Notes on

n-Channel Lattice Structures
Choice of Coefficients
Following Andersen

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Abstract - Comments:

These notes are to justify the extension to n-channel lattices of the Andersen method for scalars, this extension being given in Section V. The notes give all the pertinent background details but do not give a good textual discussion; they are the development giving the basis for a planned text.

The end motivation is to obtain multiport circuits that can synthesize simulations of n-channel processes, possibly in real-time. These notes develop the necessary mathematical formulation.

I. Motivation – Conceptual Framework

The desire is to take experimental data from an n-channel system, such as several speakers or two ears on a human, and create a digital filter model for prediction error which on inversion yields synthesis of the system.

Thus, with $1/z$ = delay, $x(t)$ = input (at sample [integer] time t), and $f_{m,t}$ = forward prediction error of mth order predictor such that (1_n = $n \times n$ identity)

$$f_{m,t} = x(t) - \sum_{j=1}^{m} a_{m,j} \cdot x(t-j)$$

$$= \left[1_n - \sum_{j=1}^{m} a_{m,j} \cdot z^{-j} \right] x(t)$$

we wish to find the $a_{m,j}$ and a structure in lattice form to yield

$$x(t) = \left[1_n - \sum_{j=1}^{m} a_{m,j} \, z^{-j} \right]^{-1} f_{m,t}$$

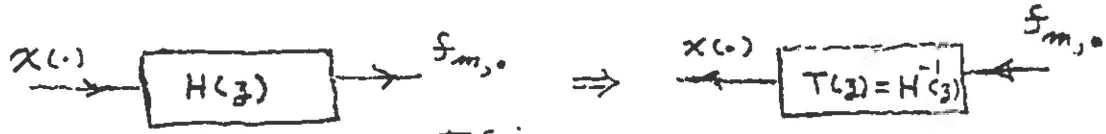

Figure 1

To be most useful we would want $f_{m,t}$ to be like a unit pulse and we wish to find an optimum size $m = M$. The latter does not seem to have been really considered for $n > 1$ in the literature.

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For the former we follow the classical probabilistic theory to set up the working framework in which we assume available full data to construct covariance matrices, etc.. after the framework is set we can then specialize to the situation where only limited data is available (that is, only a few sample signals over limited times).

Here an Andersen type of technique is developed.

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II. Theoretical Background Developments

We assume n vectors of data $x(t)$, i.e. n "channels" being sampled. In normalized (integer) time, t, form this as

$$\ldots\ x(-k), \ldots, x(-1), x(0), x(1), \ldots, x(k), \ldots$$

where each is an n-vector, i.e. $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}$

At the mth stage of our consideration we will work with $m+1$ of these n-vectors. Pictorially

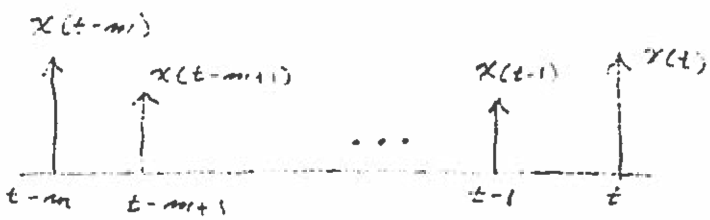

Figure 2

Thus, at the mth stage of consideration we will primarily be working at time t with the $(m+1)$ n-vectors

$$x(t-m),\ x(t-m+1),\ \ldots,\ x(t-1),\ x(t)$$

We will assume all data is real, but if complex quantities are needed the results below will hold by replacing the transpose denoted by superscript T, by the transpose complex conjugate.

We are interested in forward and backward prediction, and, thus, in the differences between the true values at time t (or $t-m$) and those values predicted in terms of m previous (or future) values, the prediction being via a digital network of coefficient $(n \times n)$ matrices $a_{m,1}, a_{m,2} \cdots a_{m,m}$ (or $\underline{a}_{m,1}, \cdots, \underline{a}_{m,m}$). That is we want

$$x(t) \approx \sum_{j=1}^{m} a_{m,j}\, x(t-j), \tag{10a}$$

$$x(t-m) \approx \sum_{j=1}^{m} \underline{a}_{m,j}\, x([t-m]+j) \tag{10b}$$

Pictorially

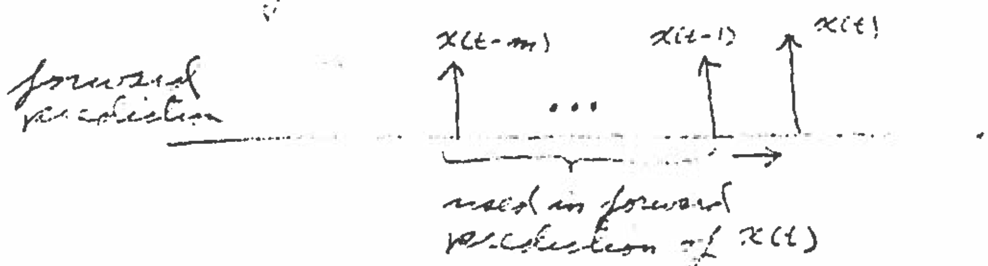

Figure 3 a)

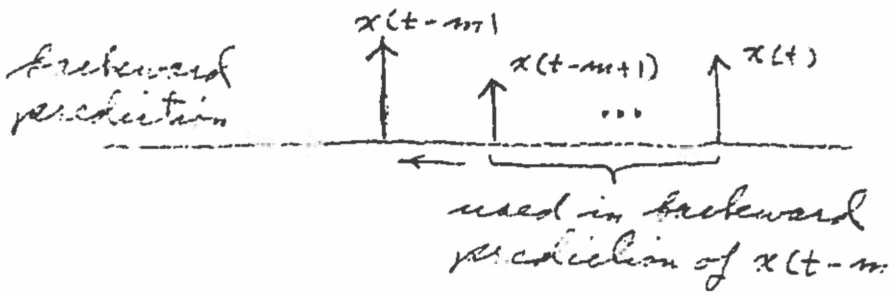

Figure 3 b)

We are interested in the error in the predicted values and true values in both the forward and backward predictions and thus define

$$f_{m,t} = x(t) - \sum_{j=1}^{m} a_{m,j}\, x(t-j) \qquad (10c)$$

$$b_{m,t} = x(t-m) - \sum_{j=1}^{m} \underline{a}_{m,j}\, x([t-m]+j) \qquad (10d)$$

which more uniformally can be written, by defining

$$a_{m,0} = \underline{a}_{m,0} = -1_n \qquad (10e)$$

[where 1_n is the $n \times n$ identity], as

$$f_{m,t} = -\sum_{j=0}^{m} a_{m,j}\, x(t-j) \qquad (10f)$$

$$b_{m,t} = -\sum_{j=0}^{m} \underline{a}_{m,j}\, x([t-m]+j) \qquad (10g)$$

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We will assume that the coefficient matrices are chosen such that the prediction errors are uncorrelated with the other signal values, thus

$$E[f_{m,t}\ x^T(t-j)] = 0_n, \quad j = 1, \cdots, m \tag{20a}$$

$$E[b_{m,t}\ x^T(t-m+j)] = 0_n, \quad j = 1, \cdots, m \tag{20b}$$

Here $E[\cdot]$ denotes expected value and 0_n is the $n \times n$ zero matrix. These give

$$E[f_{m,t}\ f^T_{m,t-j}] = 0_n \quad j = 1, \cdots, m \tag{20c}$$

$$E[b_{m,t}\ b^T_{m,t-j}] = 0_n \quad j = 1, \cdots, m \tag{20d}$$

Demonstration of (20c):

$$E[f_{m,t}\ f^T_{m,t-j}] = E\Big[f_{m,t}\Big(-\sum_{k=0}^{m} a_{m,k}\, x(t-k-j)\Big)^T\Big]$$

(by 10s) $\qquad = -\sum_{k=0}^{m} E[f_{m,t}\, x^T(t-k-j)]\, a^T_{m,k}$

which for $j = 1, \cdots, m$ is zero, as $k+j > 0$, by (20a).

Demonstration of (20d):

$$E[b_{m,t}\, b^T_{m,t-j}] = -\sum_{k=0}^{m} E[b_{m,t}\, x^T([t-j-m]-k)]\, \underline{a}^T_{m,k}$$

which is zero for $j = 1, \cdots, m$ by (20b)

It should be noted that the assumptions of (20a,b) are the key ones for the theory and what force the results into the nice form finally obtained. They are an orthogonality assumption which is equivalent to choice of filter coefficients via minimization of mean square error.

For $j=0$ the prediction errors are of course correlated, so we define

$$E[f_{m,t}\, f_{m,t}^T] = P_m$$

(20e)

$$E[b_{m,t}\, b_{m,t}^T] = \underline{P}_m$$

(20f)

both of which are symmetric, i.e. $P_m = P_m^T$, $\underline{P}_m = \underline{P}_m^T$, and positive semidefinite. We note that

$$E[f_{0,t}\, f_{0,t}^T] = E[b_{0,t}\, b_{0,t}^T] = E[x(t)\, x^T(t)] = P_0 = \underline{P}_0 = R(0) \quad (20$$

where $R(0)$ is defined as the autocorrelation $E[x(t)\, x^T($

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Next we consider correlation matrices in real (non-digital) time

$$R(\tau) = E\left[x(t)\, x^T(t-\tau)\right] \qquad (30\,a)$$

$$\underline{R}(\tau) = E\left[x(t)\, x^T(t+\tau)\right] \qquad (30\,b)$$

where, as above, $E[\cdot]$ denotes expected value.

We have, by inspection of (30), (let $t+\tau = \lambda \Rightarrow t$ for (40a))

$$\underline{R}(\tau) = R(-\tau) \qquad (40a)$$

$$R^T(\tau) = R(-\tau) \qquad (40b)$$

The key equations from which we will work are

$$\sum_{j=0}^{m} (-a_{m,j}) R(i-j) = \begin{cases} 0_n & i = 1, \ldots, m \qquad (50a) \\ P_m & i = 0 \qquad (50b) \end{cases}$$

$$\sum_{j=0}^{m} (-\underline{a}_{m,m-j}) R(i-j) = \begin{cases} 0_n & i = 0, \ldots, m-1 \qquad (50c) \\ \underline{P}_m & i = m \qquad (50d) \end{cases}$$

where, from (10e), $a_{m,0} = \underline{a}_{m,0} = -1_n$ is used.

Demonstration of (50a):

From (20a) and using (10b), & (30a), we have

(50a₁) $$0_n = E[\varepsilon_{m,t}\, x^T(t-i)] \qquad \text{for } i \neq 0$$

(50a₂) $$= E\Big[-\sum_{j=0}^{m} a_{m,j}\, x(t-j)\, x^T(t-i)\Big]$$

(50a₃) $$= -\sum_{j=0}^{m} a_{m,j}\, E[x(t-j)\, x^T(t-i)] \quad ; \text{ let } \lambda = t-i \Rightarrow t = \lambda + i$$

(50a₄) $$= -\sum_{j=0}^{m} a_{m,j}\, E[x(\lambda+i-j)\, x^T(\lambda)]$$

(50a₅) $$= -\sum_{j=0}^{m} a_{m,j}\, R(i-j)$$

which is (50a).

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Demonstration of (50b):

We first note, from (20e) and (10c)

(50b$_1$) $$P_m = E[f_{m,t}\, f_{m,t}^T] = E[f_{m,t},\ (x(t) - \sum_{j=1}^{m} a_{m,j}\, x(t-j))^T]$$

(50b$_2$) $$= E[f_{m,t}\, x^T(t)] - \sum_{j=1}^{m} E[f_{m,t}\, x^T(t-j)\, a_{m,j}^T]$$

(50b$_3$) $$= E[f_{m,t}\, x^T(t)] \qquad (\text{by } (20e))$$

In this we substitute (10c) to get

(50b$_4$) $$P_m = E[(-\sum_{j=0}^{m} a_{m,j}\, x(t-j))\, x^T(t)]$$

(50b$_5$) $$= -\sum_{j=0}^{m} a_{m,j}\, E[x(t-j)\, x^T(t)]$$

which is (50b).

Demonstration of (50c):

(50c$_1$) $$C_m = E[b_{m,t}\, x^T(t-i)] \qquad \text{for } i \neq 0$$

(50c$_2$) $$= E[-\sum_{k=0}^{m} \underline{a}_{m,k}\, x([t-m]+k)\, x^T(t-i)]$$

(50c$_3$) $$= -\sum_{k=0}^{m} \underline{a}_{m,k}\, E[x([t-m]+k)\, x^T(t-i)] \qquad \text{let } j = m-$$

(50c$_4$) $$= -\sum_{j=m}^{j=0} \underline{a}_{m,m-j}\, E[x(t-j)\, x^T(t-i)]$$

(50c$_5$) $$= -\sum_{j=0}^{m} \underline{a}_{m,m-j}\, R(i-j)$$

which is (50c).

