On Sensitivity in Semistate Described Linear Systems

B. Dziurla and R.W. Newcomb

Microsystems Laboratory
Electrical Engineering Department
University of Maryland
College Park, Maryland 20742
USA

Phone: (301) 454-6869

Supported in part by NSF Grant ECS-85-06924, and the US-Spain Joint Committee for Scientific and Technological Cooperation, Grant CCB-84-002-002.

Semistate Equations in Canonical Form:

$$\mathcal{A}\dot{x} + \mathcal{B}(x,t) = \mathcal{D}u$$
$$y = \mathcal{F}x$$

x = semistate, k-vector $\dot{} = d/dt$
u = input, n-vector
y = output, m-vector

$\mathcal{A}$, $\mathcal{D}$, $\mathcal{F}$ constant & generally singular.

Linear, time-invariant

$$\mathcal{B}(x,t) = \mathcal{B}\cdot x$$

with $\mathcal{B}$ constant

Variation with a parameter p

$$\mathcal{A}(p)\cdot\dot{x} + \mathcal{B}(p)\cdot x = \mathcal{D}(p)\cdot u$$
$$y = \mathcal{F}(p)\cdot x$$

assume u is independent of p, then

$$x = x(t, p)$$
$$y = y(t, p)$$

p may be a system element, such as an amplifier gain or resistor resistance, or it may be something like temperature or radiation that affects system elements.

Sensitivity

In sensitivity analysis we are interested in the effects of changes in the parameter p.

Time Domain

$$\frac{\partial y(t,p)}{\partial p} \qquad \text{(a function of } t\text{)}$$

Frequency Domain

Scalar transfer function sensitivity

$$T(s,p) = Y(s,p)/U(s)$$

$$S_p^T = \frac{p}{T}\frac{\partial T}{\partial p}$$

(a function of s giving % change)

Time Domain Sensitivity

Given

$$\mathcal{A}\dot{x} + \mathcal{B}\cdot x = \mathcal{D}u$$
$$y = \mathcal{F}x$$

we wish to find $y' = \dfrac{\partial y}{\partial p}$

$$\mathcal{A}'\dot{x} + \mathcal{A}\dot{x}' + \mathcal{B}'\cdot x + \mathcal{B}\cdot x' = \mathcal{D}'u$$
$$y' = \mathcal{F}'x + \mathcal{F}\cdot x'$$

Therefore simply resolve the original-like equations

$$\mathcal{A}(\dot{x'}) + \mathcal{B}(x') = \mathcal{D}'u - \underbrace{\mathcal{A}'\dot{x} - \mathcal{B}'x}_{\text{known from solving original equations}}$$

$[x'(0) \equiv 0 \text{ as } x(0,p) = x(0)]$

$$y' = \mathcal{F}x' + \mathcal{F}'x$$

Note that this process would be "simplified" if $\mathcal{A}$, $\mathcal{D}$, $\mathcal{F}$ were independent of p.

Transformation to Simplified Form

Given

$$\mathcal{A}\dot{x} + \mathcal{B}x = \mathcal{D}u$$
$$y = \mathcal{F}x$$

Find nonsingular P, Q for

$$P\mathcal{A}Q = 1_r \dotplus 0_{k-r}$$

$\dotplus$ = direct sum
1_r = $r \times r$ identity
0_{k-r} = $(k-r)\times(k-r)$ zero

Let

$$x = Q\hat{x} \quad , \quad \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$$

$$\hat{\mathcal{B}} = P\mathcal{B}Q = \begin{bmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{12} \\ \hat{\mathcal{B}}_{21} & \hat{\mathcal{B}}_{22} \end{bmatrix}$$

$$\hat{\mathcal{D}} = P\mathcal{D} = \begin{bmatrix} \hat{\mathcal{D}}_1 \\ \hat{\mathcal{D}}_2 \end{bmatrix}$$

$$\hat{\mathcal{F}} = \mathcal{F}Q = [\hat{\mathcal{F}}_1 \ , \ \hat{\mathcal{F}}_2]$$

