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SYNTHESIS OF NETWORKS PASSIVE AT p_o

by

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ABSTRACT

If a network N has $q_{-}(p_o) \geq 0$ then it is known that N is passive at p_o. In this report we show that if N is a two-port or a member of a general class of n-ports, there exists a finite passive network N_P such that N and N_P have the same admittance matrix at p_o. A simple test for $q_{-} \geq 0$ is given, for most N, and it is shown that the complex transformer isn't passive when Re $p_o > 0$.

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I. INTRODUCTION

In a previous report,[1*] Desoer and Kuh have treated the passivity and activity of an n-port network N. Besides the normal concepts of "passive" and "active" two other concepts are defined. These are "active at p_o" and "passive at p_o," the last of which will concern us here. In Ref. 1 (p. 13) it is shown that if N has an admittance matrix a necessary and sufficient condition for N to be passive at p_o is that $q_-(p_o) \geq 0$. Since N may be active and still be passive at p_o for some p_o, the following problem arises.

"If N is passive at p_o, does there exist a finite passive network N_P such that, at p_o, N and N_P are described by the same admittance matrix?"

This is the problem we will consider. Its importance stems from the fact that in certain situations it may be possible to replace an active network by a purely passive one.

In Section II we will review the definitions of passivity and q_- and extend the concepts to networks without a Y matrix. Also a simple necessity test for passivity will be given. In Sections III and IV we will give synthesis methods for a general class of n-ports and all two-ports, assuming a Y matrix exists. In Appendix 3 the non-realizability of the complex transformer for $\sigma > 0$ will be shown.

II. Q_- AND A PASSIVITY TEST

Consider an n-port N which is assumed to possess an admittance matrix Y(p) which is rational with real coefficients. N is called passive if for any voltage excitation the energy input is non-negative (evaluated at any time). Now let N be excited at t = 0 by the voltage vector $v(t) = \text{Re}\, V e^{pt}$ where V is a vector of complex constants and $p = \sigma + j\omega$. N is said to be passive at p_o if, for all

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*Refers to the bibliography.

such v(t) with $p = p_o$ and appropriate initial conditions chosen to give no transients, the energy input is non-negative for all $t > 0$, (Ref. 1, p. 5).

Let a superscript tilde, ~ , denote matrix transposition, a superscript asterisk, *, denote complex conjugation and $Y_H(p)$ denote the Hermitian part of Y. Then define, for $\sigma \geq 0$,

$$Q_-(V, p) = \begin{cases} \tilde{V}^* Y_H(p) V - (\sigma / |p|) \, |\tilde{V} Y(p) V| & \text{if } \omega \neq 0 \\ \tilde{V}^* Y_H(p) V & \text{if } \omega = 0 \end{cases} \qquad \text{(II. 1)}$$

here | | denotes the absolute value of a complex number. Physically, if $\sigma \neq 0$, $\sigma^{-1} e^{2\sigma t} Q_-$ represents the lower limit on energy into N for a given v(t) at a given instant. Instead of Q_-, Desoer and Kuh work with, (Ref. 1, p. 13),

$$q_-(p) = \min_{\|V\|=1} Q_-(V, p) \qquad \text{(II. 2)}$$

where for $\tilde{V} = [V_1, \ldots, V_n]$ we have $\|V\|^2 = \Sigma |V_i|^2$. q_- then represents the smallest energy into N at a given instant for all normalized non-zero V. Clearly $Q_-(V, p_o)$ must necessarily be non-negative for every V if N is to be passive at p_o. Conversely if $q_-(p_o) \geq 0$, and as a consequence $Q_- \geq 0$ for every V, Ref. 1 (p. 13) shows that N is passive at p_o.

