

$\rho = 0$:

VI.2) $$\nabla \begin{bmatrix} \wp^i \\ \wp^r \end{bmatrix} = \begin{bmatrix} -\gamma & 0 \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} \wp^i \\ \wp^r \end{bmatrix}$$

or

VI.3) $$\begin{bmatrix} \wp^i(t, x_k + \Delta x) \\ \wp^r(t, x_k + \Delta x) \end{bmatrix} = \begin{bmatrix} e^{-\gamma \Delta x} & \\ 0 & e^{\gamma \Delta x} \end{bmatrix} \begin{bmatrix} \wp^i(t, x_k) \\ \wp^r(t, x_k) \end{bmatrix}$$

$\gamma = 0$

VI.4) $$\nabla \begin{bmatrix} \wp^i \\ \wp^r \end{bmatrix} = \begin{bmatrix} \rho & -\rho \\ -\rho & \rho \end{bmatrix} \begin{bmatrix} \wp^i \\ \wp^r \end{bmatrix}$$

which we write as

VI.5a) $$\begin{bmatrix} \wp^i(t, x_k + \Delta x) \\ \wp^r(t, x_k + \Delta x) \end{bmatrix} - \begin{bmatrix} \wp^i(t, x_k) \\ \wp^r(t, x_k) \end{bmatrix} = \begin{bmatrix} \rho_k \Delta x \, \wp^i(t, x_k + \Delta x) & -\rho \Delta x \, \wp^r(t, x_k) \\ -\rho_k \Delta x \, \wp^i(t, x_k) & \rho \Delta x \, \wp^r(t, x_k + \Delta x) \end{bmatrix}$$

VI.5b) or $$\begin{bmatrix} 1 - \rho \Delta x & 0 \\ 0 & 1 - \rho \Delta x \end{bmatrix} \begin{bmatrix} \wp^i \\ \wp^r \end{bmatrix}_{x_{k+1}} = \begin{bmatrix} 1 & -\rho \Delta x \\ -\rho \Delta x & 1 \end{bmatrix} \begin{bmatrix} \wp^i \\ \wp^r \end{bmatrix}_{x_k}$$

or

VI.6) $$\begin{bmatrix} \wp^i \\ \wp^r \end{bmatrix}_{x_{k+1}} = \frac{1}{1 - \rho \Delta x} \begin{bmatrix} 1 & -\rho \Delta x \\ -\rho \Delta x & 1 \end{bmatrix} \begin{bmatrix} \wp^i \\ \wp^r \end{bmatrix}_{x_k}$$

Putting these two sections in cascade yields

VI.7a) $$\begin{bmatrix} p^i \\ p^r \end{bmatrix}_{k+1} = \frac{1}{\tau_k}\begin{bmatrix} 1 & -\rho_k \\ -\rho_k & 1 \end{bmatrix}\begin{bmatrix} e^{-\gamma_k} & 0 \\ 0 & e^{\gamma_k} \end{bmatrix}\begin{bmatrix} p^i \\ p^r \end{bmatrix}_k$$

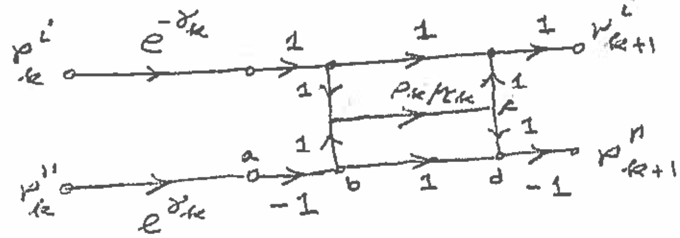
where

VI.7b) $$\gamma_k = \gamma(x_k, a)\,\Delta x$$

VI.7c) $$\rho_k = \rho(x_k, a)\,\Delta x$$

VI.7d) $$\tau_k = 1 - \rho_k$$

schematically this kth section in signal-flow graph form is



However we are interested in the reflected signal p^r_k as an output and p^r_{k+1} as an input, so express these equations with input vector $\begin{bmatrix} p^i_k \\ p^r_{k+1} \end{bmatrix}$ and output vector $\begin{bmatrix} p^i_{k+1} \\ p^r_k \end{bmatrix}$. For this

VI.7a) is

VI.8a) $$p^i_{k+1} = \frac{1}{\tau_k} e^{-\gamma_k} p^i_k - \frac{\rho_k}{\tau_k} e^{\gamma_k} p^r_k$$

VI.8b) $$p^r_{k+1} = -\frac{\rho_k}{\tau_k} e^{-\gamma_k} p^i_k + \frac{1}{\tau_k} e^{\gamma_k} p^r_k$$

The latter is

VI.8c) $$p^r_k = \rho_k e^{-2\gamma_k} p^i_k + \tau_k e^{-\gamma_k} p^r_{k+1}$$

which in the former gives

VI.8d) $$p^i_{k+1} = \frac{1}{\tau_k}(1-\rho_k^2) e^{-\gamma_k} p^i_k - \frac{\rho_k}{\tau_k^2} p^r_{k+1}$$