Giving

$$\begin{bmatrix} 1_r & \\ & 0 \end{bmatrix}\dot{\hat{x}} + \hat{\mathcal{B}}\hat{x} = \hat{\mathcal{D}}u$$

$$y = \hat{\mathcal{F}}\hat{x}$$

Let

$$\underline{x}_3 = u \qquad \underline{x}_1 = \hat{x}_1$$

$$\underline{x}_4 = \hat{\mathcal{F}}_1 \hat{x} \qquad \underline{x}_2 = \hat{x}_2$$

and $\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \\ \underline{x}_4 \end{bmatrix}$

Then

$$\begin{bmatrix} 1_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{\underline{x}} + \begin{bmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{12} & -\hat{\mathcal{D}}_1 & 0 \\ \hat{\mathcal{B}}_{21} & \hat{\mathcal{B}}_{22} & -\hat{\mathcal{D}}_2 & 0 \\ 0 & 0 & 1_n & 0 \\ -\hat{\mathcal{F}}_1 & -\hat{\mathcal{F}}_2 & 0 & 1_m \end{bmatrix} \underline{x} = \begin{bmatrix} 0 \\ 0 \\ 1_n \\ 0 \end{bmatrix} u$$

$$y = [0, 0, 0, 1_m]\, \underline{x}$$

This has all p dependence in $\underline{\mathcal{B}}$, hence

$$\underline{\mathcal{A}}\dot{\underline{x}} + \underline{\mathcal{B}}\underline{x} = \underline{\mathcal{D}}u \quad = f(t)$$

$$y = \underline{\mathcal{F}}\underline{x}$$

gives

$$\underline{a}(\dot{\underline{x}}') + \underline{\mathcal{B}}(\underline{x}') = -\underline{\mathcal{B}}' \cdot \underline{x} \quad = \underline{f}(t)$$

$$y' = \underline{\mathcal{F}} \cdot \underline{x}'$$

[note $\underline{x}'(0,p) \not\equiv 0$ though, unless $x(0) = 0$]

Therefore can solve via the Drazin inverse

$$\mathcal{A}\dot{x} + \mathcal{B}(\gamma_0)x = \mathcal{D}u$$
$$y = \mathcal{F}x$$

Then resolve

$$\mathcal{A}\dot{x}' + \mathcal{B}(\gamma_0)x' = -\mathcal{B}'x$$
$$y' = \mathcal{F}x'$$

to get y'

Drazin Inverse Solution

$$\mathcal{A}\dot{x} + \mathcal{B}x = f(t)$$
$$y = \mathcal{F}x$$

has the solution

$$x(t) = \tilde{\mathcal{A}}\tilde{\mathcal{A}}^D e^{-\tilde{\mathcal{A}}^D\tilde{\mathcal{B}}t} x_0 + \tilde{\mathcal{B}}^D(1_k - \tilde{\mathcal{A}}\tilde{\mathcal{A}}^D)\sum_{j=0}^{I-1}(-1)^j(\tilde{\mathcal{A}}\tilde{\mathcal{B}}^D)^j\tilde{f}^{(j)}(t)$$
$$+ e^{-\tilde{\mathcal{A}}^D\tilde{\mathcal{B}}t} 1(t) * (\tilde{\mathcal{A}}^D\tilde{f}(t))$$

where

$$\tilde{\mathcal{A}} = (s\mathcal{A}+\mathcal{B})^{-1}\mathcal{A}, \quad \tilde{\mathcal{B}} = (s\mathcal{A}+\mathcal{B})^{-1}\mathcal{B}, \quad \tilde{f} = (s\mathcal{A}+\mathcal{B})^{-1}f$$

and ^{D} = Drazin inverse, I = index of $\mathcal{A}$
$*$ = convolution, $1(\cdot)$ = unit step

If $A = T\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}T^{-1}$, C = nonsingular part
N = nilpotent part

then $A^D = T\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}T^{-1}$; index = smallest I for $N^I = 0$

Note: Can form ′ of $x(t) = x(t,\infty)$ from above expression = big mess.