Demonstration of (50d):

again, by (20b)

(50d₁) $$\underline{P}_m = E[b_{m,t}\, b_{m,t}^T] = E[b_{m,t},\ (x(t-m) - \sum_{k=1}^{m} \underline{a}_{m,k}\, x([t-m]+k))^T]$$

(50d₂) $$= E[b_{m,t},\ x^T(t-m)]$$

and in this we substitute (10g)

(50d₃) $$\underline{P}_m = E[-\sum_{k=0}^{m} \underline{a}_{m,k}\, x([t-m]+k),\ x^T(t-m)]$$

(50d₄) $$= -\sum_{k=0}^{m} \underline{a}_{m,k}\, E[x([t-m]+k),\ x^T(t-m)]$$

(50d₅) $$= -\sum_{k=0}^{m} \underline{a}_{m,k}\, R(k)$$

Now replace $k = m-j$, then

(50d₆) $$\underline{P}_m = -\sum_{j=m}^{0} \underline{a}_{m,m-j}\, R(m-j)$$

which is (50d).

Here P_m and $\underline{P}_m$ can be thought of as forward and reverse power types of $n \times n$ quantities.

When written out in matrix form (50) is

Putting (50) into matrix form & using $R(-j) = R^T(j)$

$$\begin{bmatrix} R(0) & R(1) & \cdots & R(m) \\ R(-1) & R(0) & \cdots & R(m-1) \\ \vdots & \vdots & \ddots & \vdots \\ R(-m) & R(1-m) & \cdots & R(0) \end{bmatrix} \begin{bmatrix} 1_n & -\underline{a}^T_{m,m} \\ -a^T_{m,1} & \vdots \\ \vdots & \vdots \\ & -\underline{a}^T_{m,1} \\ -a^T_{m,m} & 1_n \end{bmatrix} = \begin{bmatrix} P_m & 0_n \\ 0_n & \vdots \\ \vdots & \vdots \\ & 0_n \\ 0_n & \underline{P}_m \end{bmatrix} \tag{60}$$

Thus, if the $R(i)$ and P_m & $\underline{P}_m$ are known we can find the coefficients $a_{m,j}$, $\underline{a}_{m,j}$ from (60) though we wish to proceed differently via the use of prediction errors.

Assuming the validity of (60) we obtain some important properties.

First, defining

$$a_{m,k} = \underline{a}_{m,k} = 0_n \quad \text{for } k > m \tag{70a}$$

$$a_{m,0} = \underline{a}_{m,0} = -1_n \quad (\text{for all } m) \tag{70b) = (10e}$$

we obtain (analogous to (5) of Andersen)

$$\boxed{a^T_{m,k} = a^T_{m-1,k} - \underline{a}^T_{m-1,m-k}\, a^T_{m,m}} \tag{70c}$$

$$\boxed{\underline{a}^T_{m,k} = \underline{a}^T_{m-1,k} - a^T_{m-1,m-k}\, \underline{a}^T_{m,m}} \tag{70d}$$

which hold for all integer m & k (we only seem to need for non-negative integers). To obtain (70c) & (70d) from (60) takes a little space, but follows.

Derivation of (70c.):

We consider (60) for m-1 and enlarge to m:

(70c₁)
$$\begin{bmatrix} R(0) & R(1) & \cdots & R(m-2) & R(m-1) \\ R(-1) & R(0) & & R(m-3) & R(m-2) \\ \vdots & \vdots & & \vdots & \vdots \\ R(1-m) & R(2-m) & & R(-1) & R(0) \end{bmatrix} \begin{bmatrix} 1_n \\ -a^T_{m-1,1} \\ \vdots \\ -a^T_{m-1,m-1} \end{bmatrix} = \begin{bmatrix} P_{m-1} \\ 0_n \\ \vdots \\ 0_n \end{bmatrix} \equiv \begin{bmatrix} R_0 & R_{m-1} \\ R^T_{m-1} & R_{m-1,m-1} \end{bmatrix} \begin{bmatrix} 1_n \\ -A^{(m-1)}_{m-1} \end{bmatrix}$$

where some block matrix notation is introduced on the right

Going to m:

(70c₂)
$$\begin{bmatrix} R(0) & R(1) & \cdots & R(m-1) & R(m) \\ R(-1) & R(0) & & R(m-2) & R(m-1) \\ \vdots & \vdots & & \vdots & \vdots \\ R(1-m) & R(2-m) & & R(0) & R(1) \\ R(-m) & R(1-m) & & R(-1) & R(0) \end{bmatrix} \begin{bmatrix} 1_n \\ -a^T_{m,1} \\ \vdots \\ -a^T_{m,m-1} \\ -a^T_{m,m} \end{bmatrix} = \begin{bmatrix} P_m \\ 0_n \\ \vdots \\ 0_n \\ 0_n \end{bmatrix}$$

or by blocks (here super r means reverse order of blocks)

(70c₃)
$$\begin{bmatrix} R_0 & R_{m-1} & R(m) \\ R^T_{m-1} & R_{m-1,m-1} & R^r_{m-1} \\ R^T(m) & (R^T_{m-1})^r & R_0 \end{bmatrix} \begin{bmatrix} 1_n \\ -A^{(m)}_{m-1} \\ -a^T_{m,m} \end{bmatrix} = \begin{bmatrix} P_m \\ 0_n \\ \vdots \\ 0_n \\ 0_n \end{bmatrix}$$

Now the last (block) row of (70c₁) is

(70c₄) $$R^T_{m-1} = R_{m-1,m-1} A^{(m-1)}_{m-1}$$

while for the same term the middle row of (70c₃) is

(70c₅) $$R^T_{m-1} = R_{m-1,m-1} A^{(m)}_{m-1} + R^r_{m-1} a^T_{m,m}$$

Equating these last two and assuming $R_{m-1,m-1}$ <u>nonsingular</u> (I think this means good data) gives

Derivation of (70c) cont.

$$(70c_6) \qquad A_{m-1}^{(m)} = A_{m-1}^{(m-1)} - R_{m-1,m-1}^{-1} R_{m-1}^{r} a_{m,m}^{T}$$

Next we need to replace the R terms in this, so go to the reversed portion of (60), again first for m-1:

$$(70c_7) \qquad \begin{bmatrix} R(0) & R(1) & \cdots & R(m-1) \\ R(-1) & R(0) & & R(m-2) \\ \vdots & \vdots & & \vdots \\ R(1-m) & R(2-m) & & R(0) \end{bmatrix} \begin{bmatrix} -\underline{a}^T_{m-1,m-1} \\ -\underline{a}^T_{m-1,m-2} \\ \vdots \\ -\underline{a}^T_{m-1,1} \\ 1_m \end{bmatrix} = \begin{bmatrix} 0_n \\ 0_n \\ \vdots \\ 0_n \\ \underline{P}_{m-1} \end{bmatrix}$$

The first m-1 (again block) rows of this are on multiplying the last column by 1_m & separating

$$(70c_8) \qquad \begin{bmatrix} R(m-1) \\ R(m-2) \\ \vdots \\ R(1) \end{bmatrix} = \begin{bmatrix} R(0) & R(1) & \cdots & R(m-2) \\ R(-1) & R(0) & & R(m-3) \\ \vdots & \vdots & & \vdots \\ R(2-m) & R(3-m) & \cdots & R(0) \end{bmatrix} \begin{bmatrix} \underline{a}^T_{m-1,m-1} \\ \underline{a}^T_{m-1,m-2} \\ \vdots \\ \underline{a}^T_{m-1,1} \end{bmatrix}$$

or in the notation above & solving for the vector of $\underline{a}$'s on the right of $(70c_8)$ it is

$$(70c_9) \qquad \begin{bmatrix} \underline{a}^T_{m-1,m-1} \\ \underline{a}^T_{m-1,m-2} \\ \vdots \\ \underline{a}^T_{m-1,1} \end{bmatrix} = R_{m-1,m-1}^{-1} R_{m-1}^{r}$$

Substituting in $(70c_6)$ and writing in blocks

$$(70c_{10}) \qquad \begin{bmatrix} a^T_{m,1} \\ a^T_{m,2} \\ \vdots \\ a^T_{m,m-1} \end{bmatrix} = \begin{bmatrix} a^T_{m-1,1} \\ a^T_{m-1,2} \\ \vdots \\ a^T_{m-1,m-1} \end{bmatrix} - \begin{bmatrix} \underline{a}^T_{m-1,m-1} \\ \underline{a}^T_{m-1,m-2} \\ \vdots \\ \underline{a}^T_{m-1,1} \end{bmatrix} a^T_{m,m}$$

which is (70c) when (70a, b) are used. We next get (70d)

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Derivation of (70d)

We extend the "reverse set" of equations written out as ($70c_7$) from m-1 to m:

$$(70d_1)\quad \left[\begin{array}{c|cc|c} R(0) & R(1) \;\cdots & R(m-1) & R(m) \\ \hline R(-1) & R(0) & R(m-2) & R(m-1) \\ \vdots & \vdots & \vdots & \vdots \\ R(1-m) & R(2-m) & R(0) & R(1) \\ \hline R(-m) & R(1-m) & R(-1) & R(0) \end{array}\right] \begin{bmatrix} -\underline{a}^T_{m,m} \\ -\underline{a}^T_{m,m-1} \\ \vdots \\ -\underline{a}^T_{m,1} \\ 1_m \end{bmatrix} = \begin{bmatrix} 0_n \\ 0_n \\ \vdots \\ 0_n \\ \underline{P}_m \end{bmatrix}$$

We equate the middle of this

$$(70d_2)\quad \begin{bmatrix} R(m-1) \\ \vdots \\ R(1) \end{bmatrix} = \begin{bmatrix} R(0) & & R(m-2) \\ \vdots & & \vdots \\ R(2-m) & \cdots & R(0) \end{bmatrix} \begin{bmatrix} \underline{a}^T_{m,m-1} \\ \vdots \\ \underline{a}^T_{m,1} \end{bmatrix} + \begin{bmatrix} R(-1) \\ \vdots \\ R(1-m) \end{bmatrix} \underline{a}^T_{m,m}$$

to ($70c_3$) to get

$$(70d_3)\quad \begin{bmatrix} \underline{a}^T_{m,m-1} \\ \vdots \\ \underline{a}^T_{m,1} \end{bmatrix} = \begin{bmatrix} \underline{a}^T_{m-1,m-1} \\ \vdots \\ \underline{a}^T_{m-1,1} \end{bmatrix} - R^{-1}_{m-1,m-1} R^T_{m-1} \underline{a}^T_{m,m}$$

in which we substitute from ($70c_4$) to get

$$(70d_4)\quad \begin{bmatrix} \underline{a}^T_{m,m-1} \\ \vdots \\ \underline{a}^T_{m,1} \end{bmatrix} = \begin{bmatrix} \underline{a}^T_{m-1,m-1} \\ \vdots \\ \underline{a}^T_{m-1,1} \end{bmatrix} - \begin{bmatrix} a_{m-1,1} \\ \vdots \\ a^T_{m-1,m-1} \end{bmatrix} \underline{a}^T_{m,m}$$

which is (70d)

Derivation of (80a) cont.

or

$$(80a_2)\qquad \underbrace{R(-m-1)(-[-1_m])}_{j=0 \text{ term}} - \underbrace{R(0)\, a^T_{m+1,m+1}}_{j=m+1 \text{ term}} - \sum_{j=1}^{m} R(j-m-1)\, a^T_{m+1,j} = C$$

In this we will substitute (70c) updated to $m+1$ (with $k \to j$)

$$(80a_3)\qquad a^T_{m+1,j} = a^T_{m,j} - \underline{a}^T_{m,m+1-j}\, a^T_{m+1,m+1}$$

Thus

$$(80a_4)\qquad R(-m-1) - \sum_{j=1}^{m} R(j-m-1)\left\{a^T_{m,j} - \underline{a}^T_{m,m+1-j}\, a^T_{m+1,m+1}\right\} = R(0)\, a^T_{m+1,m+1}$$

Rearranging

$$(80a_5)\qquad R(-m-1) - \sum_{j=1}^{m} R(j-m-1)\, a^T_{m,j} = \left\{R(0) - \sum_{j=1}^{m} R(j-m-1)\, \underline{a}^T_{m,m+1-j}\right\} a^T_{m+1,m+1}$$

and solving for $a^T_{m+1,m+1}$ (noting $a_{m,0} = -1_m = \underline{a}_{m,0}$)

$$(80a_6)\qquad a^T_{m+1,m+1} = \left\{\sum_{j=1}^{m+1} R(j-m-1)\, \underline{a}^T_{m,m+1-j}\right\}^{-1} \left\{\sum_{j=0}^{m} R(j-m-1)\, a^T_{m,j}\right\}$$

Now shift indices in the inverse (i.e. $j \to k+1$)

$$(80a_7)\qquad \sum_{j=1}^{m+1} R(j-m-1)\, \underline{a}^T_{m,m+1-j} = \sum_{k=0}^{m} R(k-m)\, \underline{a}^T_{m,m-k}$$

and use (50j) to see that this latter is $-\underline{P}_m$. The other term on the right of $(80a_6)$ is what we defined at (80c) to be $-\Delta^T_m$, in which case $(80a_6)$ is (80a). Note that Δ_m is (50a) evaluated (out of its range) at $i = m+1$.