From the physical meaning of Q_- it should be clear that Q_- is independent of the description of the device. If N has an impedance matrix Q_- is defined in a manner dual to Eq. (II. 1). For devices with no Z or Y matrix the required quantity is

$$\mathcal{Q}_-(V, I, p) = \begin{cases} (1/2)[\tilde{V}^* I + \tilde{I}^* V] - (\sigma / |p|) \, |\tilde{V} I| & \text{if } \omega \neq 0 \\ (1/2)[\tilde{V}^* I + \tilde{I}^* V] & \text{if } \omega = 0 \end{cases} \qquad \text{(II. 3)}$$

We will assume that a given network has $q_-(p_o) \geq 0$. Then at p_o Y is a matrix of complex numbers and can be written as

$$Y = Y_{RS} + Y_{RSS} + jY_{IS} + jY_{ISS} \qquad \text{(II. 4)}$$

where the subscripts R and I refer to real and imaginary parts and S and SS refer to symmetric and skew-symmetric matrices. The relation $q_-(p_o)$ puts a constraint on the terms of Eq. (II. 4) which we will now determine. We first observe that Q_- is independent of Y_{RSS}; a fact that is not surprising since Y_{RSS} can easily be realized by gyrators. Clearly the Hermitian part, $Y_{RS} + jY_{ISS}$, is necessarily positive semi-definite. If $\sigma_o = 0$ or $\omega_o = 0$ this is the only constraint. Otherwise we observe that

$$(\sigma_o^2 + \omega_o^2)[\tilde{V}^* Y_H V]^2 \geq \sigma_o^2 [\tilde{V}^* Y^* V^*][\tilde{V} Y V] \tag{II. 5}$$

Now choose

$$V = V_o$$

where V_o is a purely real vector. Then

$$\begin{aligned} \tilde{V}_o^* Y_H V_o &= \tilde{V}_o Y_{RS} V_o \\ \tilde{V}_o Y V_o &= \tilde{V}_o Y_{RS} V_o + j\tilde{V}_o Y_{IS} V_o \end{aligned} \tag{II. 6}$$

Substituting Eq. (II. 6) into Eq. (II. 5) and combining terms gives

$$\omega_o^2 [\tilde{V}_o Y_{RS} V_o]^2 \geq \sigma_o^2 [\tilde{V}_o Y_{IS} V_o]^2$$

Consequently

$$\tilde{V}_o [\omega_o Y_{RS} \pm \sigma_o Y_{IS}] V_o \geq 0 \tag{II. 7}$$

where the inequality holds for either choice of sign and it is to be remembered that $\omega_o > 0$. We have then proven the following.

Theorem 1: If N is passive at p_o, with $\sigma_o \geq 0$, $\omega_o > 0$, then the real symmetric matrices

$$\omega_o Y_{RS} + \sigma_o Y_{IS}$$

$$\omega_o Y_{RS} - \sigma_o Y_{IS}$$

are both necessarily positive semi-definite. If N is strictly passive, i.e., $Q_-(V, p_o) > 0$ for all non-zero V, they are both positive definite.

The synthesis method given in Section III will prove the following.

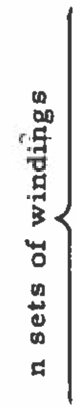
Theorem 2: If, at p_o with $\sigma_o \geq 0$, $\omega_o > 0$, N has

a) $Y_{ISS} = 0$ and

b) $\omega_o Y_{RS} \pm \sigma_o Y_{IS}$ positive semi-definite (for both signs)

then there exists a finite passive network N_P such that N and N_P have the same admittance matrix at p_o.

As a consequence of Theorems 1 and 2 we see that, if $Y_{ISS} = 0$, we only need to test Q_- with real V. More important, we see that the positive semi-definiteness of $\omega_o Y_{RS} \pm \sigma_o Y_{IS}$ is a necessary and sufficient condition for N to be passive at p_o (since N_P necessarily has $q_- \geq 0$). This is summarized in the simple test of Theorem 3, (Ref. 2, p. 307).



Theorem 3: If $Y_{ISS} = 0$, N is passive at p_o with $\sigma_o \geq 0$, $\omega_o > 0$ if and only if every principal minor of

$$\omega_o Y_{RS} \pm \sigma_o Y_{IS} \qquad \text{(for both signs)}$$

is non-negative.