Thus

VI. 8e) $$\begin{bmatrix} p^i_{k+1} \\ p^r_k \end{bmatrix} = \begin{bmatrix} \frac{1}{\tau_k}(1-\rho_k^2)e^{-\gamma_k} & -\frac{\rho_k}{\tau_k^2} \\ \rho_k e^{-2\gamma_k} & +\tau_k e^{-\gamma_k} \end{bmatrix} \begin{bmatrix} p^i_k \\ p^r_{k+1} \end{bmatrix}$$

By inverting signals in the last figure according to the labelled signals as

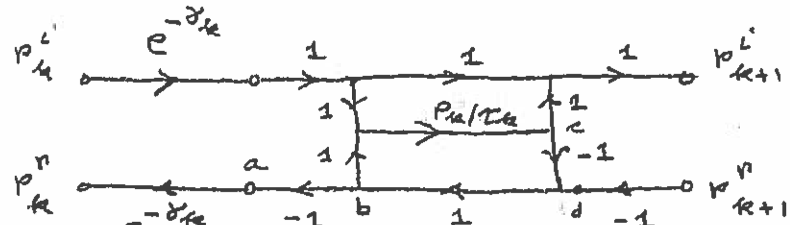
VI. 9a,b) $a = e^{\gamma_k} p^r_k$ or $p^r_k = e^{-\gamma_k} a$

VI. 9c,d) $b = -a$ or $a = -b$

VI. 9e,f) $d = b + c$ or $b = d - c$

VI. 9g,h) $p^r_{k+1} = -d$ or $d = -p^r_{k+1}$

we get the following useful signal-flow graph



For this we note at the kth junction of sections $R(x_k, a)$ is the reference, this being different for the two ends of the same section.

Terminations

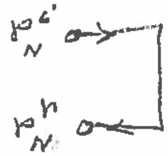
The desired representation is a cascade of N sections of the kind just exhibited along with terminations. At the helicotrema end

VII.1a) $v_v = 0 \quad \equiv u_v = 0$

which from the transformation to scattering variables of V.1b) is

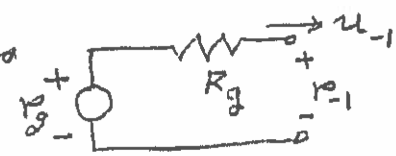
VII.1b) $p_N^i = p_N^r$

which in a signal-flow graph is a direct connection:

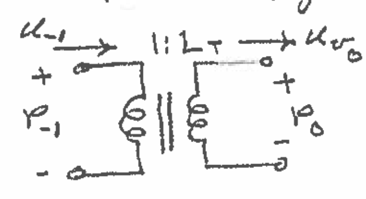


In a circuit diagram, since v_v is analogous to a current, this is an open circuit.

At the source end, we excite the outer ear with a speaker-like transducer which we take as a pressure source; in a circuit diagram this is



The coupling to the round window we represent by an ideal transformer in a circuit diagram



where

VII.2a) $p_0 = L_T p_{-1}$

VII.2b) $u_{-1} = L_T u_{v_0}$

and $L_T > 1$ since the middle ear acts as a lever arm to increase the force (and decrease velocity).

Here R_g is the impedance of the outer ear, which in the literature taken to be real. We take

VII.3) $$R_{-1} = R_g$$

in which case, from V.1b)

VII.4a,b) $$p_{-1}^{i} = \tfrac{1}{2}(p_{-1} + R_{-1}u_{-1}) = \tfrac{1}{2}([p_g - R_g u_{-1}] + R_{-1}u_{-1})$$

VII.4c) $$= \tfrac{1}{2}p_g$$

That is the incident pressure at stage zero is $\frac{1}{2}$ the input transducer pressure while the reflected pressure is

VII.5a) $$p_{-1}^{r} = \tfrac{1}{2}(p_{-1} - R_{-1}u_{-1})$$

VII.5b) $$= \tfrac{1}{2}p_g - R_g u_{-1}$$

In a signal-flow graph representation,

$$p_g \xrightarrow{1/2} p_{-1}^{i}$$

Note that if the load (cochlea) as reflected into the primary side of the transformer were to equal R_g, then p_{-1} would equal $R_{-1}u_{-1}$ and the output p_{-1}^{r} would be 0. However, such matching does not occur, giving a reflected pressure to be measured.

The scattering description of the transformer is needed: see V.1a) with VII.2)

VII.6a,b,c) $$p_0^{i} = \tfrac{1}{2}(p_0 + R(0,\alpha)u_{v_0}) = \tfrac{1}{2}(L_T p_{-1} + R_0 u_{-1}/L_T)\ ;\ R_0 = R(0,\alpha)$$

VII.6d) $$= \tfrac{1}{2}\left(L_T[p_{-1}^{i} + p_{-1}^{r}] + \frac{R_0}{L_T}[G_{-1}p_{-1}^{i} - G_{-1}p_{-1}^{r}]\right)$$