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We also have need for the "Durbin" relations

$$a_{m+1,m+1} = \Delta_m \underline{P}_m^{-1} \qquad (80a)$$

$$\underline{a}_{m+1,m+1} = \underline{\Delta}_m P_m^{-1} \qquad (80b)$$

where

$$\Delta_m = \sum_{j=0}^{m} (-a_{m,j}) R(m+1-j) \qquad (80c)$$

$$\underline{\Delta}_m = \sum_{j=0}^{m} (-\underline{a}_{m,m-j}) R(-j-1) \qquad (80d)$$

A key result is

$$\underline{\Delta}_m = \Delta_m^T \qquad (80e)$$

while also we can rewrite (80c, d) in alternate forms

$$\Delta_m = \sum_{j=0}^{m} (-a_{m,m-j}) R(j+1) \qquad (80f)$$

$$\underline{\Delta}_m = \sum_{j=0}^{m} (-\underline{a}_{m,j}) R(j-m-1) \qquad (80g)$$

Note that if we know $\underline{P}_m$ & P_m these allow the update of the diagonal a & $\underline{a}$ blocks.

Derivation of (80a):

We update (50a) to m+1 and use the $i = m+1$ term:

(80a$_1$)
$$\sum_{j=0}^{m+1} R^T(m+1-j)(-a^T_{m+1,j}) = 0_n$$

Derivation of (80b):

For (80b) the steps are about the same. For $i = 0$ in (50a) for m replaced by $m+1$

(80b$_1$) $$\sum_{j=0}^{m+1} R(j)\left(-\underline{a}^T_{m+1,\,m+1-j}\right) = 0_n$$

or

(80b$_2$) $$R(m+1) - R(0)\,\underline{a}^T_{m+1,\,m+1} = \sum_{j=1}^{m} R(j)\,\underline{a}^T_{m+1,\,m+1-j}$$

Use (70d) in the form

(80b$_3$) $$\underline{a}^T_{m+1,\,m+1-j} = \underline{a}^T_{m,\,m+1-j} - a^T_{m,\,j}\,\underline{a}^T_{m+1,\,m+1}$$

Thus, from (80b$_3$) into (80b$_2$)

(80b$_4$) $$R(m+1) - \sum_{j=1}^{m} R(j)\,\underline{a}^T_{m,\,m+1-j} = \left(R(0) - \sum_{j=1}^{m} R(j)\,a^T_{m,\,j}\right)\underline{a}^T_{m+1,\,m+1}$$

or

(80b$_5$) $$\underline{a}^T_{m+1,\,m+1} = \left\{\sum_{j=0}^{m} R(j)\,a^T_{m,\,j}\right\}^{-1}\left\{\sum_{j=1}^{m+1} R(j)\,\underline{a}^T_{m,\,m+1-j}\right\}$$

(80b$_6$) $$= -P_m^{-1}\left\{\sum_{\substack{k=0\\ k=j-1}}^{m} R(k+1)\,\underline{a}^T_{m,\,m-k}\right\}$$

from (50b). But (80b$_6$) is (80b) by the definition of $\underline{\Delta}^T_m$ at (80d), which in fact is (50c) evaluated at $i = -1$.

The alternate forms for Δ_m & $\underline{\Delta}_m$ of (80f, g) are obtained by setting $k = m - j$ (and then replacing k by j) in (80c, d).

Demonstration of (80e)

We have

$$
\begin{bmatrix} R(0) & R(1) & \cdots & R(m) & R(m+1) \\ R(-1) & R(0) & & R(m-1) & R(m) \\ \vdots & & & & \\ R(-m-1) & R(-m) & & \cdots & R(0) \end{bmatrix}
\begin{bmatrix} 1_m & 0_m \\ -a^T_{m,1} & -\underline{a}^T_{m,m} \\ \vdots & \vdots \\ -a^T_{m,m} & -\underline{a}^T_{m,1} \\ 0_m & 1_m \end{bmatrix}
=
\begin{bmatrix} P_m & \underline{\Delta}_m \\ 0_m & 0_m \\ \vdots & \vdots \\ 0_m & 0_m \\ \Delta_m & \underline{P}_m \end{bmatrix}
\tag{80e$_1$}
$$

Now multiply on the left by

$$
\begin{bmatrix} 0_m & -\underline{a}_{m,m}, & \cdots, & -\underline{a}_{m,1}, & 1_m \\ 1_m, & -a_{m,1}, & \cdots, & -a_{m,m}, & 0_m \end{bmatrix}
\tag{80e$_2$}
$$

First by multiplying by the matrix of R's, to give the columns of the very right matrix transposed and in opposite block order, i.e.

$$
\begin{bmatrix} \underline{\Delta}^T_m, & 0_m, & \cdots, & \underline{P}^T_m \\ P^T_m, & 0_m, & \cdots, & \Delta^T_m \end{bmatrix}
\begin{bmatrix} 1_m & 0_m \\ -a^T_{m,1} & -\underline{a}^T_{m,m} \\ \vdots & \vdots \\ -a^T_{m,m} & -\underline{a}^T_{m,1} \\ 0_m & 1_m \end{bmatrix}
=
\begin{bmatrix} \underline{\Delta}^T_m, & \underline{P}_m \\ P_m, & \Delta^T_m \end{bmatrix}
\tag{80e$_3$}
$$

and then multiplying (80e$_2$) by the right side of (80e$_1$) gives

$$
\begin{bmatrix} 0_m, & -\underline{a}_{m,m}, & \cdots, & -\underline{a}_{m,1}, & 1_m \\ 1_m, & -a_{m,1}, & \cdots, & -a_{m,m}, & 0_m \end{bmatrix}
\begin{bmatrix} P_m & \underline{\Delta}_m \\ 0_m & 0_m \\ \vdots & \vdots \\ 0_m & 0_m \\ \Delta_m & \underline{P}_m \end{bmatrix}
=
\begin{bmatrix} \Delta_m & \underline{P}_m \\ P_m & \underline{\Delta}_m \end{bmatrix}
\tag{80e$_4$}
$$

Equating (80e$_3$) & (80e$_4$) shows $\underline{\Delta}_m = \Delta^T_m$.

We also have the following result which should prove useful for investigating the a's in terms of measured quantities

$$\Delta_m = E[\, b_{m,t-1} \, f^T_{m,t} \,] \qquad (80h)$$

Demonstration of (80h):

(80h$_1$) $E[\, b_{m,t-1} \, f^T_{m,t} \,] = E[\, b_{m,t} \, f^T_{m,t+1} \,]$

(80h$_2$) $= E[\, b_{m,t} \, (-\sum_{k=0}^{m} a_{m,k} \, x(t+1-k))^T \,]$ by (10f)

(80h$_3$) $= -\sum_{k=0}^{m} E[\, b_{m,t} \, x^T(t+1-k) \,] \, a^T_{m,k}$

(80h$_4$) $= -E[\, b_{m,t} \, x^T(t+1) \,] \, a^T_{m,0}$ by (20b)

(80h$_5$) $= E[\, b_{m,t} \, x^T(t+1) \,]$ by $a_{m,0} = -1_n$

(80h$_6$) $= -\sum_{j=0}^{m} \underline{a}_{m,j} \, E[\, x([t-m]+j) \, x^T(t+1) \,]$ by (10g)

(80h$_7$) $= -\sum_{j=0}^{m} \underline{a}_{m,j} \, R(j-m-1)$ by (30a)

(80h$_8$) $= \Delta_m$ by (80g)

Likewise

$$\Delta_m = E[\, f_{m,t} \, b^T_{m,t-1} \,] \qquad (80i)$$

which follows by transposing (80h) but as a direct check is shown as follows

Demonstration of (80i)

(80i₁) $E[f_{m,t}\, b^T_{m,t-1}] = E\left[f_{m,t}\left(-\sum_{k=0}^{m} \underline{a}_{m,k}\, x([t-1-m]+k)\right)^T\right]$

(80i₂) $= -\sum_{k=0}^{m} E[f_{m,t}\, x^T([t-1-m]+k)]\, \underline{a}^T_{m,k}$

(80i₃) $= -\ E[f_{m,t}\, x^T(t-1-m)]\, \underline{a}^T_{m,0}$ by (20a)

(80i₄) $= E[f_{m,t}\, x^T(t-1-m)]$

(80i₅) $= -\sum_{j=0}^{m} a_{m,k}\, E[x(t-k)\, x^T(t-1-m)]$

(80i₆) $= -\sum_{j=0}^{m} a_{m,k}\, R(m+1-k)$

(80i₇) $= \Delta_m$ by (80c)

Therefore the a's can be given directly in terms of measured quantities by:

$$a_{m+1,m+1} = \Delta_m \underline{P}_m^{-1} = E[f_{m,t}\, b^T_{m,t-1}]\left\{E[b_{m,t}\, b^T_{m,t}]\right\}^{-1} \quad (8\ldots$$

$$\underline{a}_{m+1,m+1} = \underline{\Delta}_m P_m^{-1} = E[b_{m,t-1}\, f^T_{m,t}]\left\{E[f_{m,t}\, f^T_{m,t}]\right\}^{-1} \quad (8\ldots$$

Next we look at an updating of P_m and $\underline{P}_m$ in going from m to $m+1$. Here

$$P_{m+1} = P_m - \Delta_m \underline{P}_m^{-1} \underline{\Delta}_m \tag{81a}$$

$$\underline{P}_{m+1} = \underline{P}_m - \underline{\Delta}_m P_m^{-1} \Delta_m \tag{81b}$$

Demonstration of (81a)

Into (50b) we insert (70c)

(81a$_1$) $$P_{m+1} = \sum_{j=0}^{m+1} (-a_{m+1,j}) R(-j)$$

(81a$_2$) $$= \sum_{j=0}^{m+1} (-a_{m,j} + a_{m+1,m+1} \underline{a}_{m,m+1-j}) R(-j)$$

(81a$_3$) $$= \sum_{j=0}^{m+1} (-a_{m,j}) R(-j) - a_{m+1,m+1} \sum_{j=0}^{m+1} (-\underline{a}_{m,m+1-j}) R(-j)$$

(81a$_4$) $$= \sum_{j=0}^{m} (-a_{m,j}) R(-j) \qquad \text{by (70a), let } k = m+1-j$$

$$- a_{m+1,m+1} \sum_{k=0}^{m+1} (-\underline{a}_{m,k}) R(k-1-m)$$

(81a$_5$) $$= P_m - a_{m+1,m+1} \underline{\Delta}_m \qquad \text{by (70a) \& (80g)}$$

which is (81a) in view of $a_{m+1,m+1} = \Delta_m \underline{P}_m^{-1}$

Demonstration of (81b)

Into (50d) we insert (70d)

(81b$_1$) $$\underline{P}_{m+1} = \sum_{j=0}^{m+1} (-\underline{a}_{m+1,m+1-j}) R(m+1-j)$$

Demonstration of (81b) cont.

$$\underline{P}_{m+1} = \sum_{j=0}^{m+1} (-\underline{a}_{m,m+1-j} + \underline{a}_{m+1,m+1} a_{m,j}) R(m+1-j) \qquad (81b_2)$$

$$= \sum_{k=0}^{m+1} (-\underline{a}_{m,k}) R(k) \qquad \text{by } k = m+1-j \qquad (81b_3)$$

$$- \underline{a}_{m+1,m+1} \sum_{j=0}^{m+1} (-a_{m,j}) R(m+1-j)$$

$$= \sum_{k=0}^{m} (-\underline{a}_{m,k}) R(k) - \underline{a}_{m+1,m+1} \sum_{j=0}^{m} (-a_{m,j}) R(m+1-j) \qquad (81b_4)$$

$$= \underline{P}_m - \underline{a}_{m+1,m+1} \Delta_m \quad \text{by (50d) with } m-j=k \text{ \& (80c)} \qquad (81b_5)$$

which is (81b) in view of $\underline{a}_{m+1,m+1} = \underline{\Delta}_m P_m^{-1}$