III. SYNTHESIS OF N_P WHEN $Y_{ISS} = 0$

In the synthesis we will generally use the notation of Ref. 3 (p. 81) for transformer networks. Since Ref. 3 isn't readily available, this is repeated in Fig. 1 for reference. Here, if the transformer network is loaded at its series terminals by a network of admittance matrix Y_2, we find for the admittance Y_1 looking into the shunt terminals

$$Y_1 = \tilde{T} Y_2 T \qquad \text{(III. 1)}$$

We will have two regions to consider corresponding to the division of Q_- in Eq. (II. 1). For $\omega_o > 0$ we have two cases depending upon the differences in rank of the two matrices of Theorem 1. Further several possible synthesis methods are available when

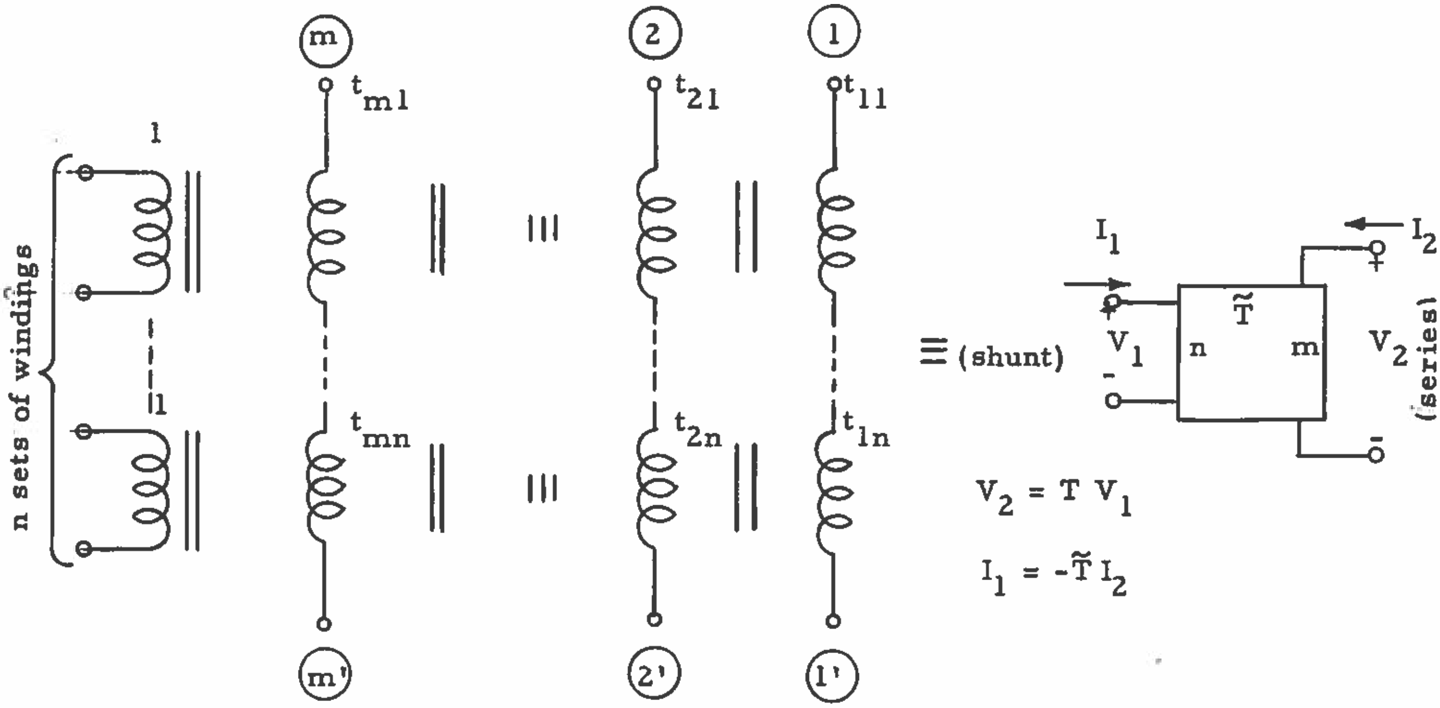


FIG. 1.--Transformer notation, $T = [t_{ij}]$ = real.

$\omega_o > 0$. The main step of the synthesis is the simultaneous diagonalization of Y_{RS} and Y_{IS} by the use of Appendix 1. These diagonalized matrices are then realized by obtaining recognizable combinations of p_o.