Transfer Function

Scalar case

Put p in $\mathcal{B}$

$$\mathcal{A}\dot{x} + \mathcal{B}(p)x = \mathcal{D}u$$
$$y = \mathcal{F}x$$

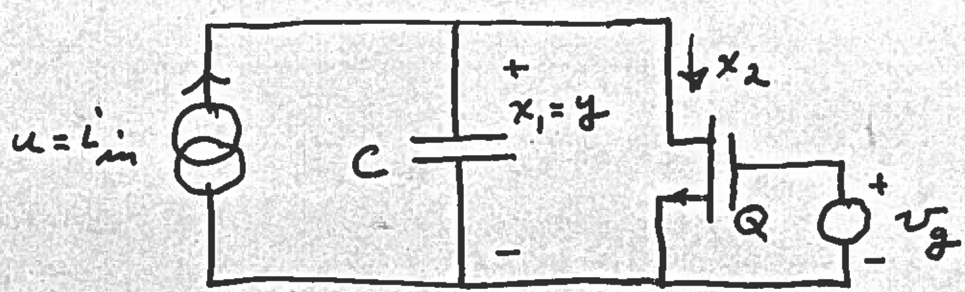
$$Y = T(s,p)\,U$$

$$T(s,p) = \mathcal{F}(\mathcal{A}s + \mathcal{B})^{-1}\mathcal{D}$$

$$\frac{\partial T}{\partial p} = \mathcal{F}(\mathcal{A}s+\mathcal{B})^{-1}(-\mathcal{B}')(\mathcal{A}s+\mathcal{B})^{-1}\mathcal{D}$$

$$S_p^T = \frac{p\{\mathcal{F}(\mathcal{A}s+\mathcal{B})^{-1}(-\mathcal{B}')(\mathcal{A}s+\mathcal{B})^{-1}\mathcal{D}\}}{\mathcal{F}(\mathcal{A}s+\mathcal{B})^{-1}\mathcal{D}}$$

Example:



$$C\dot{x}_1 + x_2 = u$$

$$\left.x_2 - \beta(v_g - V_T)x_1 = 0\right\} \text{ assume Q in resistive region } (v_g \approx V_T,\ v_g > V_T)$$

$$y = x_1$$

or

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\dot{x} + \begin{bmatrix} 0 & 1/C \\ -\beta(v_g - V_T) & 1 \end{bmatrix} x = \begin{bmatrix} 1/C \\ 0 \end{bmatrix} u$$

$$y = [1,\ 0]x$$

Let $p = v_g$. Therefore are interested in the output voltage sensitivity with respect to the gate voltage (whose variation varies the transistor resistance)

Now normalize:

multiply 2nd row by $\frac{1}{\beta V_T}$ and then let $\underline{\underline{x}}_2 = \frac{1}{\beta V_T} x_2$. Multiply 1st row by $\frac{C}{\beta V_T}$. Let $\underline{\underline{u}} = \frac{1}{\beta V_T} u$. Change time scale by $\tau = \frac{\beta V_T}{C} t$ and change v_g scale by $V_g = v_g / V_T$. Let $\underline{\underline{x}}_1(\tau) = x_1(t)$, $\underline{\underline{x}}_2(\tau) = \underline{\underline{x}}_2(t)$, $\underline{\underline{u}}(\tau) = \underline{\underline{u}}(t)$. Then replace all variables by their original symbols to get

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 0 & 1 \\ -(V_g - 1) & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [1 \, , \, 0] x$$

Then

$$(s\mathcal{A} + \mathcal{B})^{-1} = \frac{1}{s + (V_g - 1)} \begin{bmatrix} 1 & -1 \\ (V_g - 1) & s \end{bmatrix}$$

$$\tilde{\mathcal{A}} = (s\mathcal{A} + \mathcal{B})^{-1} \mathcal{A} = \frac{1}{s + (V_g - 1)} \begin{bmatrix} 1 & 0 \\ (V_g - 1) & 0 \end{bmatrix}$$

$$\tilde{\mathcal{B}} = \frac{1}{s + (V_g - 1)} \begin{bmatrix} V_g - 1 & 0 \\ -s(V_g - 1) & s + (V_g - 1) \end{bmatrix}$$

$$\tilde{\mathcal{D}} = \frac{1}{s + (V_g - 1)} \begin{bmatrix} 1 \\ (V_g - 1) \end{bmatrix} = (s\mathcal{A} + \mathcal{B})^{-1} \mathcal{D}$$