In view of the relations (80a,b) rearranged as $\Delta_m = a_{m+1,m+1} \underline{P}_m$ and $\underline{\Delta}_m = \underline{a}_{m+1,m+1} P_m$ we have the important relationships, from (81a,b)

$$P_{m+1} = (1_n - a_{m+1,m+1} \underline{a}_{m+1,m+1}) P_m \qquad (82a)$$

$$\underline{P}_{m+1} = (1_n - \underline{a}_{m+1,m+1} a_{m+1,m+1}) \underline{P}_m \qquad (82b)$$

From (20e,f) we know that P_m and P_{m+1}, as well as $\underline{P}_{m+1}$ & $\underline{P}_m$, are positive semidefinite. This gives some magnitude constraints on the a's.

alternate derivations of (80a, b)(81a, b):
We introduce new notation and let

$$(80_1) \qquad \mathcal{P}_m = [P_m, 0_m, \ldots, 0_m], \quad \underline{\mathcal{P}}_m = [0_m, \ldots, 0_m, \underline{P}_m]$$

$$(80_2) \qquad a_m = [-1_m, a_{m,1}, \ldots, a_{m,m}], \quad \underline{a}_m = [\underline{a}_{m,m}, \ldots, \underline{a}_{m,1}, -1_m]$$

$$(80_3) \qquad \mathcal{R}_m = \begin{bmatrix} R(0) & R(1) & \cdots & R(m) \\ R(-1) & R(0) & & R(m-1) \\ \vdots & \vdots & & \vdots \\ R(-m) & R(1-m) & & R(0) \end{bmatrix}, \quad R_m = [R(1), \ldots, R(m)]$$

Then (60) is

$$(80_4) \qquad \mathcal{R}_m [-a_m^T, -\underline{a}_m^T] = [\mathcal{P}_m^T, \underline{\mathcal{P}}_m^T]$$

while, using (80c, d, 5, g)

$$(80_5) \qquad \mathcal{R}_{m+1} \left[\begin{bmatrix} -a_m^T \\ \hline 0_m \end{bmatrix}, \begin{bmatrix} 0_m \\ \hline -\underline{a}_m^T \end{bmatrix} \right] = \left[\begin{bmatrix} \mathcal{P}_m^T \\ \hline \Delta_m^T \end{bmatrix}, \begin{bmatrix} \underline{\Delta}_m^T \\ \hline \underline{\mathcal{P}}_m^T \end{bmatrix} \right]$$

also, on updating (80_4),

$$(80_6) \qquad \mathcal{R}_{m+1} [-a_{m+1}^T, -\underline{a}_{m+1}^T] = [\mathcal{P}_{m+1}^T, \underline{\mathcal{P}}_{m+1}^T]$$

which on using (70c, d) becomes

$$(80_7) \qquad \mathcal{R}_{m+1} \left[\begin{bmatrix} -a_m^T \\ \hline 0_m \end{bmatrix} - \begin{bmatrix} 0_m \\ \hline -\underline{a}_m^T \end{bmatrix} a_{m+1,m+1}^T, \begin{bmatrix} 0_m \\ \hline -\underline{a}_m^T \end{bmatrix} - \begin{bmatrix} -a_m^T \\ \hline 0_m \end{bmatrix} \underline{a}_{m+1,m+1}^T \right] = [\mathcal{P}_{m+1}^T, \underline{\mathcal{P}}_{m+1}^T]$$

$$(80_8) \qquad = \left[\begin{bmatrix} \mathcal{P}_m^T \\ \hline \Delta_m^T \end{bmatrix} - \begin{bmatrix} \underline{\Delta}_m^T \\ \hline \underline{\mathcal{P}}_m^T \end{bmatrix} a_{m+1,m+1}^T, \begin{bmatrix} \underline{\Delta}_m^T \\ \hline \underline{\mathcal{P}}_m^T \end{bmatrix} - \begin{bmatrix} \mathcal{P}_m^T \\ \hline \Delta_m^T \end{bmatrix} \underline{a}_{m+1,m+1}^T \right]$$

Equating the four sub-blocks of (80_7) with (80_8) gives

$$(80_{9a,b}) \qquad P_{m+1}^T = P_m^T - \underline{\Delta}_m^T a_{m+1,m+1}^T, \qquad 0_m = \underline{\Delta}_m^T - P_m^T \underline{a}_{m+1,m+1}^T$$

$$(80_{10a,b}) \qquad 0_m = \Delta_m^T - \underline{P}_m^T a_{m+1,m+1}^T, \qquad \underline{P}_{m+1}^T = \underline{P}_m^T - \Delta_m^T \underline{a}_{m+1,m+1}^T$$

(80_{9a}) is $(81a_5)$ for P_{m+1}, (80_{9b}) is (80b) for $\underline{a}_{m+1,m+1}$ while (80_{10a}) is (80a) for $a_{m+1,m+1}$ and (80_{10b}) is $(81b_5)$ for $\underline{P}_{m+1}$.

From (80_{9a})

$$(80_{11}) \qquad P_{m+1} = P_m - \Delta_m \underline{P}_m^{-1} \underline{\Delta}_m \qquad \text{using } (80_{10a})$$

$$(80_{12}) \qquad = P_m - a_{m+1,m+1} \underline{\Delta}_m P_m^{-1} P_m$$

$$(80_{13}) \qquad = P_m - a_{m+1,m+1} \underline{a}_{m+1,m+1} \qquad \text{using } (80_{9b})$$

$$(80_{14}) \qquad = P_m - a_{m+1,m+1} \underline{P}_m \underline{P}_m^{-1} \underline{\Delta}_m$$

$$(80_{15}) \qquad = P_m - a_{m+1,m+1} \underline{P}_m a^T_{m+1,m+1} \qquad \text{using } \underline{\Delta}_m = \Delta_m^T$$

From (80_{10b})

$$(80_{16}) \qquad \underline{P}_{m+1} = \underline{P}_m - \underline{\Delta}_m P_m^{-1} \Delta_m$$

$$(80_{17}) \qquad = \underline{P}_m - \underline{a}_{m+1,m+1} P_m \underline{a}^T_{m+1,m+1}$$

by the symmetry and insertion of —'s in (80_{15})

The latter results can be rewritten in the interesting form

$$\begin{bmatrix} P_{m+1} & 0_n \\ 0_n & \underline{P}_{m+1} \end{bmatrix} = \begin{bmatrix} 1_n & -a_{m+1,m+1} \\ \underline{a}_{m+1,m+1} & 1_n \end{bmatrix} \begin{bmatrix} P_m & 0_n \\ 0_n & \underline{P}_m \end{bmatrix} \begin{bmatrix} 1_n & \underline{a}^T_{m+1,m+1} \\ -a^T_{m+1,m+1} & 1_n \end{bmatrix} \qquad (82c)$$

which can be used to obtain that $(P_m \dotplus \underline{P}_m)$, with $\dotplus$ = direct sum, is a congruency transformation on $P_0 \dotplus \underline{P}_0 = R_0 \dotplus R_0$. In fact

$$P_m \dotplus \underline{P}_m = \mathcal{J}^T [R_0 \dotplus R_0] \mathcal{J} \qquad (82d)$$

$$\mathcal{J}^T = \begin{bmatrix} 1_n & -a_{m,m} \\ \underline{a}_{m,m} & 1_n \end{bmatrix} \begin{bmatrix} 1_n & -a_{m-1,m-1} \\ \underline{a}_{m-1,m-1} & 1_n \end{bmatrix} \cdots \begin{bmatrix} 1_n & -a_{1,1} \\ \underline{a}_{1,1} & 1_n \end{bmatrix} \qquad (82$$

This confirms the positive semidefinite nature of P_m & $\underline{P}_m$.

We have some further relations of most importance for the physical structure.

In (10a,d) let $k' = m-k$ and use

$$f_{m,t} = -\sum_{k=0}^{m} a_{m,k}\, x(t-k) = -\sum_{k'=0}^{m} a_{m,m-k'}\, x(t-m+k') \tag{90a}$$

$$b_{m,t} = -\sum_{k=0}^{m} \underline{a}_{m,k}\, x(t-m+k) = -\sum_{k'=0}^{m} \underline{a}_{m,m-k'}\, x(t-k') \tag{90b}$$

Next use (70) to get relations among the f's and b's:

$$\boxed{f_{m,t} = f_{m-1,t} - a_{m,m}\, b_{m-1,t-1}} \tag{100a}$$

$$\boxed{b_{m,t} = -\underline{a}_{m,m}\, f_{m-1,t} + b_{m-1,t-1}} \tag{100b}$$

These are key equations, for recursions and for circuit constructions. Initial values are (from (90) using $a_{0,0} = -1_n = \underline{a}_{0,0}$)

$$\boxed{f_{0,t} = x(t)} \tag{100c}$$

$$\boxed{b_{0,t} = x(t)} \tag{100d}$$

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Derivation of (100):

we calculate

(100a$_1$) $$f_{m,t} - f_{m-1,t} = -\sum_{k=0}^{m} a_{m,k}\, x(t-k) + \sum_{k=0}^{m-1} a_{m-1,k}\, x(t-k)$$

(100a$_2$) $$= -\sum_{k=0}^{m} \left[a_{m,k} - a_{m-1,k}\right] x(t-k) \quad \left(\text{as } a_{m-1,m} = 0_n \text{ by } (70a)\right)$$

(100a$_3$) $$= \sum_{k=0}^{m} + \left[a_{m,m}\, \underline{a}_{m-1,m-k}\right] x(t-k) \quad (\text{by } (70c))$$

(100a$_4$) $$= a_{m,m} \sum_{k=0}^{m} \underline{a}_{m-1,m-k}\, x(t-k)$$

(100a$_5$) $$= a_{m,m}\left(\sum_{k'=0}^{m-1} a_{m-1,m-1-k'}\, x(t-1-k')\right) \quad \left(\text{by } \sigma,\ k' = k \ \& \ (70d)\right)$$

(100a$_6$) $$= -a_{m,m}\, b_{m-1,t-1}$$

which is (100a).

(100b$_1$) $$b_{m,t} - b_{m-1,t-1} = -\sum_{k=0}^{m} \underline{a}_{m,k}\, x(t-m+k) + \sum_{k=0}^{m-1} \underline{a}_{m-1,k}\, x(t-1-(m-1)+k)$$

(100b$_2$) $$= -\sum_{k=0}^{m} \left[\underline{a}_{m,k} - \underline{a}_{m-1,k}\right] x(t-m+k)$$

(100b$_3$) $$= \sum_{k=0}^{m} \left[\underline{a}_{m,m}\, a_{m-1,m-k}\right] x(t-m+k) \quad (\text{by } (70d))$$

(100b$_4$) $$= \underline{a}_{m,m} \sum_{k=0}^{m} a_{m-1,m-k}\, x(t-m+k)$$

(100b$_5$) $$= \underline{a}_{m,m} \sum_{k'=0}^{m-1} a_{m-1,m-1-k'}\, x(t-(m-1)+k') \quad \left(\text{by } \ldots\right)$$

(100b$_6$) $$= -\underline{a}_{m,m}\, f_{m-1,t}$$

which is (100b).

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III. Lattice Structure (follows Lim [LI 84])

At this point we can lay down the physical structures of interest.

We consider the $(2n+2n) = 4n$ terminal flow-graph for the mth stage

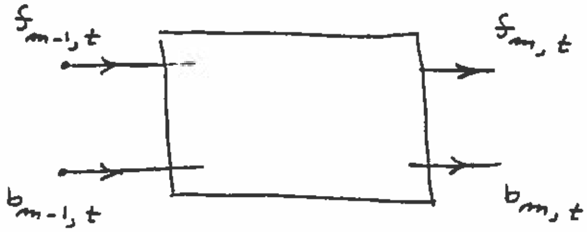

Figure 4

The governing equations are (100), repeated here

$$f_{m,t} = f_{m-1,t} - a_{m,m}\, b_{m-1,t-1} \qquad (120a)$$

$$b_{m,t} = -\underline{a}_{m,m}\, f_{m-1,t} + b_{m-1,t-1} \qquad (120b)$$

$$f_{0,t} = b_{0,t} = x(t) \qquad (120c,d)$$

Thus, the system flow-graph, of $n \times n$ matrix transmittances, is [with $\frac{1}{z}$ = unit delay in time]

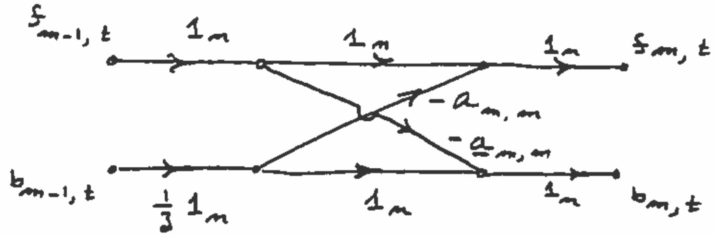

Figure 5 a)

with the zeroth stage, from (100c,d)

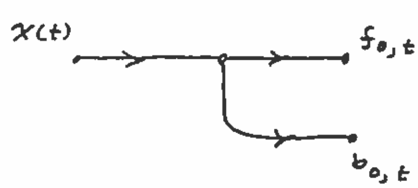

Figure 5 b)

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III. Lattice Structure (follows Lim [LI 84])

At this point we can lay down the physical structures of interest.