VII.6e) $$= \tfrac{1}{2}\left[L_T + \frac{R_0 G_{-1}}{L_T}\right]p_{-1}^{i} + \tfrac{1}{2}\left[L_T - \frac{R_0 G_{-1}}{L_T}\right]p_{-1}^{r}$$

and

VII.7a,b) $p_0^r = \frac{1}{2}\left(p_0 - R_0 u_{v_0}\right) = \frac{1}{2}\left(L_T p_{-1} - \frac{R_0}{L_T} u_{-1}\right)$

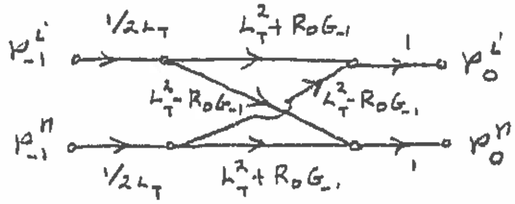
VII.7c) $= \frac{1}{2}\left(L_T[p_{-1}^i + p_{-1}^r] - \frac{R_0}{L_T}[G_{-1}p_{-1}^i - G_{-1}p_{-1}^r]\right)$

VII.7d) $= \frac{1}{2}\left[L_T - \frac{R_0 G_{-1}}{L_T}\right]p_{-1}^i + \frac{1}{2}\left[L_T + \frac{R_0 G_{-1}}{L_T}\right]p_{-1}^r$

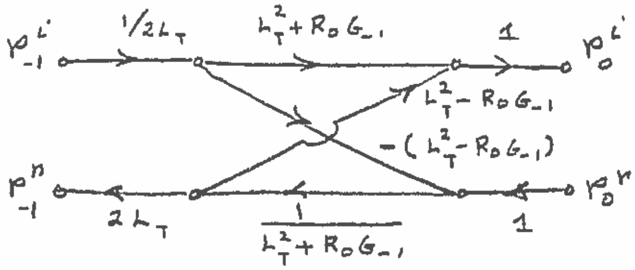
or

VII.8) $$\begin{bmatrix} p_0^i \\ p_0^r \end{bmatrix} = \frac{1}{2L}\begin{bmatrix} L_T^2 + R_0G_{-1} & L_T^2 - R_0G_{-1} \\ L_T^2 - R_0G_{-1} & L_T^2 + R_0G_{-1} \end{bmatrix}\begin{bmatrix} p_{-1}^i \\ p_{-1}^r \end{bmatrix}$$

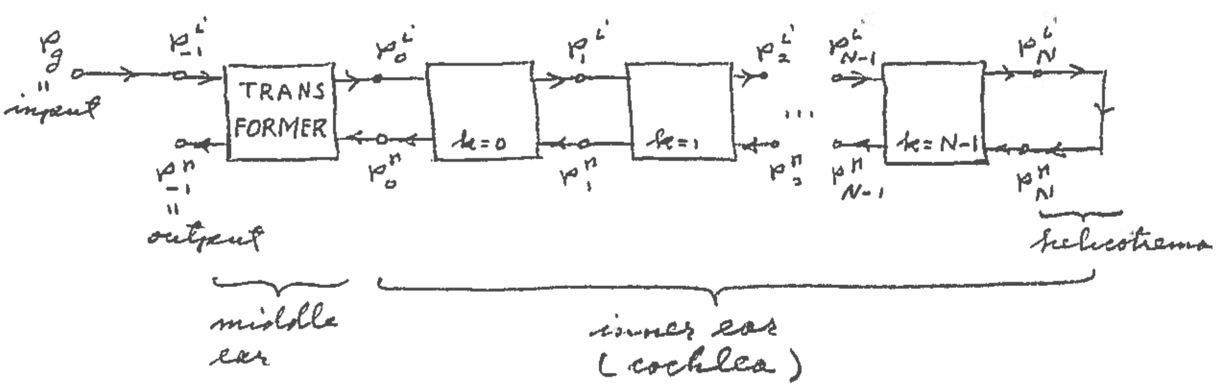
which in signal-flow graph form is



on taking p_0^r as an input and p_{-1}^r as an output this is



The full system cascades the various sections



VIII. Circuit Realization – Linear Case

For a circuit realization we return to IV.7) to get the internal sections.

VIII.1a) $$\frac{p(x+\Delta x)-p(x)}{\Delta x} = -P(x,s)\,u_v$$

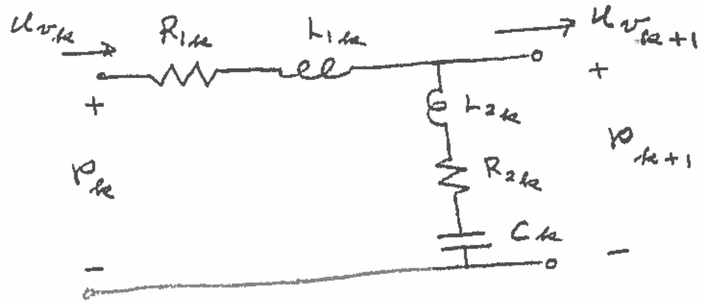
VIII.1b) $$\frac{u_v(x+\Delta x)-u_v(x)}{\Delta x} = -\frac{1}{Q(x,s)}\,p$$

or

VIII.2a) $$p(x_{k+1}) = p(x_k) - \frac{2\Delta x}{S_v(x_k)}\left[R_v+\rho s\right]u_v(x_k)$$