Now

$$\tilde{a} = \frac{1}{s+(V_g-1)}\begin{bmatrix} 1 & 0 \\ V_g-1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ V_g-1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{s+(V_g-1)} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -(V_g-1) & 1 \end{bmatrix}$$

$$= T\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}T^{-1} \quad ; I = 1$$

or $$\tilde{a}^D = \begin{bmatrix} 1 & 0 \\ V_g-1 & 1 \end{bmatrix}\begin{bmatrix} s+(V_g-1) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -(V_g-1) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s+(V_g-1) & 0 \\ (V_g-1)(s+(V_g-1)) & 0 \end{bmatrix}$$

$$\tilde{\mathcal{B}}^D = \tilde{\mathcal{B}}^{-1} = \begin{bmatrix} \frac{s+(V_g-1)}{(V_g-1)} & 0 \\ -s & 1 \end{bmatrix}$$

$$\tilde{a}\tilde{a}^D = \begin{bmatrix} 1 & 0 \\ V_g-1 & 0 \end{bmatrix} \qquad \tilde{a}^D\tilde{\mathcal{B}} = \begin{bmatrix} V_g-1 & 0 \\ (V_g-1)^2 & 0 \end{bmatrix}$$

$$\tilde{\mathcal{B}}^D[1_n - \tilde{a}\tilde{a}^D] = \begin{bmatrix} 0 & 0 \\ -(V_g-1) & 1 \end{bmatrix}, \quad \tilde{a}\tilde{\mathcal{B}}^D = \begin{bmatrix} \frac{1}{V_g-1} & 0 \\ 1 & 0 \end{bmatrix}$$

$$\tilde{a}^D\tilde{\mathcal{D}} = \begin{bmatrix} 1 \\ V_g-1 \end{bmatrix}$$

$x_0 = \begin{bmatrix} x_{0_1} \\ x_{0_2} \end{bmatrix}$ and here $x_0' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\tilde{\mathcal{B}}^D[1_n - \tilde{a}\tilde{a}^D]\times\tilde{\mathcal{D}} = \begin{bmatrix} 0 & 0 \\ -(V_g-1) & 1 \end{bmatrix}\cdot\frac{1}{s+(V_g-1)}\begin{bmatrix} 1 \\ V_g-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x(t,V_g) = \begin{bmatrix} 1 & 0 \\ V_g-1 & 0 \end{bmatrix} e^{-\begin{bmatrix} V_g-1 & 0 \\ (V_g-1)^2 & 0 \end{bmatrix} t} x_0 + e^{-\begin{bmatrix} V_g-1 & 0 \\ (V_g-1)^2 & 0 \end{bmatrix} t} 1(t) * \begin{bmatrix} 1 \\ V_g-1 \end{bmatrix} u(t)$$

But

$$e^{-\begin{bmatrix} V_g-1 & 0 \\ (V_g-1)^2 & 0 \end{bmatrix} t} = \begin{bmatrix} e^{-(V_g-1)t} & 0 \\ (V_g-1)(e^{-(V_g-1)t}-1) & 1 \end{bmatrix}$$

or

$$x(t,V_g) = \begin{bmatrix} e^{-(V_g-1)t} & 0 \\ (V_g-1)e^{-(V_g-1)t} & 0 \end{bmatrix} \begin{bmatrix} x_{o_1} \\ x_{o_2} \end{bmatrix} +$$

$$+ \begin{bmatrix} e^{-(V_g-1)t} \\ (V_g-1)e^{-(V_g-1)t} \end{bmatrix} 1(t) * u(t)$$

$$= \begin{bmatrix} 1 \\ (V_g-1) \end{bmatrix} \left\{ e^{-(V_g-1)t} x_{o_1} + e^{-(V_g-1)t} 1(t) * u(t) \right\}$$