We consider the $(2n+2n) = 4n$ terminal flow-graph for the mth stage

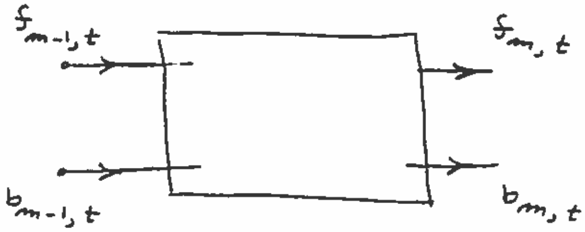

Figure 4

The governing equations are (100), repeated here

$$f_{m,t} = f_{m-1,t} - a_{m,m}\, b_{m-1,t-1} \tag{122a}$$

$$b_{m,t} = -\underline{a}_{m,m}\, f_{m-1,t} + b_{m-1,t-1} \tag{122b}$$

$$f_{0,t} = b_{0,t} = x(t) \tag{122c,d}$$

Thus, the system flow-graph, of $n \times n$ matrix transmittances, is [with $\frac{1}{z}$ = unit delay in time]

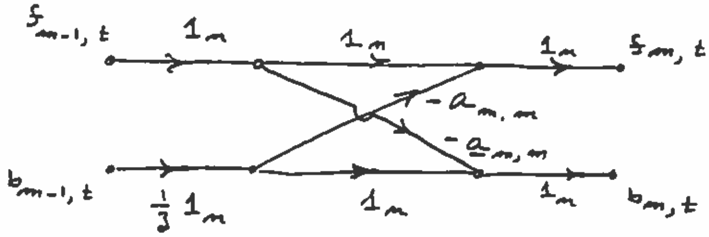

Figure 5 a)

with the zeroth stage, from (100c,d)

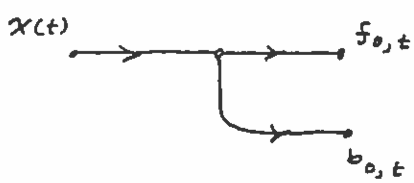

Figure 5 b)

Of interest are also relationships among the $a_{m,m}$ and $\underline{a}_{m,m}$.

Using $\underline{\Delta}_m = \Delta_m^T$, (80e), we obtain directly from (80a), $\Delta_m = a_{m+1,m+1}\underline{P}_m$, and (80b), $\underline{\Delta}_m = \underline{a}_{m+1,m+1} P_m$, along with $\underline{P}_m^T = \underline{P}_m$

$$\underline{a}_{m+1,m+1} = \underline{P}_m a^T_{m+1,m+1} P_m^{-1} \tag{110}$$

assuming all inverses exist equations (80) can also be written as

$$\underline{P}_m \Delta_m^{-1} a_{m+1,m+1} = a_{m+1,m+1} \underline{P}_m \Delta_m^{-1} = 1_m \tag{112a}$$

$$= P_m \underline{\Delta}_m^{-1} \underline{a}_{m+1,m+1} = \underline{a}_{m+1,m+1} P_m \underline{\Delta}_m^{-1} \tag{112b}$$

$$= \Delta_m^{-1} a_{m+1,m+1} \underline{P}_m = \underline{\Delta}_m^{-1} \underline{a}_{m+1,m+1} P_m \tag{112c}$$

which leads to the following relationships among forward and backward coefficients

$$\underline{a}_{m+1,m+1} = (\underline{\Delta}_m P_m^{-1} \underline{P}_m \Delta_m^{-1})\, a_{m+1,m+1} \tag{113a}$$

$$= a_{m+1,m+1} (\underline{P}_m \Delta_m^{-1} \underline{\Delta}_m P_m^{-1}) \tag{113b}$$

$$= (\underline{\Delta}_m P_m^{-1})\, a_{m+1,m+1} (\underline{P}_m \Delta_m^{-1}) \tag{113c}$$

$$= (\underline{\Delta}_m \underline{P}_m \Delta_m^{-1})\, a_{m+1,m+1} (P_m^{-1}) \tag{113d}$$

$$= (\Delta_m^{-1})\, a_{m+1,m+1} (\underline{P}_m \underline{\Delta}_m P_m^{-1}) \tag{113e}$$

III. Lattice Structure (follows Lin [LI 84])

At this point we can lay down the physical structures of interest.

We consider the $(2n+2n) = 4n$ terminal flow-graph for the mth stage

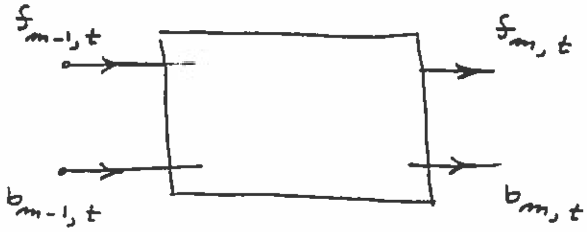

Figure 4

The governing equations are (100), repeated here

$$f_{m,t} = f_{m-1,t} - a_{m,m}\, b_{m-1,t-1} \qquad (120a)$$

$$b_{m,t} = -\underline{a}_{m,m}\, f_{m-1,t} + b_{m-1,t-1} \qquad (120b)$$

$$f_{0,t} = b_{0,t} = x(t) \qquad (120c,d)$$

Thus, the system flow-graph, of $n \times n$ matrix transmittances, is [with $\frac{1}{z}$ = unit delay in time]

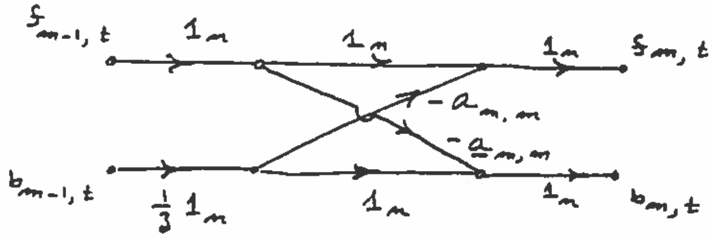

Figure 5 a)

with the zeroth stage, from (100c,d)

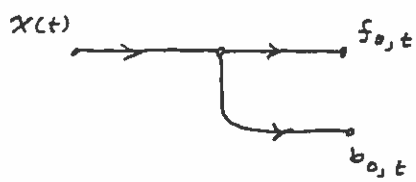

Figure 5 b)

By rewriting the equations (120) as

$$f_{m-1,t} = f_{m,t} + a_{m,m} b_{m-1,t-1} \qquad (130a)$$

$$b_{m,t} = -\underline{a}_{m,m} f_{m-1,t} + b_{m-1,t-1} \qquad (130b)$$

we have an equivalent structure

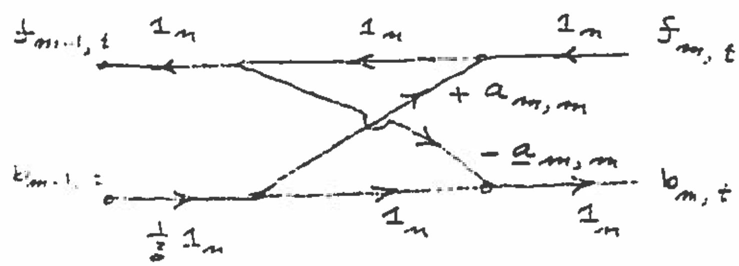

Figure 6 a)

where the zeroth stage is, again from (100c,d)

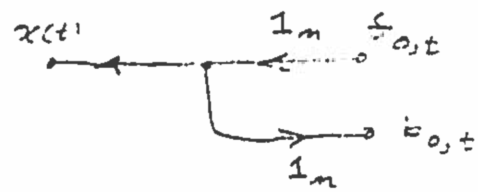

Figure 6 b)

For Figures 5 and 6 we have

$$\begin{bmatrix} f_{m,t} \\ b_{m,t} \end{bmatrix} = \begin{bmatrix} 1_n & -\frac{1}{2} a_{m,m} \\ -\underline{a}_{m,m} & \frac{1}{2} 1_n \end{bmatrix} \begin{bmatrix} f_{m-1,t} \\ b_{m-1,t} \end{bmatrix} \qquad (140a)$$

with

$$\begin{bmatrix} f_{0,t} \\ b_{0,t} \end{bmatrix} = \begin{bmatrix} 1_n \\ 1_n \end{bmatrix} x(t) \qquad (140b)$$

which is useful for cascading sections of the form of those in Fig. 5. For Fig. 5 we will choose

$$f_{0,t} = \text{input} = x, \quad f_{m,t} = \text{output} = y \qquad (140c)$$

For a form useful for cascading sections of the nature of Fig. 6 where $f_{m,t}$ and $b_{m-1,t}$ are inputs, we first substitute (130a) into (130b)

$$b_{m,t} = -\underline{a}_{m,m}\left[f_{m,t} + a_{m,m} b_{m-1,t-1}\right] + b_{m-1,t-1} \qquad (150a)$$

$$= -\underline{a}_{m,m} f_{m,t} + \left[1_n - \underline{a}_{m,m} a_{m,m}\right] b_{m-1,t-1} \qquad (150b)$$

This gives

$$\begin{bmatrix} f_{m-1,t} \\ b_{m,t} \end{bmatrix} = \begin{bmatrix} 1_n & \frac{1}{z} a_{m,m} \\ -\underline{a}_{m,m} & \frac{1}{z}(1_n - \underline{a}_{m,m} a_{m,m}) \end{bmatrix} \begin{bmatrix} f_{m,t} \\ b_{m-1,t} \end{bmatrix} \qquad (160a)$$

with

$$f_{0,t} = b_{0,t} \qquad (160b)$$

For Fig. 6 we will choose

$$f_{m,t} = \text{input} = u, \quad f_{0,t} = \text{output} = y \qquad (160c)$$

Therefore, for a cascade of the prediction error sections of Fig. 5 we multiply the coefficient matrices of (140) while for a cascade of the synthesis sections of Fig. 6 we multiply the coefficient matrices of (160). In the former case our overall transfer function is, see Fig. 1,

$$H_m(z) = \sum_{j=0}^{m} (-a_{m,j}) z^{-j} \tag{170a}$$

while for the synthesis structure the transfer function is

$$T_m(z) = H_m^{-1}(z) = \left[\sum_{j=0}^{m} (-a_{m,j}) z^{-j} \right]^{-1} \tag{170b}$$

Further, to obtain the $a_{m,j}$ in terms of the $a_{k,k}$ & $\underline{a}_{k,k}$ of the sections we can multiply the coefficient matrices of (140a), ignore $b_{m,t}$ (i.e. project onto the f space), and use (140b) for the zeroth section

$$f_{m,t} = [1, 0] \begin{bmatrix} 1_n & -\frac{1}{z} a_{m,m} \\ -\underline{a}_{m,m} & \frac{1}{z} 1_n \end{bmatrix} \cdots \begin{bmatrix} 1_n & -\frac{1}{z} a_{1,1} \\ -\underline{a}_{1,1} & \frac{1}{z} 1_n \end{bmatrix} \begin{bmatrix} 1_n \\ 1_n \end{bmatrix} x(t) \tag{180}$$