VIII.2b) $$u_v(x_{k+1}) = u_v(x_k) - \frac{D(x_{k+1})\Delta x}{\mu(x_{k+1})s+\sigma(x_{k+1})+\frac{\phi(x_{k+1})}{s}}\cdot p(x_{k+1})$$

These latter come from the following circuit

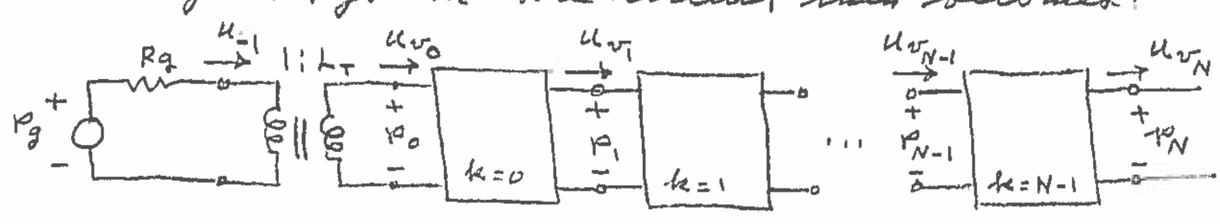


where

VIII.3a,b) $$R_{1k} = \frac{2R_v}{S_v(x_k)}\cdot\Delta x\,, \qquad L_{1k} = \frac{2\rho}{S_v(x_k)}\Delta x$$

VIII.3c,d,e) $$L_{2k} = \frac{\mu(x_{k+1})}{D(x_{k+1})\Delta x}\,, \quad R_{2k} = \frac{\sigma(x_{k+1})}{D(x_{k+1})\Delta x}\,, \quad C_k = \frac{D(x_{k+1})\Delta x}{\phi(x_{k+1})}$$

The full system as a circuit then becomes:



Here the input is p_g and the output u_{-1}.

IX Improvement in Nonlinearities

In the previous $S_v(x,t) \approx S_v(x)$ was assumed at II.5). Here the corrections are incorporated when the time variation of $S_v(t,x)$ is included. From p. 4

IX-1) $$S_v(t,x) = S_v(x) - \frac{1}{2} D(x)\, \xi_m(t,x)$$

Thus

IX.2a) $$\frac{\partial (S_v v_v)}{\partial t} = S_v \frac{\partial v_v}{\partial t} + v_v \frac{\partial S_v}{\partial t} = S_v \frac{\partial v}{\partial t} - v_v \left(\frac{D}{2}\dot{\xi}_m\right)$$

or

IX.2b) $$\frac{\partial v_v}{\partial t} = \frac{1}{S_v}\frac{\partial u_v}{\partial t} + \frac{v_v}{S_v}\cdot \frac{1}{2} D \dot{\xi}_m$$

and II.4), $\nabla p = -2R_v v_v - 2\rho \frac{\partial v_v}{\partial t} = -\frac{2R_v}{S_v} u_v - 2\rho \left[\frac{1}{S_v} s u_v + \frac{u_v}{S_v^2}\cdot\left(\frac{D}{2}\dot{\xi}_m\right)\right]$

or

IX.3) = II.5') $$\nabla p = -\frac{2}{S_v}\left[R_v u_v + \rho \frac{u_v}{S_v}\cdot \frac{1}{2} D \dot{\xi}_m + \rho \frac{\partial u_v}{\partial t}\right] \qquad /\rho$$

$$= -\frac{2}{S_v}\left[R_v + \rho s\right] u_v - \frac{2\rho}{S_v^2} u_v s \left(\frac{1}{2} D \xi_m\right) \qquad /\rho$$

Using IV.4), $D\xi_m/2 = -\frac{1}{sQ} p$, IV.6a) is

IX.4) = IV.6a') $$\nabla p = -P(x,s)\, u_v + \frac{2\rho}{S_v^2} u_v \cdot \frac{1}{Q} p \qquad /\rho$$

The final nonlinear equations would then be

IX.5a) $$\nabla p = -P u_v + \frac{2\rho}{S_v^2} u_v \cdot \frac{1}{Q}\cdot p \qquad /\rho$$

IX.5b) $$\nabla u_v = -\left[1 + \frac{8}{D^4}\left(\frac{1}{sQ}\cdot p\right)^2\right]\cdot \frac{1}{Q}\cdot p$$