Region I: $\omega_o = 0$

Here Y is purely real, $Y = Y_{RS} + Y_{RSS}$, with Y_{RS} positive semi-definite. Let

$$E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{III. 2}$$

and let 1_n denote the unit matrix of order n. Then we can write (Ref. 3, p. 117),

$$\begin{aligned} Y_{RS} &= \tilde{T}_1 \, 1_{r_1} \, T_1 & r_1 &= \text{rank } Y_{RS} \\ Y_{RSS} &= \tilde{T}_2 \underbrace{[E \dotplus \cdots \dotplus E]}_{r_2/2} T_2 & r_2 &= \text{rank } Y_{RSS} \end{aligned} \tag{III. 3}$$

Here $\dot{+}$ denotes the direct sum and T_1 and T_2 are real matrices of order $r_1 \times n$ and $r_2 \times n$, respectively. $Y(p_o)$ is then realized by Fig. 2 where a gyrator of the given polarity has an admittance matrix equal to E of Eq. (III. 2).

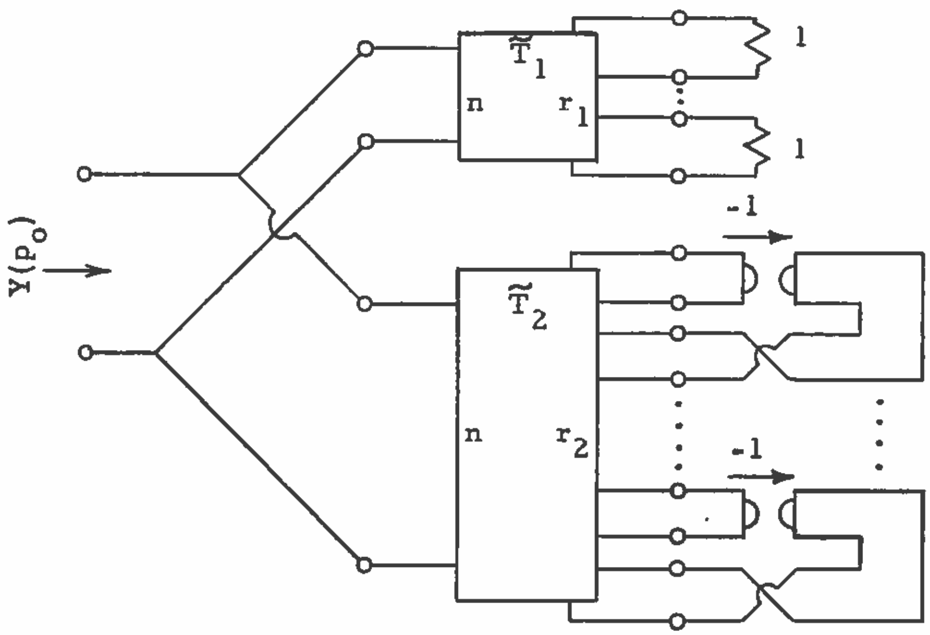


FIG. 2. --Realization of $Y(p_o)$ when $\omega_o = 0$.

Region II: $\omega_o > 0$

If $\sigma_o = 0$ we know that Y_{RS} is positive semi-definite. The decomposition of Eq. (III. 3) can then be used to realize Y_{RS} and Y_{RSS}. Y_{IS} can be also diagonalized to

$$Y_{IS} = \tilde{T}[1_r \dot{+} (-1_s)]\,T \tag{III. 4}$$

This is realized by connecting r capacitances and s inductances through a transformer network.

If $\sigma_o > 0$ then Theorem 1 shows that $\omega_o Y_{RS} + \sigma_o Y_{IS}$ and $\omega_o Y_{RS} - \sigma_o Y_{IS}$ are both positive semi-definite. By Theorem A. 1 (Appendix 1) they can be simultaneously diagonalized. Let them have ranks r_+ and r_-, respectively. We then have two cases depending upon $r_+ \leq r_-$ or $r_+ > r_-$. The results for these two situations are almost identical with Fig. 8 exhibiting one realization for the first case.