For small m this is:

$$f_{1,t} = [1_n - a_{1,1} z^{-1}] x \tag{190a}$$

$$f_{2,t} = [1, 0] \begin{bmatrix} 1_n + \frac{1}{z} a_{2,2} \underline{a}_{1,1} - \frac{1}{z} a_{1,1} - \frac{1}{z^2} a_{2,2} \\ -\underline{a}_{2,2} - \frac{1}{z} \underline{a}_{1,1} + \frac{1}{z} \underline{a}_{2,2} a_{1,1} + \frac{1}{z^2} 1_n \end{bmatrix} x(t)$$

$$= [1_n + (a_{2,2} \underline{a}_{1,1} - a_{1,1}) z^{-1} - a_{2,2} z^{-2}] x(t) \tag{190b}$$

$$f_{3,t} = [1, 0] \begin{bmatrix} 1_n + \frac{1}{z} a_{2,2} \underline{a}_{1,1} - \frac{1}{z} a_{1,1} - \frac{1}{z^2} a_{2,2} + \frac{1}{z} a_{3,3} \underline{a}_{2,2} + \frac{1}{z^2} a_{3,3} \underline{a}_{1,1} \\ - \frac{1}{z^2} a_{3,3} \underline{a}_{2,2} a_{1,1} - \frac{1}{z^3} a_{3,3} \\ -\underline{a}_{3,3} - \frac{1}{z} \underline{a}_{3,3} a_{2,2} \underline{a}_{1,1} + \frac{1}{z} \underline{a}_{3,3} a_{1,1} + \frac{1}{z^2} \underline{a}_{3,3} a_{2,2} \\ - \frac{1}{z} \underline{a}_{2,2} - \frac{1}{z} \underline{a}_{1,1} + \frac{1}{z} \underline{a}_{2,2} a_{1,1} + \frac{1}{z^3} 1_n \end{bmatrix} x(t)$$

$$= [1_n + (a_{3,3} \underline{a}_{2,2} + a_{2,2} \underline{a}_{1,1} - a_{1,1}) z^{-1} + (-a_{3,3} \underline{a}_{2,2} a_{1,1} + a_{3,3} \underline{a}_{1,1} - a_{2,2}) z^{-2} - a_{3,3} z^{-3}] x(t) \tag{190c}$$

IV. Normalization. (follows Morf [MO78])

To proceed to obtain a means for calculating the a's we consider a normalization that will erase the differences between the a's and the $\underline{a}$'s. Thus, first factor the positive semi-definite P_m's and $\underline{P}_m$'s, which we now assume nonsingular,

$$P_m = p_m^T p_m \tag{200a}$$

$$\underline{P}_m = \underline{p}_m^T \underline{p}_m \tag{200b}$$

Then we have, by (20e,f)

$$(p_m^T)^{-1} E[f_{m,t} f_{m,t}^T](p_m)^{-1} = 1_n \tag{200c}$$

$$(\underline{p}_m^T)^{-1} E[b_{m,t} b_{m,t}^T](\underline{p}_m)^{-1} = 1_n \tag{200d}$$

which on pulling the outer matrices inside the E[ ] leads us to define

$$\mathcal{F}_{m,t} = (p_m^T)^{-1} f_{m,t} \tag{201a}$$

$$\mathcal{B}_{m,t} = (\underline{p}_m^T)^{-1} b_{m,t} \tag{201b}$$

Next define

$$\alpha_{m+1} = (p_m^T)^{-1} \Delta_m (\underline{p}_m)^{-1} \tag{202}$$

which, by (80e,j,k) leads to

$$a_{m+1,m+1} = \Delta_m \underline{P}_m^{-1} = p_m^T \alpha_{m+1} (\underline{p}_m^T)^{-1} \tag{203a}$$

$$\underline{a}_{m+1,m+1} = \underline{\Delta}_m P_m^{-1} = \Delta_m^T P_m^{-1} = (\underline{p}_m^T)(\underline{p}_m^{-T} \Delta_m^T p_m^{-1})(p_m^T)^{-1}$$
$$= \underline{p}_m^T \alpha_{m+1}^T (p_m^T)^{-1} \tag{203b}$$

With these choices the normalized prediction errors become

$$\mathcal{F}_{m,t} = (p_m^T)^{-1}\left[p_{m-1}^T \mathcal{F}_{m-1,t} - a_{m,m}\, \underline{p}_{m-1}^T \mathcal{B}_{m-1,t-1} \right]$$

$$= (p_m^T)^{-1}(p_{m-1}^T)\left[\mathcal{F}_{m-1,t} - (p_{m-1}^T)^{-1} a_{m,m} (\underline{p}_{m-1}^T) \mathcal{B}_{m-1,t-1} \right]$$

$$= (p_m^T)^{-1}(p_{m-1}^T)\left[\mathcal{F}_{m-1,t} - \alpha_m \mathcal{B}_{m-1,t-1} \right] \qquad (204a)$$

$$\mathcal{B}_{m,t} = (\underline{p}_m^T)^{-1}\left[-\underline{a}_{m,m} (p_{m-1}^T) \mathcal{F}_{m-1,t} + (\underline{p}_{m-1}^T) \mathcal{B}_{m-1,t-1} \right]$$

$$= (\underline{p}_m^T)^{-1}(\underline{p}_{m-1}^T)\left[-(\underline{p}_{m-1}^T)^{-1} \underline{a}_{m,m} (p_{m-1}^T) \mathcal{F}_{m-1,t} + \mathcal{B}_{m-1,t-1} \right]$$

$$= (\underline{p}_m^T)^{-1}(\underline{p}_{m-1}^T)\left[-\alpha_m^T \mathcal{F}_{m-1,t} + \mathcal{B}_{m-1,t-1} \right] \qquad (204b)$$

Here, as desired, the same term, α_m, enters into both expressions.

By straightforward calculation we have, for (82a,b)

$$P_{m+1} = p_m^T \left[1_n - \alpha_{m+1} \alpha_{m+1}^T \right] p_m \qquad (205a)$$

$$\underline{P}_{m+1} = \underline{p}_m^T \left[1_n - \alpha_{m+1}^T \alpha_{m+1} \right] \underline{p}_m \qquad (205b)$$

which leads us to let

$$\mathcal{P}_{m+1} = (p_m^T)^{-1} P_{m+1} (p_m)^{-1} \qquad (206a)$$

$$\underline{\mathcal{P}}_{m+1} = (\underline{p}_m^T)^{-1} \underline{P}_{m+1} (\underline{p}_m)^{-1} \qquad (206b)$$

giving

$$\mathcal{P}_{m+1} = 1_n - \alpha_{m+1} \alpha_{m+1}^T \qquad (206c)$$

$$\underline{\mathcal{P}}_{m+1} = 1_n - \alpha_{m+1}^T \alpha_{m+1} \qquad (206d)$$

From (206) we see that the eigenvalues of $\alpha_i \alpha_i^T$, and hence of α_i, are less than 1 in magnitude.

I. Choice of Coefficients with Real Data – Anderson Extension
(follows Andersen [AN74])

Next we turn to the situation of experimental data. In this case the matrices $R(\cdot)$ are not fully available and what we wish to do is construct them using a method like Burg's where first we determine the $a_{m,m}$ and $\underline{a}_{m,m}$ and then $R(m)$, P_m to preserve the properties desired.

Toward this we assume that everything has been determined for the $m-1$st step, $m>0$, and we wish to find quantities for the mth step. Thus, we primarily wish to find $a_{m,m}$ and $\underline{a}_{m,m}$ for which we return to the normalization of (201a). Define, following Andersen

$$\Pi_m = \frac{1}{2}\,\frac{1}{N-m}\sum_{t=1}^{N+m}\left\{ f_{m,t}^T P_{m-1}^{-1} f_{m,t} + b_{m,t}^T \underline{P}_{m-1}^{-1} b_{m,t}\right\} \quad (210$$

where N is the length of samples available and Π_m is normalized power (output) in the predictive errors (normalized by input power [coming out of the $(m-1)$st stage])

We wish to express this as a function of α_m and then minimize it with respect to α_m.

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We have, using (201) & (204)

$$f_{m,t}^T P_{m-1}^{-1} f_{m,t} + b_{m,t}^T \underline{P}_{m-1}^{-1} b_{m,t}$$

$$= \mathcal{F}_{m,t}^T (\rho_m) P_{m-1}^{-1} (\rho_m^T) \mathcal{F}_{m,t} + \mathcal{B}_{m,t}^T (\underline{\rho}_m) \underline{P}_{m-1}^{-1} (\underline{\rho}_m^T) \mathcal{B}_{m,t}$$

$$= [\mathcal{F}_{m-1,t} - \alpha_m \mathcal{B}_{m-1,t-1}]^T [\mathcal{F}_{m-1,t} - \alpha_m \mathcal{B}_{m-1,t-1}]$$
$$+ [-\alpha_m^T \mathcal{F}_{m-1,t} + \mathcal{B}_{m-1,t-1}]^T [-\alpha_m^T \mathcal{F}_{m-1,t} + \mathcal{B}_{m-1,t-1}] \quad (211)$$

Differentiating Π_m with respect to the entries of α_m and setting equal to zero yields

$$\alpha_m \left[\sum_{t=m}^{N} (\mathcal{B}_{m-1,t-1} \mathcal{B}_{m-1,t-1}^T)\right] + \left[\sum_{t=m}^{N} (\mathcal{F}_{m-1,t} \mathcal{F}_{m-1,t}^T)\right] \alpha_m = 2 \left[\sum_{t=m}^{N} (\mathcal{F}_{m-1,t} \mathcal{B}_{m-1,t-1}^T)\right] \quad (212)$$

which is a linear equation to be solved for α_m to minimize Π_m, given the $\mathcal{F}_{m-1,t}$, $\mathcal{B}_{m-1,t-1}$.

Demonstration of (212):

Setting the partial derivatives of Π_m with respect to entries of α_m equal to zero is the same as setting the sum of partial derivatives of (211) to zero. For this we shorten notation and consider one term in the sum, written as

$$(212_1) \qquad \Pi = (\mathcal{F} - \alpha\mathcal{B})^T (\mathcal{F} - \alpha\mathcal{B}), \qquad \alpha = [\alpha_{ij}]$$

Then

$$(212_2) \qquad \frac{\partial \Pi}{\partial \alpha_{ij}} = -\mathcal{B}_j (\mathcal{F} - \alpha\mathcal{B})_i - (\mathcal{F} - \alpha\mathcal{B})_i^T \mathcal{B}_j$$

where the subscripts indicate the matrix entry.

Now

$$(212_3) \quad (\alpha \mathcal{B})^T_i = [\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{im}] \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \\ \vdots \\ \mathcal{B}_m \end{bmatrix} = (\alpha \mathcal{B})_i$$

and, hence, (212_2) is

$$(212_4) \quad \frac{\partial \Pi}{\partial \alpha_{ij}} = -2\, \mathcal{F}_i \cdot \mathcal{B}_j + 2\, [\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{im}] \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \\ \vdots \\ \mathcal{B}_m \end{bmatrix} \mathcal{B}_j$$

Put into matrix form, this is

$$(212_5) \quad \left[\frac{\partial \Pi}{\partial \alpha_{ij}}\right] = -2 \begin{bmatrix} \mathcal{F}_1 \mathcal{B}_1 & \mathcal{F}_1 \mathcal{B}_2 & \cdots & \mathcal{F}_1 \mathcal{B}_m \\ \mathcal{F}_2 \mathcal{B}_1 & \mathcal{F}_2 \mathcal{B}_2 & & \mathcal{F}_2 \mathcal{B}_m \\ \vdots & & & \vdots \\ \mathcal{F}_m \mathcal{B}_1 & \cdots & & \mathcal{F}_m \mathcal{B}_m \end{bmatrix} + 2 \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \cdots & \vdots \\ \vdots & & & \vdots \\ \alpha_{m1} & & \cdots & \alpha_{mm} \end{bmatrix} \begin{bmatrix} \mathcal{B}_1 \mathcal{B}_1 & \mathcal{B}_1 \mathcal{B}_2 & \cdots \\ \vdots & & \\ \mathcal{B}_m \mathcal{B}_1 & \cdots & \mathcal{B}_m \mathcal{B}_m \end{bmatrix}$$

$$(212_6) \quad = -2\, \mathcal{F} \mathcal{B}^T + 2\, \alpha \mathcal{B} \mathcal{B}^T.$$

Similarly, for

$$(212_7) \quad \frac{\partial}{\partial \alpha_{ij}} (-\alpha^T \mathcal{F} + \mathcal{B})^T (-\alpha^T \mathcal{F} + \mathcal{B}) = -2\, \mathcal{F}_i \cdot \mathcal{B}_j + 2\, \mathcal{F}_i\, (\alpha^T \mathcal{F})_j$$

we have

$$(212_8) \quad \mathcal{F}_i\, (\alpha^T \mathcal{F})_j = \mathcal{F}_i \cdot [\alpha_{1j}, \alpha_{2j}, \ldots, \alpha_{mj}] \begin{bmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ \vdots \\ \mathcal{F}_m \end{bmatrix}$$

or

$$(212_9) \quad [-2\, \mathcal{F}_i \mathcal{B}_j + 2\, \mathcal{F}_i \cdot (\alpha^T \mathcal{F})_j] = -2\, \mathcal{F} \mathcal{B}^T + 2\, \mathcal{F} \mathcal{F}^T \alpha$$

Combining (212_6) & (212_9), inserting the sum on t of Π_m and setting equal to zero yields (212).