X. Circuit Realization – Nonlinear Case

From IV.3c) we have

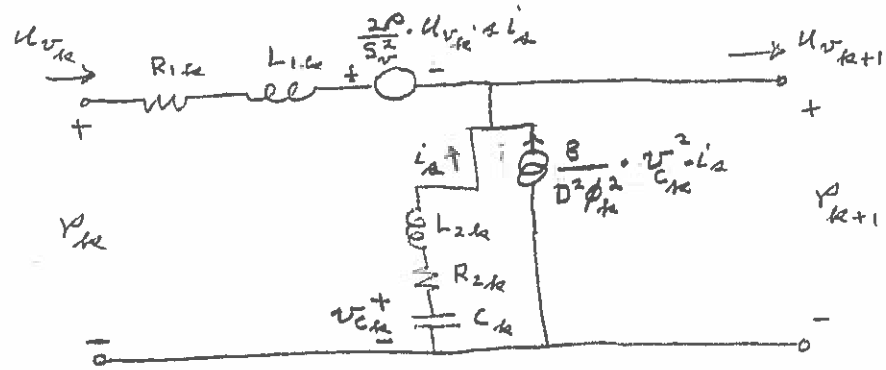
X.1) $$\frac{D\xi_m}{2}\Delta x = \frac{-D\Delta x}{s(\mu s + \sigma + \phi/s)}\cdot\varphi = \frac{1}{s} i_s = -\frac{\Delta x}{\Delta Q}\cdot\varphi \quad (= C v_c)$$

which is seen to be the integral of the current i_s going up in the shunt arm branch, and by IV.3a)

X.2a) $$u_v(x_{k+1}) = u_v(x_k) + \left[1 + \frac{8}{D^4\Delta x^2}\left(\frac{D\xi_m}{2}\Delta x\right)^2\right]\cdot s\left(\frac{D\xi_m}{2}\Delta x\right)$$

while IV.6a') is written as

X.2b) $$\varphi(x_{k+1}) = \varphi(x_k) - P(x_k, s)\Delta x\, u_v(x_k) - \frac{2\rho}{S_v^2}u_v(x_k)\cdot s\left(\frac{D\xi_m}{2}\Delta x\right)$$ $/\rho$



$/\rho$

$/v_{c_k}^2$

$/v_{c_k}$

Here, since $C_k = D\Delta x/\phi$ by VIII.3a),

X.3a,b,c) $$\frac{8}{D^4\Delta x^2}\left(\frac{D}{2}\xi_m\Delta x\right)^2 = \frac{8}{D^4\Delta x^2}\left(\frac{1}{s}i_s\right)^2 = \frac{8}{D^4\Delta x^2}\left(C_k v_{c_k}\right)^2 = \frac{8}{D^2\phi^2}v_{c_k}^2$$

The terminations are as in the linear case so the figure before this remains valid.

XI Table of Values

To implement the previous circuits we need parameter values for a typical ear, as is collected here.

$L = 35$ mm /mm

$D(x) = ax + b$; $a = 0.086$, $b = 0.2$ mm /

$\rho = 1$ gm/cm^3

$R_v = 0.056$ gm/cm-sec$^{1/2}$

$S_v(x) = c e^{-dx}$; $c = 0.025$cm, $d = 0.5$/cm

$R_g = 193$ dyne-sec/cm^3

$\mu(x) = 0.1$ gm/cm^2

$\sigma(x) = f e^{-gx}$; $f = 600$ dyne-sec/cm^3, $g = 1.7$/cm

$\phi(x) = h e^{-kx}$; $h = 2\times10^9$ dyne/cm^3, $k = 3.4$/cm

$L_T = 1.3$

Note that $S_v(L) = 0.00869$ cm^2, $D(L) = 0.5$mm ; since $S_v(L) = \frac{1}{2}(\pi r_L^2) \Rightarrow r_L = 0.744$mm and $2r_L = 1.49 > 0.5 = D(L)$. //

XII. Justification of $v_t = -v_v$

In actual fact this result should be

XII. 1) $$u_t = -u_v$$

where

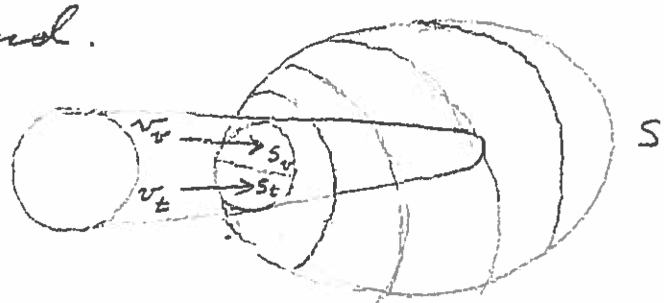
XII. 2) $$u_t = S_t v_t \; ; \quad u_v = S_v v_v$$

or

XII. 3) $$v_t = -\frac{S_v}{S_t} v_v$$

But if $S_t = S_v$ then $v_t = -v_v$ as we have used.

To see that $u_t = -u_v$ pass a closed surface S that cuts the cochlea by a transverse plane at x and surrounds the helicotrema end.