To get $x'(t,p)$ we repeat with Du replaced by $-\mathcal{B}'x$. Here

$$\mathcal{B}=\begin{bmatrix}0 & 1\\ -(V_g-1) & 1\end{bmatrix} \text{ or } \mathcal{B}'=\begin{bmatrix}0 & 0\\ -1 & 0\end{bmatrix} \text{ and}$$

$$-\mathcal{B}'x(t,V_g)=\begin{bmatrix}0\\1\end{bmatrix}\{e^{-(V_g-1)t}x_{0_1}+e^{-(V_g-1)t}1(t)*u(t)\}$$

and

$$\tilde{f}\rightarrow-(s\mathcal{a}+\mathcal{B})^{-1}\mathcal{B}'x$$

$$=\frac{1}{s+(V_g-1)}\begin{bmatrix}-1\\s\end{bmatrix}\{e^{-(V_g-1)t}x_{0_1}+e^{-(V_g-1)t}1(t)*u(t)\}$$

$$\tilde{\mathcal{a}}^{0}\tilde{f}=\begin{bmatrix}-1\\-(V_g-1)\end{bmatrix}\{e^{-(V_g-1)t}x_{0_1}+e^{-(V_g-1)t}1(t)*u(t)\}$$

and

$$x'(t,V_g)=\begin{bmatrix}0\\0\end{bmatrix}+\overbrace{\begin{bmatrix}0 & 0\\ -(V_g-1) & 1\end{bmatrix}\frac{1}{s+(V_g-1)}\begin{bmatrix}-1\\s\end{bmatrix}}^{\text{note} \neq 0 \text{ so jumps in } x' \text{ at } t=0+}\{e^{-(V_g-1)t}x_{0_1}+e^{-(V_g-1)t}1(t)*u(t)\}$$

$$+\begin{bmatrix}e^{-(V_g-1)t} & 0\\ (V_g-1)e^{-(V_g-1)t} & 1\end{bmatrix}1(t)*\begin{bmatrix}-1\\-(V_g-1)\end{bmatrix}\{e^{-(V_g-1)t}x_{0_1}+e^{-(V_g-1)t}1(t)*u(t)\}$$

as $t>0$ has a $1(t)$ multipl[ier]

$$=\begin{bmatrix}0\\1\end{bmatrix}\{e^{-(V_g-1)t}x_{0_1}+e^{-(V_g-1)t}1(t)*u(t)\}$$

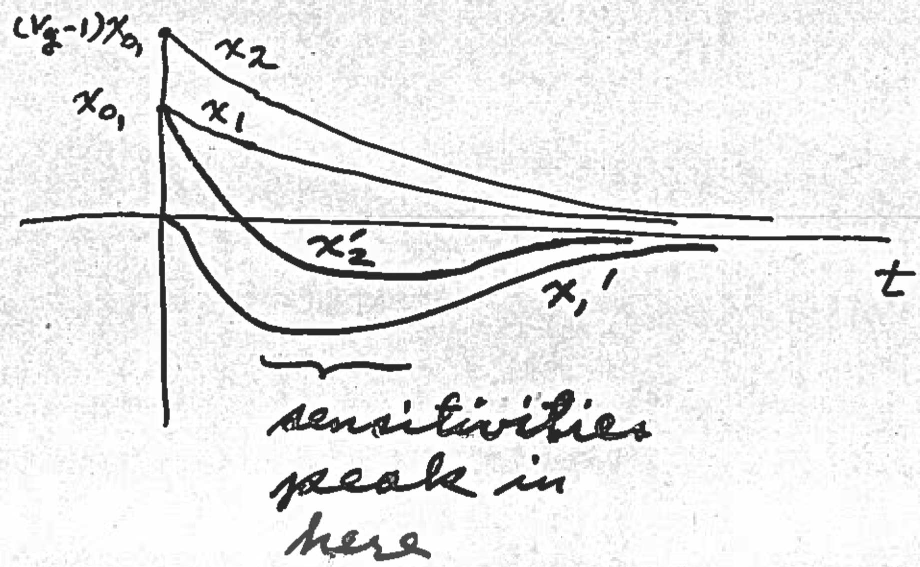
$$-\begin{bmatrix}e^{-(V_g-1)t}\\ (V_g-1)(e^{-(V_g-1)t}-1)\end{bmatrix}1(t)*\{e^{-(V_g-1)t}x_{0_1}1(t)+e^{-(V_g-1)t}1(t)*u(t)\}$$