Demonstration of at a minimum

From (212_4) & (212_8) we have

$$\frac{\partial^2\left[(\mathcal{F}-\alpha\mathcal{B})^2(\mathcal{F}-\alpha\mathcal{B})+(-\alpha^T\mathcal{F}+\mathcal{B})^T(-\alpha^T\mathcal{F}+\mathcal{B})\right]}{\partial\alpha_{ij}\,\partial\alpha_{kl}}$$

(212_{10}) $$= 2\,\delta_{ik}\,\mathcal{B}_{k}\mathcal{B}_{j} + 2\,\delta_{jl}\,\mathcal{F}_{i}\mathcal{F}_{k}$$

where $\delta_{ik} = \begin{cases} 0 & i \neq k \\ 1 & \text{if } i = k \end{cases}$. This has

(212_{11}) $$\frac{\partial^2}{\partial\alpha_{ij}^2} = 2\,\mathcal{B}_j^2 + 2\,\mathcal{F}_i^2$$

or

(212_{12}) $$\frac{\partial^2 \Pi_m}{\partial\alpha_{ij}^2} > 0$$

which shows the solution to (212) minimizes, rather than maximizes, Π_m.

The solution for α_m is unique by:
A. Feintuch & M. Rubin, "The Matrix Equation AX−XB=C," The American Mathematical Monthly, Vol. 91, No. 8, October 1984, pp. 506–507.

Here, α_m is formed using only previous data. Once we have it then we can form the new P_m & $\underline{P}_m$ using (206) and (205). We are guaranteed a stable filter if we can show the P_m and $\underline{P}_m$ are positive semidefinite or that α_m has singular less than one.

But this latter is true as shown next. First we recall that the singular values of a (real) matrix a are the positive square roots of the eigenvalues of $a^T a$, which are equal to those of $a a^T$ for a square; these are what are needed to show P_m and $\underline{P}_m$ are positive definite, by (205).

Demonstration of singular values of $\alpha_m < 1$

Any matrix has a singular value decomposition. Writing that for α_m as

(212$_{10}$) $$\alpha_m = U S V$$

we have V and V orthogonal and S diagonal with its diagonal entries the singular values or zero and ordered such that the (1,1) entry, s_1, is the largest singular value.

[For singular value decomposition see:
F. M. Callier & C. A. Desoer, "Multivariable Feedback Systems," Springer-Verlag, 1982, pp. 2-10.]

We next insert (212_{10}) in (212) and premultiply by $U^T = U^{-1}$ & postmultiply by $V^T = V^{-1}$ to get

(212_{11}) $$S\left[\sum_{t=m}^{NMM}\left(V\mathcal{B}_{m-1,t-1}\cdot\{V\mathcal{B}_{m-1,t-1}\}^T\right)\right] + \left[\sum_{t=m}^{NMM}\left(U^T\mathcal{F}_{m-1,t}\cdot\{U^T\mathcal{F}_{m-1,t}\}^T\right)\right]S = 2\left[\sum_{t=m}^{NMM}\left(U^T\mathcal{F}_{m-1,t}\cdot\{V\mathcal{B}_{m-1,t-1}\}^T\right)\right]$$

Now premultiply by the n-vector

(212_{12}) $$\xi^T = [1, 0, \ldots, 0]$$

and postmultiply by its transpose to get

(212_{13}) $$s_1 = 2\,\frac{\sum_{t=m}^{NMM}(\xi^T U^T\mathcal{F}_{m-1,t})(\mathcal{B}^T_{m-1,t-1}V^T\xi)}{\sum_{t=m}^{NMM}\left[(\xi^T V\mathcal{B}_{m-1,t-1})(\xi^T V\mathcal{B}_{m-1,t-1})^T + (\xi^T U^T\mathcal{F}_{m-1,t})(\xi^T U^T\mathcal{F}_{m-1,t})^T\right]}$$

The right hand side is less (or equal) to 1 since, for every term under the summations, we apply the inequality

(212_{14}) $$(y-x)^T(y-x) = y^Ty^T + x^Tx - 2x^Ty \geq 0$$

Equality in (212_{14}) holds if and only if $y = x$, that is

(212_{15}) $$s_1 = 1 \text{ iff } \mathcal{B}^T_{m-1,t-1}V^T\xi = \mathcal{F}^T_{m-1,t}U\xi$$

which for experimental data would scarcely occur.

Given α_m, found using (212) we can then up-date P_m and $\underline{P}_m$ to P_{m+1} and $\underline{P}_{m+1}$ via (82a,b) while also updating to $f_{m+1,t}$ and $b_{m+1,t}$ via (100a,b). This also gives an implied up-date to $R(m+1)$ via (50). Hence, the cycle can be repeated. The starting conditions would be

$$P_0 = \underline{P}_0 = \frac{1}{N}\sum_{t=1}^{N} x(t)x^T(t) = R_d(0) \tag{213a}$$

$$f_{0,t} = b_{0,t} = x(t). \tag{213b}$$

Note that the data should be such that P_0 is positive definite. Here $R_d(0)$, perhaps best called a data $R(0)$, will replace $R(0)$ as used in the previous sections.

At each step we obtain a positive definite P_m which guarantees, following Burg (see two sections below) a stable transfer function $T_m(z)$.

VI. How to Use the above.

Given an n-vector signal $x(t)$ for $-N \le t \le N$ we

1. Form (from (213), (30), (10c, f))

(214a) $$P_0 = \underline{P}_0 = \frac{1}{N}\sum_{t=1}^{N} x(t)x^T(t) \triangleq R_d(0)$$ (a data $R(0)$ but to be used like a true $R(0)$)

(214b) $$a_{0,0} = \underline{a}_{0,0} = -1_n$$

(214c) $$f_{0,t} = b_{0,t} = x(t)$$

2) Factor (from (200))

(214d) $$P_0 = \underline{P}_0 = \rho_0^T \rho_0 = \underline{\rho}_0^T \rho_0$$

3) Form (from 201)

(214e) $$\mathcal{F}_{0,t} = (\rho_0^T)^{-1} f_{0,t}$$

(214f) $$\mathcal{B}_{0,t} = (\underline{\rho}_0^T)^{-1} b_{0,t}$$

4. Solve the linear equation for α_1 (from (212))

(214g) $$\alpha_1\left[\sum_{t=1}^{N} \mathcal{B}_{0,t-1}\mathcal{B}_{0,t-1}^T\right] + \left[\sum_{t=1}^{N} \mathcal{F}_{0,t}\mathcal{F}_{0,t}^T\right]\alpha_1 = 2\left[\sum_{t=1}^{N} \mathcal{F}_{0,t}\mathcal{B}_{0,t-1}^T\right]$$

5. Form (from (203))

(214h) $$a_{1,1} = \rho_0^T \alpha_1 (\underline{\rho}_0^T)^{-1}$$

(214i) $$\underline{a}_{1,1} = \underline{\rho}_0^T \alpha_1^T (\rho_0^T)^{-1}$$

which go in the circuit of Figs. 5 or 6.

6. Start a new cycle by forming (from (205) and factoring (from (200))

(214j) $$P_1 = \rho_0^T[1_n - \alpha_1\alpha_1^T]\rho_0 = \rho_1^T\rho_1$$

(214k) $$\underline{P}_1 = \underline{\rho}_0^T[1_n - \alpha_1^T\alpha_1]\underline{\rho}_0 = \underline{\rho}_1^T\underline{\rho}_1$$

and from (100)

(214l) $$f_{1,t} = f_{0,t} - a_{1,1}\, b_{0,t-1}$$

(214m) $$b_{1,t} = -\underline{a}_{1,1}\, f_{0,t} + b_{0,t-1}$$

giving

(214n) $$\mathcal{F}_{1,t} = (\rho_1^T)^{-1} f_{1,t}$$

(214o) $$\mathcal{B}_{1,t} = (\underline{\rho}_1^T)^{-1} b_{1,t}$$

for forming α_2 [~~[illegible] $\mathcal{B}_{1,0}$ [illegible] $b_{0,-1}$ [illegible]~~]

7. Continue; for the cycle in going from $m-1$ to m we form

(215a) $$P_{m-1} = \rho_{m-2}^T\left[1_n - \alpha_{m-1}\alpha_{m-1}^T\right]\rho_{m-2} = \rho_{m-1}^T\,\rho_{m-1}$$

(215b) $$\underline{P}_{m-1} = \underline{\rho}_{m-2}^T\left[1_n - \alpha_{m-1}^T\alpha_{m-1}\right]\underline{\rho}_{m-2} = \underline{\rho}_{m-1}^T\,\underline{\rho}_{m-1}$$

(215c) $$f_{m-1,t} = f_{m-2,t} - a_{m-1,m-1}\, b_{m-2,t-1}$$

(216d) $$b_{m-1,t} = -\underline{a}_{m-1,m-1}\, f_{m-2,t} + b_{m-2,t-2}$$

(216e) $$\mathcal{F}_{m-1,t} = \left(\rho_{m-1}^T\right)^{-1} f_{m-1,t}$$

(216f) $$\mathcal{B}_{m-1,t} = \left(\underline{\rho}_{m-1}^T\right)^{-1} b_{m-1,t}$$

and solve

(216g) $$\alpha_m\left[\sum_{t=m}^{NMAX}\mathcal{B}_{m-1,t-1}\,\mathcal{B}_{m-1,t-1}^T\right] + \left[\sum_{t=m}^{NMAX}\mathcal{F}_{m-1,t}\,\mathcal{F}_{m-1,t}^T\right]\alpha_m = 2\left[\sum_{t=m}^{NMAX}\mathcal{F}_{m-1,t}\,\mathcal{B}_{m-1,t-1}^T\right]$$

to obtain

(216h) $$a_{m,m} = \rho_{m-1}^T\,\alpha_m\left(\underline{\rho}_{m-1}^T\right)^{-1}$$

(216i) $$\underline{a}_{m,m} = \underline{\rho}_{m-1}^T\,\alpha_m^T\left(\rho_{m-1}^T\right)^{-1}$$

8. Stops at steps M where M is either predetermined or determined by some criteria of the user.

We note the following.

a) There is nothing statistical about the procedure, except possibly the choice of the stopping step M. In setting up the procedure statistical quantities, such as $R(j)$, were used, but in the end only nonstatistical properties of these quantities were used, such as the positive definiteness of $R(0)$.

b) t can actually be real or sampled (discretized) time. Thus, the implementation can be in terms of analog structures, using delay lines, or digital structures, using digital delays.

c) The choice of stopping size M is of concern. One would like the unit pulse (or unit impulse) response of the digital (or analog) synthesis structure to reproduce $x(t)$. Or, looked at in terms of the prediction error structure, one desires $\xi_{M,t}$ to be a unit pulse with $x(t)$ as the input. Thus, one can set limits on $\xi_{M,t}$ to be close to what is desired. If one were to actually excite the synthesis structure with the $\xi_{M,t}$ resulting from the prediction error filter (of input $x(t)$) then $x(t)$ would be the output (of the synthesis structure) [since at (170) $H_M^{-1} = T_M$]. Thus, using $\xi_{M,t}$ rather than a unit pulse would give a much better "synthesis," if one had the freedom to choose $\xi_{M,t}$ as the input.

VII. Demonstration That $T_M(z)$ is Stable and Maximum Entropy Extension

Here it is shown that the technique presented yields a transfer function $T_M(z)$ having all its poles within $|z| < 1$. The proof is essentially that of Burg [BU75, pp. 85-8] with a few details filled in.