Since the velocity over S_v is to be taken uniform it is in one direction as is that over S_t. By conservation of mass

XII. 4 $$\iint_S \rho \vec{v} \cdot dS \equiv 0 = \rho(v_v S_v + v_t S_t)$$

which is XII. 1

If the two areas S_v and S_t are not equal (which strictly they will not be under perturbation) then II, 3b) becomes

XII, 5a) $$\frac{\partial v_t}{\partial t} + v_t \nabla v_t = -\frac{\partial\left(\frac{S_v}{S_t} v_v\right)}{\partial t} + \left(\frac{S_v}{S_t} v_v\right)\nabla\left(\frac{S_v}{S_t} v_v\right)$$

XII, 5b) $$= -\frac{1}{\rho}\left(\nabla p_t - R_v \frac{S_v}{S_t} v_v\right)$$

or

XII, 5c) $$\nabla p_t = R_v \frac{S_v}{S_t} v_v + \rho \frac{\partial\left(\frac{S_v}{S_t} v_v\right)}{\partial t} - \rho\left(\frac{S_v}{S_t} v_v\right)\nabla\left(\frac{S_v}{S_t} v_v\right)$$

while, from II, 3a)

II, 3a) $$\nabla p_v = -R_v v_v - \rho \frac{\partial v_v}{\partial t} - \rho v_v \nabla v_v$$

Then, as $p = p_v - p_t$,

XII, 6a) $$\nabla p = -R_v\left(1 + \frac{S_v}{S_t}\right) v_v - \rho \frac{\partial\left[\left(1+\frac{S_v}{S_t}\right) v_v\right]}{\partial t} - \rho\left\{v_v \nabla v_v - \frac{S_v}{S_t} v_v \nabla\left(\frac{S_v}{S_t} v_v\right)\right\}$$

XII, 6b) $$= -R_v\left(\frac{1}{S_v} + \frac{1}{S_t}\right) u_v - \rho \frac{\partial\left[\left(\frac{1}{S_v}+\frac{1}{S_t}\right) u_v\right]}{\partial t} - \rho\left\{\frac{u_v}{S_v}\nabla\left(\frac{u_v}{S_v}\right) - \frac{u_v}{S_t}\nabla\left(\frac{u_v}{S_t}\right)\right\}$$

For which we will define the parallel area as

XII, 7) $$\frac{1}{S_p} = \frac{1}{S_v} + \frac{1}{S_t} = \frac{S_v + S_t}{S_v S_t} = \frac{2 S_v(x)}{S_v(t,x) S_t(t,x)} \quad \left(\approx \frac{2}{S_v} \text{ if } S_v = S_t\right)$$

and note that

XII, 8a,b) $$\nabla\left(\frac{u_v}{S_v}\right) = \frac{1}{S_v}\nabla u_v - \frac{u_v}{S_v^2}\nabla S_v \;;\; \nabla\left(\frac{u_v}{S_t}\right) = \frac{1}{S_t}\nabla u_v - \frac{u_v}{S_t^2}\nabla S_t$$

making XII, 6b)

XII, 9) $$\nabla p = -\frac{R_v}{S_p} u_v - \frac{\rho}{S_p}\frac{\partial u_v}{\partial t} - \rho\left\{\left(\frac{1}{S_v^2} - \frac{1}{S_t^2}\right) u_v \nabla u_v\right\}$$
$$- \rho u_v \frac{\partial(1/S_p)}{\partial t} - \rho\left\{\frac{u_v^2}{S_t^3}\nabla S_t - \frac{u_v^2}{S_v^3}\nabla S_v\right\}$$

Now, from XII.7),

XII.10a) $\dfrac{\partial(1/S_M)}{\partial t} = 2\,S_v(x)\cdot\dfrac{-1}{S_v^2(t,x)\,S_t^2(t,x)}\left\{S_t\dfrac{\partial S_v}{\partial t} + S_v\dfrac{\partial S_t}{\partial t}\right\}$

XII.10b,c) $= \dfrac{-2S_v(x)}{S_v\cdot S_t}\left\{\dfrac{1}{S_v}\cdot\left(-\dfrac{D}{2}\dot{\xi}_m\right) + \dfrac{1}{S_t}\left(+\dfrac{D}{2}\dot{\xi}_m\right)\right\} = \dfrac{-2S_v(x)}{S_vS_t}\left(\dfrac{1}{S_t}-\dfrac{1}{S_v}\right)\left(\dfrac{D}{2}\dot{\xi}_m\right)$

XII.10d) $= \dfrac{4\,S_v(x)}{(S_vS_t)^2}\left(\dfrac{D}{2}\xi_m\right)\left(\dfrac{D}{2}\dot{\xi}_m\right)$

since

XII.11a,b) $S_v(t,x) = S_v(x) - \dfrac{D(x)}{2}\xi_m(t,x)$; $S_t(t,x) = S_t(x) + \dfrac{D(x)}{2}\xi_m(t,x)$

Further

XII.12a,b) $\nabla S_v = S_v'(x) - \dfrac{D'}{2}\xi_m - \dfrac{D}{2}\nabla\xi_m$; $\nabla S_t = S_t'(x) + \dfrac{D'}{2}\xi_m + \dfrac{D}{2}\nabla\xi_m$

giving

XII.13a,b) $S_v^3\nabla S_t - S_t^3\nabla S_v = \left[S_v^3S_t'(x) - S_t^3S_v'(x)\right] + (S_v^3+S_t^3)\nabla\left(\dfrac{D}{2}\xi_m\right)$

and

XII.14a,b) $\dfrac{1}{S_v^2} - \dfrac{1}{S_t^2} = \dfrac{S_t - S_v}{S_vS_t}\cdot\dfrac{S_t+S_v}{S_vS_t} = \dfrac{S_t(x) - S_v(x) + D\,\xi_m}{S_vS_t}\cdot\dfrac{S_v(x)+S_t(x)}{S_vS_t}$