Special Case - Initial Conditions Only

$u(t) \equiv 0$

$$x'(t, V_g) = x_{0_1} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-(V_g-1)t} 1(t) - \begin{bmatrix} e^{-(V_g-1)t} \\ (V_g-1)\left(e^{-(V_g-1)t} - 1\right) \end{bmatrix} 1(t) * e^{-(V_g-1)t} 1(t) \right\}$$

with $x(t) = \begin{bmatrix} 1 \\ V_g - 1 \end{bmatrix} e^{-(V_g-1)t} x_{0_1}$



Example Transfer Function Sensitivity

$$S^T_{v_g} = \frac{v_g\{\mathcal{F}(\mathcal{A}s+\mathcal{B})^{-1}(-\mathcal{B}')(\mathcal{A}s+\mathcal{B})^{-1}\mathcal{D}\}}{\mathcal{F}(\mathcal{A}s+\mathcal{B})^{-1}\mathcal{D}}$$

$$(\mathcal{A}s+\mathcal{B})^{-1} = \frac{1}{s+(v_g-1)}\begin{bmatrix} 1 & -1 \\ v_g-1 & s \end{bmatrix}$$

$$\mathcal{B}' = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{F} = [1, 0], \quad \mathcal{D} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(s,p) = \mathcal{F}(\mathcal{A}s+\mathcal{B})^{-1}\mathcal{D} = [1, 0]\left\{\frac{1}{s+(v_g-1)}\begin{bmatrix} 1 & -1 \\ v_g-1 & s \end{bmatrix}\right\}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{s+(v_g-1)}$$

and

$$\mathcal{F}(\mathcal{A}s+\mathcal{B})^{-1} = \frac{1}{s+(v_g-1)}\begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$-\mathcal{B}'(\mathcal{A}s+\mathcal{B})^{-1}\mathcal{D} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\frac{1}{s+(v_g-1)}\begin{bmatrix} 1 \\ v_g-1 \end{bmatrix} = \frac{1}{s+(v_g-1)} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

or

$$\mathcal{F}(\mathcal{A}s+\mathcal{B})^{-1}(-\mathcal{B}')(\mathcal{A}s+\mathcal{B})^{-1}\mathcal{D} = \frac{-1}{(s+(v_g-1))^2}$$

$$S^T_{v_g} = \frac{+v_g\left(\frac{-1}{(s+(v_g-1))^2}\right)}{\frac{1}{(s+(v_g-1))}} = \boxed{\frac{-v_g}{s+(v_g-1)}}$$

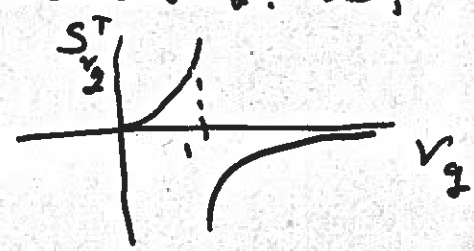
which checks $S^T_{v_g} = \frac{p}{T} \cdot \frac{\partial T}{\partial p}$

$$T = \frac{V_c}{I_{in}} = \text{input impedance}$$

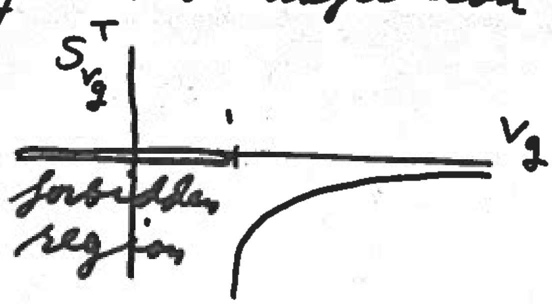
$$= \frac{1}{s + (V_g - 1)}$$

$$S_{\rho}^{T} = S_{V_g}^{V_c/I_{in}} = \frac{-V_g}{s + (V_g - 1)}$$

S_{ρ}^{T} decreases in magnitude as frequency $\omega = s/j$ increases. at DC, $\omega = 0$, $S_{\rho}^{T} = \frac{-V_g}{V_g - 1}$



useful for tuning near $V_g = 1$, but by assumption $V_g = v_g/V_T > 1$ or



Conclusion

The semistate equations can be directly used for sensitivity calculations. The calculations are somewhat messy but could be well-organized for use in design from semistate specifications