Given P_m & $\underline{P}_m$, formed as at (215b), we also form from (50a), the "data" $R(m)$, called $R_d(m)$ by

$$R_d(m)\, a^T_{m,m} = -P_m + \sum_{j=0}^{m-1} R_d(j)\, a^T_{m,j} \qquad (217a)$$

where the $a^T_{m,j}$ are obtained from (70c)

$$a^T_{m,j} = a^T_{m-1,j} - \underline{a}^T_{m-1,m-j}\, a^T_{m,m} \qquad (217b)$$

that is, $a^T_{m,j}$ and $R_d(m)$ are formed in terms of quantities already obtained. We also set, to satisfy (40b) and (70d),

$$R_d(-m) = R_d^T(m) \qquad (217c)$$

$$\underline{a}^T_{m,j} = a^T_{m-1,j} - a^T_{m-1,m-j}\, \underline{a}^T_{m,m} \qquad (217d)$$

in which case (50) and $(80_1)-(80_{10})$ hold with $R(j)$ replaced by $R_d(j)$ (found recursively via (217a)). We have for $m \le M$,

$$(217_1) \qquad \begin{bmatrix} R_d(0) & R_d(1) & \cdots & R_d(m) \\ R_d(-1) & R_d(0) & & \vdots \\ \vdots & & \ddots & \\ R_d(-m) & & \cdots & R_d(0) \end{bmatrix} \begin{bmatrix} 1_n \\ -a^T_{m,1} \\ \vdots \\ -a^T_{m,m} \end{bmatrix} = \begin{bmatrix} P_m \\ 0_n \\ \vdots \\ 0_n \end{bmatrix}$$

m

$$\begin{bmatrix} 1_n & -a_{m,1} & \cdots & -a_{m,m} \\ 0_n & 1_n & 0_n \cdots & 0_n \\ & 0_n & \ddots & \vdots \\ & & & 0_n \\ 0_n & 0_n & \cdots & 0_n \; 1_n \end{bmatrix} \begin{bmatrix} R_d(0) & R_d(1) & \cdots & R_d(m) \\ R_d(-1) & R_d(0) & & \vdots \\ \vdots & & & \\ R_d(-m) & & \cdots & R_d(0) \end{bmatrix} \begin{bmatrix} 1_n & 0_n & \cdots & 0_n \\ -a^T_{m,1} & 1_n & & 0_n \\ \vdots & 0_n & \ddots & \vdots \\ -a^T_{m,m} & 0_n & \cdots 0_n & 1_n \end{bmatrix}$$

$$(217_2) \qquad = \begin{bmatrix} P_m & c_n \cdot & 0_n \\ 0_n & R_d(0) & \cdots & R_d(m-1) \\ \vdots & \vdots & & \vdots \\ 0_n & R_d(-m+1) & \cdots & R_d(0) \end{bmatrix}$$

or, repeating for $m-1, \ldots, 0$, we have

$$\begin{bmatrix} 1_n & -a_{m,1} & -a_{m,2} & \cdots & -a_{m,m} \\ 0_n & 1_n & -a_{m-1,1} & \cdots & -a_{m-1,m-1} \\ \vdots & & 1_n & \ddots & \\ & & & & -a_{1,1} \\ 0_n & & & 0_n & 1_n \end{bmatrix} \begin{bmatrix} R_d(0) & R_d(1) & \cdots & R_d(m) \\ R_d(-1) & R_d(0) & & \vdots \\ \vdots & & & \vdots \\ R_d(-m) & & \cdots & R_d(0) \end{bmatrix} \begin{bmatrix} 1_n & 0_n & 0_n & \cdots & 0_n \\ -a^T_{m,1} & 1_n & & & \\ \vdots & -a^T_{m-1,1} & \ddots & & \vdots \\ & \vdots & & & 0_n \\ -a^T_{m,m} & -a^T_{m-1,m-1} & & & 1_n \end{bmatrix}$$

$$(217_3) \qquad = \begin{bmatrix} P_m & & & \\ & P_{m-1} & & \\ & & \ddots & \\ & & & P_0 \end{bmatrix}$$

which shows that the matrix of $_dR(j)$'s is positive definite since the P_j's are, by our choice of the $a_{m,m}$ and our choice of $R_d(m)$ to fill in the matrix of the $R_d(j)$'s, this being true for $m \le M$, M being the number of actual lattice sections.

For $m > M$ we can consider that

$$a_{m,m} = \underline{a}_{m,m} = 0_n, \quad m > M \tag{218a}$$

in which case, by $(80j, k)$ we will force

$$\Delta_m = \underline{\Delta}_m = 0_n \quad \text{for } m \geq M \tag{218b}$$

which further gives us an extension of $R_d(j)$ to $j = M+1$, via (80c), as

$$R_d(M+1) = \sum_{j=1}^{M} (-a_{M,j}) R(M+1-j) \tag{218c}$$

Forming

$$\begin{bmatrix} 1_m & -a_{M,1} & \cdots & -a_{M,M} & 0_m \\ & 1_m & 0_m & \cdots & 0_m \\ & & \ddots & & \\ & 0 & & & 0_m \\ & & & & 1_m \end{bmatrix} \begin{bmatrix} R_d(0) & \cdots & R_d(M) & R_d(M+1) \\ R_d(-1) & & & \vdots \\ \vdots & & & \\ R_d(-M) & & & \\ R_d(-M-1) & & & R_d(0) \end{bmatrix} \begin{bmatrix} 1_m & & & & \\ -a_{M,1}^T & 1_m & & 0 & \\ \vdots & 0_m & & \ddots & \\ -a_{M,M}^T & \vdots & & & \\ 0_m & 0_m & \cdots & 0_m & 1_m \end{bmatrix}$$

$$= \begin{bmatrix} P_M & 0_m & \cdots & 0_m \\ 0_m & R_d(0) & \cdots & R_d(M) \\ \vdots & \vdots & & \vdots \\ 0_m & R_d(-M) & \cdots & R_d(0) \end{bmatrix} \tag{218d}$$

shows the augmented (to M+1) block Toeplitz matrix is also positive definite. Note that, if M is replaced by $m < M$ in (218d), then Δ_m terms appear in the upper right & lower left corners.

Note that (218c) is

$$\sum_{j=0}^{M} (-a_{M,j}) R_d(s-j) = 0_m \quad , \quad s = M+1 \tag{219a}$$

where by (50a) this also holds for $s = 1, \ldots, M$.

Burg's maximum entropy extension is

equation (219a) extended to all $s>0$. Thus, we define the $R_d(j)$, $j>M$ via

$$\sum_{j=0}^{M}(-a_{M,j})R_d(s-j)=0_n\quad,\quad s>0 \tag{219b}$$

By the argument used at (218d) the block Toeplitz matrix of R_d's to any size will be positive definite. For $s<0$ the left of (219b) is generally not zero. If we call it $A(s)$, i.e.

$$\sum_{j=0}^{M}(-a_{M,j})R_d(s-j)=A(s) \tag{219c}$$

then

$$A(0)=P_M \tag{219d}$$

$$A(s)=0_n \quad \text{for } s>0 \tag{219e}$$

Next we switch to a z-transform type of formulation in order to be able to work with the transfer function. We define, in the standard z-transform way

$$R_d(z)=\sum_{k=-\infty}^{\infty}R_d(k)z^{-k} \tag{220a}$$

$$a(z)=\sum_{k=-\infty}^{\infty}A(k)z^{-k} \tag{220b}$$

while we have, from (170a)

$$H_M(z)=\sum_{j=0}^{M}(-a_{M,j})z^{-j} \tag{220c}$$

Equation (219c) is a discrete convolution which in z-transforms becomes

$$H_M(z) \cdot R_d(z) = \mathcal{A}(z) \tag{230a}$$

or

$$R_d(z) = H_M^{-1}(z)\,\mathcal{A}(z) = T_M(z)\,\mathcal{A}(z) \tag{230b}$$

Thus, $A(z)$ is a (matrix set of) inputs to $T_M(z)$ and $R_d(z)$ the outputs. The inputs stop at (discrete time) zero while the outputs continue on forever (note that (230b) shows that $A(z)$ is nonzero). Now, since the $R_d(z)$ remain bounded as z increases (because, as per (215d), these enter into nondiagonal entries of positive definite matrices and, hence, can not exponentially increase), $T_M(z)$ has no poles in $|z|>1$.

To see that $T_M(z)$ has no poles on $|z|=1$ we note that at a pole of $T_M(z)$, say z_p, there is a vector h in the null space of $H_M(z_p)$, i.e. (here $\underline{0}$ is the zero n-vector)

$$H_M(z_p)\, h = \underline{0} \qquad (240a)$$

This can be rewritten as

$$[1_n, -a_{M,1}, \ldots, -a_{M,M}] \begin{bmatrix} h \\ z_p^{-1} h \\ \vdots \\ z_p^{-M} h \end{bmatrix} = \underline{0} \qquad (240b)$$

or, on transposing & conjugating (as h & z_p may be complex)

$$[\bar{h}^T \;\; \bar{z}_p^{-1}\bar{h}^T, \ldots, \bar{z}_p^{-M}\bar{h}^T] \begin{bmatrix} 1_n \\ -a_{M,1}^T \\ \vdots \\ -a_{M,M}^T \end{bmatrix} = \underline{0}^T \qquad (240c)$$

This in turn gives

$$\begin{bmatrix} h \\ z_p^{-1} h \\ \vdots \\ z_p^{-M} h \end{bmatrix} [\bar{h}^T, \bar{z}_p^{-1}\bar{h}^T, \ldots, \bar{z}_p^{-M}\bar{h}^T] \begin{bmatrix} 1_n \\ -a_{M,1}^T \\ \vdots \\ -a_{M,M}^T \end{bmatrix} = \underline{0} \qquad (240d)$$

Because $|z_p|=1$ the z_p's cancel on the diagonal blocks of the left (product) matrix which becomes Toeplitz. We then subtract it, multiplied by a real scalar α, from the $R_d(\cdot)$ matrix to get

$$\left\{\begin{bmatrix} R_d(0) & \cdots & R_d(M) \\ \vdots & & \vdots \\ R_d(-M) & \cdots & R_d(0) \end{bmatrix} - \alpha \begin{bmatrix} h \\ z_p^{-1}h \\ \vdots \\ z_p^{-M}h \end{bmatrix}\left[\bar{h}^T, z_p\bar{h}^T, \ldots, z_p^M\bar{h}^T\right]\right\}\begin{bmatrix} 1_m \\ -a_{M,1}^T \\ \vdots \\ -a_{M,M}^T \end{bmatrix} = \begin{bmatrix} P_M \\ 0_m \\ \vdots \\ 0_m \end{bmatrix} \quad (2$$

Therefore, for any α, the $a_{M,j}$, P_M, satisfy this equation (250). From the previous we know that such an equation satisfied by the $a_{M,j}$, P_M with a Toeplitz coefficient matrix must have that coefficient matrix positive definite. But for large enough α $R_d(0) - \alpha h\bar{h}^T$ is not positive definite unless $h = \underline{0}$. Thus, $h = 0$ or the null space of $H_M(z_p)$ is the zero vector in which case $H_M(z_p)$ is invertible at z_p, i.e. there are no poles for $|z_p| = 1$.

The above arguments can be applied to termination of the filter at any stage m, i.e. any stage can be chosen final to show that $T_m(z)$ is analytic in $|z| \geq 1$ for all $m = 1, \ldots, M$.

We should also be able to use scattering matrix network theory to accomplish these results [DE81] but as yet I have not carried this out.

From (230a) we can derive the Maximum Entropy extension also solely in terms of P_M and T_M. Thus, by replacing z by $1/z$ we have

$$H_M(1/z)\, R_d(1/z) = a(1/z) \qquad (260a)$$

But from the series for $R_d(z)$, (220a), and the "asymmetry" of R_d, (217c), having $R_d(-j) = R_d^T(j)$, (260a) becomes on transposing

$$R_d^T(1/z)\, H_M^T(1/z) = R_d(z)\, H_M^T(1/z) = a^T(1/z) \qquad (260b)$$

Now multiply by $H_M(z)$ on the left to get

$$H_M(z)\, R_d(z)\, H_M^T(1/z) = P_M \qquad (260c)$$

since

$$H_M(z)\, a(1/z) = \left[\sum_{j=0}^{M} (-a_{M,j})\, z^{-j}\right] \sum_{k=0}^{\infty} \left(\sum_{l=0}^{M} [R_d^T(k-l)(-a_{M,l}^T)\left(\frac{1}{z}\right)^{k}\right)$$

$$= \sum_{j=0}^{M}\sum_{k=0}^{\infty}\sum_{l=0}^{M} a_{M,j}\, R_d(l-k)\, a_{M,l}^T\, z^{-(j+k)}$$

$$= \sum_{m=k+j=0}^{\infty}\ \sum_{l=0}^{M}\sum_{j=0}^{M} a_{M,j}\, R_d(l+m-j)\, a_{M,l}^T\, z^{-m} \qquad (260c)$$

$$= P_M$$

as $\sum_{j=0}^{M} a_{M,j}\, R_d(l+m-j) = 0$ for $l+m>0$ by (219b).

Hence, from (260c),

$$R_d(z) = T_M(z)\, P_M \left(T_M^T(1/z)\right)^{-1} \qquad (261)$$

gives all $R_d(j)$.

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