Case A: $r_+ \leq r_-$

In this case we write, by Theorem A. 1,

$$Y_{RS} + (\sigma_o/\omega_o)\, Y_{IS} = \tilde{T}[1_{r_+} \dotplus 0_{n-r_+}]\, T$$
$$Y_{RS} - (\sigma_o/\omega_o)\, Y_{IS} = \tilde{T}[\Lambda_{r_+} \dotplus 1_{r_- - r_o} \dotplus 0_{n-r_+-r_-+r_o}]\, T \qquad \text{(III.5)}$$

Here r_o is the rank of the matrix $\Lambda_{r_+} = \text{diag.}\,[\lambda_1, \ldots, \lambda_{r_+}]$, which has the first r_o $\lambda_i > 0$. Adding and subtracting these, and letting T_+ be the first $r_+ + r_- - r_o$ rows of $T/\sqrt{2}$ gives

$$Y_{RS} = \tilde{T}_+[(1_{r_+} + \Lambda_{r_+}) \dotplus 1_{r_- - r_o}]\, T_+$$
$$Y_{IS} = (\omega_o/\sigma_o)\, \tilde{T}_+[(1_{r_+} - \Lambda_{r_+}) \dotplus (-1_{r_- - r_o})]\, T_+ \qquad \text{(III. 6)}$$

The final decomposition of Y is then

$$Y(p_o) = Y_{RSS} + \tilde{T}_+ \{[(1_{r_+} + \Lambda_{r_+}) \dotplus 1_{r_- - r_o}] + j(\omega_o/\sigma_o)[(1_{r_+} - \Lambda_{r_+}) \dotplus (-1_{r_- - r_o})]\}\, T_+ \qquad \text{(III. 7)}$$

The separate terms in this decomposition can now be synthesized. As in Eq. (III. 3) we can write $Y_{RSS} = \tilde{T}_2[E \dotplus \cdots \dotplus E]\, T_2$ which is realized in Fig. 2. For the synthesis of the remaining terms of Eq. (III. 7) define

$$y_i(p_o) = \begin{cases} (1+\lambda_i) + j(\omega_o/\sigma_o)(1-\lambda_i) & \text{for } i=1, \ldots, r_+ \\ 1 - j(\omega_o/\sigma_o) & \text{for } i=r_++1, \ldots, r_++r_--r_o \text{ (if } r_--r_o>0) \end{cases} \qquad \text{(III. 8)}$$

Synthesis of $y_i(p_o)$ for $i = r_+ + 1, \ldots, r_+ + r_- - r_o$:

We can write, by rationalizing $1/y_i(p_o)$,

$$y_i(p_o) = 1 - j\frac{\omega_o}{\sigma_o} = \frac{\sigma_o^2 + \omega_o^2}{\sigma_o} \cdot \frac{1}{\sigma_o + j\omega_o} = \frac{\sigma_o^2 + \omega_o^2}{\sigma_o} \cdot \frac{1}{p_o} \qquad \text{(III. 9)}$$

This is easily extended to all p by defining

$$y_i(p) = \frac{\sigma_o^2 + \omega_o^2}{\sigma_o} \cdot \frac{1}{p} \qquad \text{(III. 10)}$$

These y_i are then realized by an inductance as shown in Fig. 3.

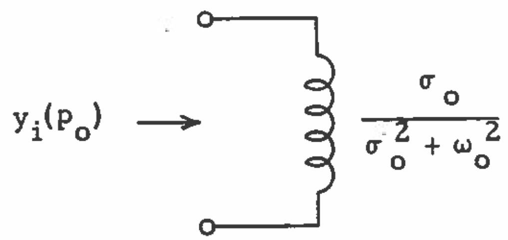


FIG. 3.--Realization of $y_i(p_o)$ for $i = r_+ + 1, \ldots, r_+ + r_- - r_o$.

Synthesis of $y_i(p_o)$ for $i = 1, \ldots, r_+$:

Several alternative methods are available. Since these lend insight into the meaning of passivity at p_o, they will all be described. The first two methods give L-C circuits while the last one gives R-C or R-L circuits.