Equations XII.10) – XII.14) allow the evaluation of the nonlinear terms in XII.9). When $S_v(x) = S_t(x)$ we get for XII.9)

XII.15) $\nabla p = -\dfrac{2R_v}{S_v}u_v - \dfrac{2\rho}{S_v}\dfrac{\partial u_v}{\partial t} - \underbrace{\dfrac{2\rho}{S_v^3}\left[\nabla\left(\left(\dfrac{D}{2}\xi_m\right)u_v^2\right) + u_v\dfrac{\partial\left(\frac{D}{2}\xi_m\right)^2}{\partial t}\right]}_{\text{nonlinear terms}}$

In this the nonlinear terms appear to be quite small. In any event the complete set of describing equations is, from IV.3a,c) & XII.15)

XII.16a) $\nabla p = -\dfrac{2\rho}{S_v}\left[\dfrac{R_v}{\rho} + \alpha\right]u_v - \dfrac{2\rho}{S_v^3}\left[\nabla\left(\left(\dfrac{D}{2}\xi_m\right)u_v^2\right) + u_v\dfrac{\partial\left(\frac{D}{2}\xi_m\right)^2}{\partial t}\right]$

XII.16b) $\nabla u_v = \left[1 + \dfrac{8}{D^4}\left(\dfrac{D}{2}\xi_m\right)^2\right]\left(\dfrac{D}{2}\dot{\xi}_m\right)$

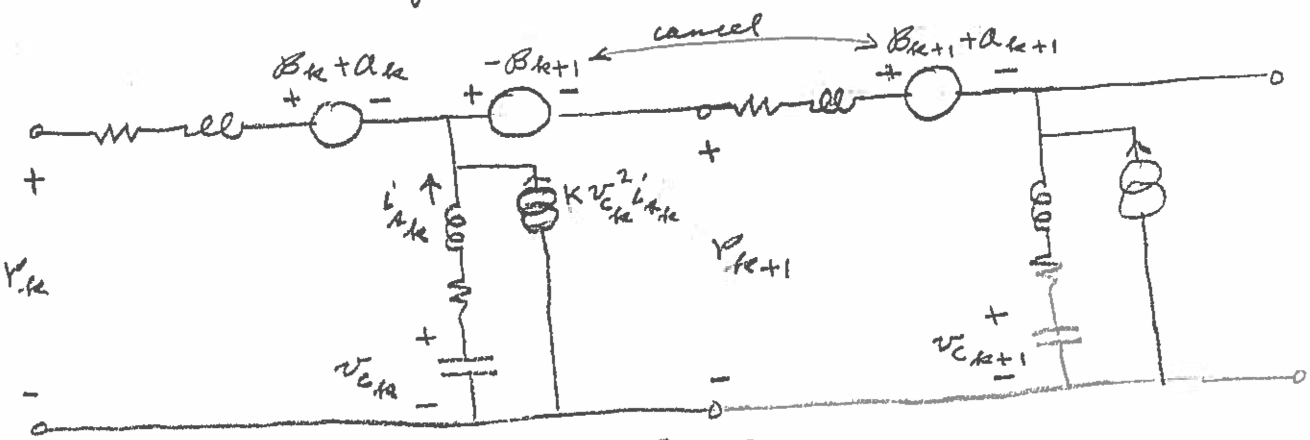
XII.16c) $-p = \dfrac{\left[\mu\alpha^2 + \sigma\alpha + \phi\right]}{D}\left(\dfrac{D}{2}\xi_m\right)$

Note that assuming $S_v = S_t$ before taking the derivatives gave $-\frac{2\rho u_v}{S_v^2}\cdot(\frac{D}{2}\dot{\xi}_m)$, see IX, 3); while the more accurate calculation for XII, 16a), gave for the same term $-\frac{2\rho u_v}{S_v^2}(\frac{D}{2}\dot{\xi}_m)\frac{(\frac{D}{2}\xi_m)}{S_v}$ which has the factor $(\frac{D}{2}\xi_m)/S_v$ different. This seems to be a significant difference.

The structure of a circuit for equations XII.16) is as on page 19 except that the nonlinear series arm voltage source depends on v_{c_k} from the same section and $v_{c_{k+1}}$ of the next section since

XII.17) $$p(x_{k+1}) = p(x_k) - \frac{2\Delta x}{S_v}(R_v+\rho a)u_v - \frac{2\rho}{S_v^3}\Big[\underbrace{2u_{v_k}\Delta x\, i_{a_k} C_k v_{c_k}}_{=a_k}\Big] - \frac{2\rho}{S_v^3}\Big[\underbrace{u^2_{v_{k+1}}C_{k+1}v_{k+1}}_{=\beta_{k+1}} - \underbrace{u^2_{v_k}C_k v_k}_{=\beta_k}\Big]$$

which is obtained from XII.16a) with $i_a = a(\frac{D}{2}\dot{\xi}_m) = -\frac{1}{aQ}p = 4v_c$