Method 1: We can write

$$y_i(p_o) = (1+\lambda_i) + j(\omega_o/\sigma_o)(1-\lambda_i) = \left(1 + j\frac{\omega_o}{\sigma_o}\right) + \lambda_i\left(1 - j\frac{\omega_o}{\sigma_o}\right)$$

$$= \frac{p_o}{\sigma_o} + \frac{\lambda_i(\sigma_o^2 + \omega_o^2)}{\sigma_o p_o} \qquad \text{(III. 11)}$$

This is extended to all p by defining

$$y_i(p) = \frac{p}{\sigma_o} + \frac{\lambda_i(\sigma_o^2 + \omega_o^2)}{\sigma_o\, p} = c_i p + \frac{1}{\ell_i p} \tag{III. 12}$$

This has

$$\omega_i^2 = \frac{1}{\ell_i c_i} = \lambda_i(\sigma_o^2 + \omega_o^2) \tag{III. 13}$$

$y_i(p)$ is realized by the shunt resonant circuit of Fig. 4 where the

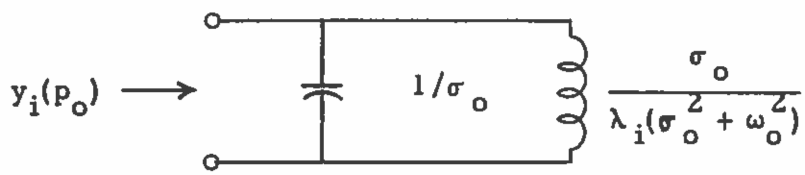


FIG. 4. --Realization of $y_i(p_o)$, $i = 1, \ldots, r_+$.

inductances are absent for $i > r_o$ (for which $\lambda_i = 0$). It should be observed that $y_i(p_o)$ acts like a resistance at $\lambda_i = 1$ even though lossless elements are used. Since parallel resonant circuits work, it is natural to look for series resonant circuits. This is covered by Method 2.

Method 2: We can write

$$y_i(p_o) = (1+\lambda_i) + j(\omega_o/\sigma_o)(1-\lambda_i) = \frac{(1+\lambda_i)^2 + (\omega_o/\sigma_o)^2(1-\lambda_i)^2}{(1+\lambda_i) - j(\omega_o/\sigma_o)(1-\lambda_i)}$$

$$= \cfrac{1}{\cfrac{\sigma_o - j\omega_o}{\sigma_o[(1+\lambda_i)^2 + (\omega_o/\sigma_o)^2(1-\lambda_i)^2]} + \cfrac{\lambda_i(\sigma_o + j\omega_o)}{\sigma_o[(1+\lambda_i)^2 + (\omega_o/\sigma_o)^2(1-\lambda_i)^2]}}$$

$$= \cfrac{1}{\cfrac{\sigma_o^2 + \omega_o^2}{\sigma_o[(1+\lambda_i)^2 + (\omega_o/\sigma_o)^2(1-\lambda_i)^2]\, p_o} + \cfrac{\lambda_i\, p_o}{\sigma_o[(1+\lambda_i)^2 + (\omega_o/\sigma_o)^2(1-\lambda_i)^2]}}$$

(III. 14)

This is extended to all p by defining

$$y_i(p) = \frac{1}{\frac{1}{pc_i} + p\ell_i} \tag{III.15}$$

where

$$\ell_i = \frac{\lambda_i}{\sigma_o[(1+\lambda_i)^2+(\omega_o/\sigma_o)^2(1-\lambda_i)^2]}, \quad c_i = \frac{\sigma_o[(1+\lambda_i)^2+(\omega_o/\sigma_o)^2(1-\lambda_i)^2]}{\sigma_o^2+\omega_o^2} \tag{III.16}$$

If $\lambda_i = 0$ then $\ell_i = 0$ and if $\lambda_i \neq 0$ then

$$\omega_i^2 = \frac{1}{\ell_i c_i} = \frac{\sigma_o^2+\omega_o^2}{\lambda_i} \tag{III.17}$$

These y_i are then realized by the series resonant circuits of Fig. 5.

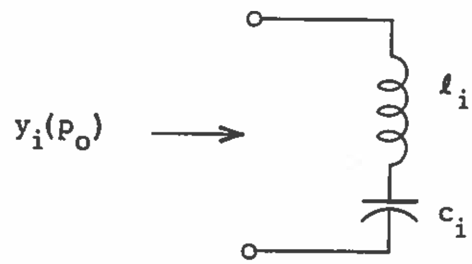


FIG. 5.--Realization of $y_i(p_o)$, $i = 1, \ldots, r_+$, ℓ_i and c_i given by Eq. (III.16).

Here the inductances are again absent if $i > r_o$.