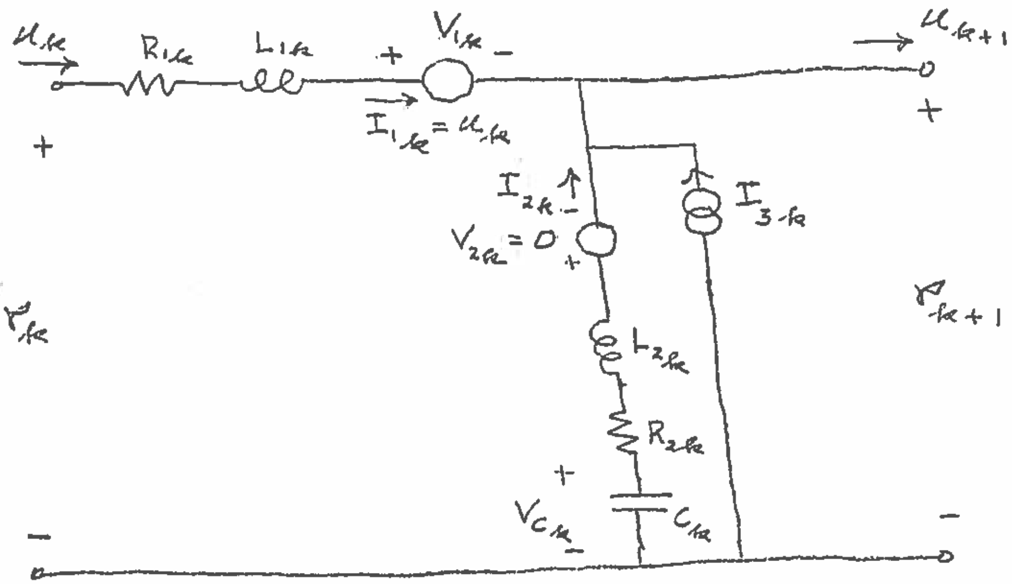


XII.18) $$K = \frac{8}{D^4\Delta x^2}\cdot C_k^2 = \frac{8}{D^2\phi^2}$$

Putting $u^2_{v_{k+1}}C_{k+1}v_{k+1}$ on the right of the kth section and $u^2_{v_k}C_k v_k$ on the left we see the former can be cancelled with the latter of the (k+1)st section. Thus the $\nabla((\frac{D}{2}\xi_m)u_v^2)$ term can be ignored and the circuit of p.19 holds with the nonlinear voltage source being

XII.19) $$a_k = \frac{4\rho u_{v_k}}{S_v^2}\Delta x\, i_{a_k} C_k v_k$$

Therefore a useful representation of the kth section for use on Spice, etc., is



where

XII.20a) $$V_{1k} = \frac{4\rho \Delta x C_k}{S_v^3(x_k)} \cdot I_{1k} \cdot I_{2k} \cdot V_{C_k}$$

XII.20b) $$I_{3k} = \frac{8 C_k}{D^4(x_k)\Delta x^2} \cdot I_{2k} \cdot V_{C_k}^2$$

and

VIII.3a,b) $$R_{1k} = \frac{2 R_v}{S_v(x_k)} \Delta x \;, \qquad L_{1k} = \frac{2\rho}{S_v(x_k)} \Delta x$$

VIII.3c,d,e) $$L_{2k} = \frac{\mu(x_{k+1})}{D(x_{k+1})\Delta x} \;, \quad R_{2k} = \frac{\sigma(x_{k+1})}{D(x_{k+1})\Delta x} \;, \quad C_k = \frac{D(x_{k+1})\Delta x}{\phi(x_{k+1})}$$

XII.20c) $\Delta x = L/N$, $N = \#$ of sections , $x_k = k\Delta x$, $k = 0, \cdots, N-1$

with parameters as given on page 20.

Note that for the linearized representation the sources present in the above circuit are absent.

XIII. Time Discretization – Linear Case

For time discretization we use backward differencing

XIII.1) $$s f = \frac{\partial f}{\partial t} \approx \frac{f(t) - f(t-\Delta t)}{\Delta t} = \frac{\left(1 - \frac{1}{z}\right)}{\Delta t} \cdot f(t)$$

where the derivative operator is replaced via the delay operator $1/z$. Thus, we use the replacement

XIII.2) $$s = \left(1 - \frac{1}{z}\right)/\Delta t$$

where Δt is the sampling interval.

For the section of p. 13, we need

XIII.3a) $$\rho_k(z) = \rho(x_k, s)\Delta x = \frac{1}{4PQ}\frac{\partial PQ}{\partial x}$$

XIII.3b) $$\gamma_k(z) = \gamma(x_k, s)\Delta x = \sqrt{\frac{P}{Q}}$$

where, IV.5),

IV.5) $$P(x,s) = \frac{2}{S_v(x)}\left[R_v + \rho s\right], \quad Q(x,s) = \frac{\mu(x) s^2 + \sigma(x) s + \phi(x)}{D(x)\, s}$$

We see that ρ_k is of degree 3, in either s or z, and γ_k is irrational of degree $(2)^{1/2}$ and occurs as an exponent. Thus, we desire to do a rational approximation of $e^{-\gamma_k(z)}$.