Method 3: We can obtain R-C and R-L networks by considering various regions of λ_i.

Case 1: $\lambda_i \leq 1$

We can write

$$y_i(p_o) = \frac{1}{\sigma_o}[\sigma_o(1+\lambda_i)+j\omega_o(1-\lambda_i)] = \frac{1}{\sigma_o}[2\sigma_o\lambda_i+\sigma_o(1-\lambda_i)+j\omega_o(1-\lambda_i)]$$

$$= 2\lambda_i + \frac{(1-\lambda_i)}{\sigma_o}p_o \tag{III.18}$$

or

$$y_i(p) = 2\lambda_i + \frac{(1-\lambda_i)}{\sigma_o} p \qquad \text{(III. 19)}$$

These y_i are then realized by the circuits of Fig. 6 where the resistance

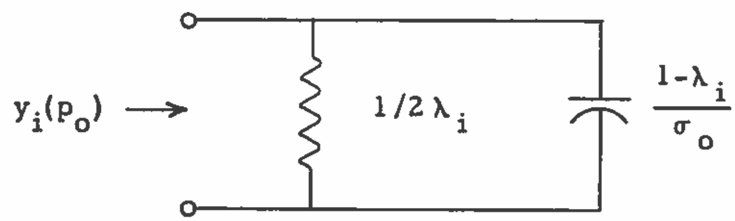


FIG. 6. --Realization of $y_i(p_o)$ for $\lambda_i \leq 1$.

is absent if $\lambda_i = 0$ and the capacitance is absent if $\lambda_i = 1$.

Case 2: $\lambda_i > 1$

We can write

$$y_i(p_o) = 2 + (\lambda_i - 1) - j(\omega_o/\sigma_o)(\lambda_i - 1)$$

$$= 2 + \frac{(\lambda_i - 1)(\sigma_o^2 + \omega_o^2)}{\sigma_o p_o} \qquad \text{(III. 20)}$$

or

$$y_i(p) = 2 + \frac{(\lambda_i - 1)(\sigma_o^2 + \omega_o^2)}{\sigma_o p} \qquad \text{(III. 21)}$$

These y_i are then realized by the circuits of Fig. 7. Note that if we were to let $\lambda_i = 1$, this circuit would reduce to that of Fig. 6.

Using any one of the above three methods we can synthesize $Y(p_o)$. For example, the network which results from Method 2 is exhibited in Fig. 8. The advantage of Method 2 is seen through Fig. 8, since the inductances can be absorbed in the transformer windings to yield coupled coils. Further it should be noted that the networks of

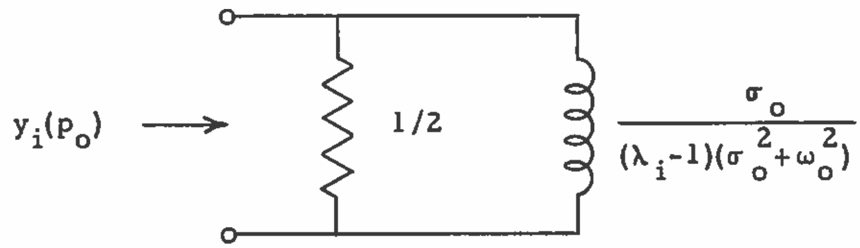


FIG. 7. --Realization of $y_i(p_o)$ for $\lambda_i > 1$.

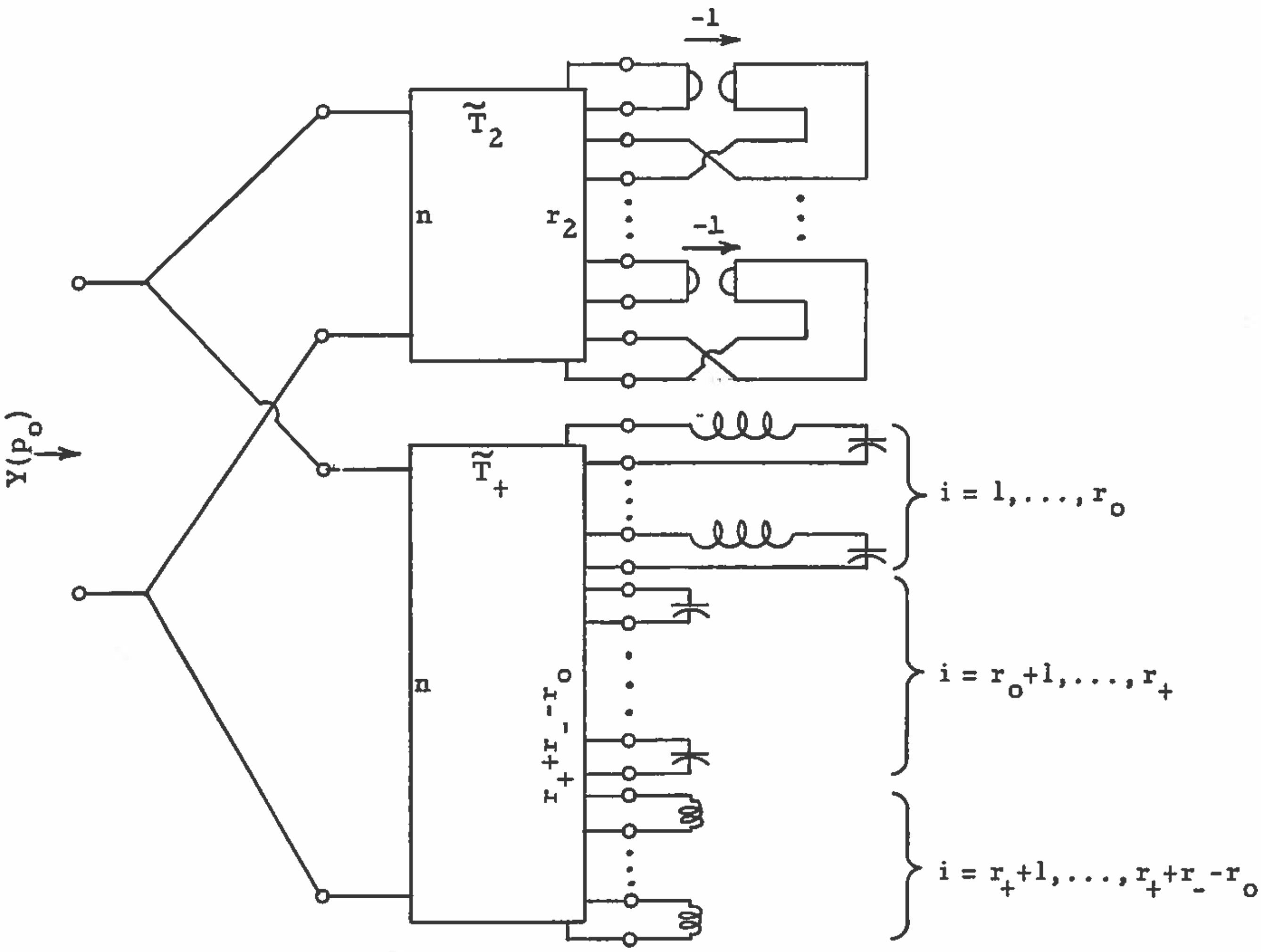


FIG. 8. --Method 2 realization of N_P, $\omega_o > 0$, $r_+ \leq r_-$.

Methods 1 and 2 are lossless while that of Method 3 isn't. Also the reader should note that only the hypothesis of Theorem 2 has been used in obtaining the passive network.

Case B: $r_+ > r_-$

We now write by Theorem A. 1

$$Y_{RS} - (\sigma_o/\omega_o)\, Y_{IS} = \widetilde{T}\,[1_{r_-} \dotplus 0_{n-r_-}]\, T$$
$$Y_{RS} + (\sigma_o/\omega_o)\, Y_{IS} = \widetilde{T}[\Lambda_{r_-} \dotplus 1_{r_+-r_o} \dotplus 0_{n-r_--r_++r_o}]\, T \qquad \text{(III. 22)}$$

Performing the same operations as in Case A, we obtain

$$Y = Y_{RSS} + \widetilde{T}_-\{[(\Lambda_{r_-} + 1_{r_-}) \dotplus 1_{r_--r_o}] + j(\omega_o/\sigma_o)[(\Lambda_{r_-} - 1_{r_-}) \dotplus 1_{r_--r_o}]\}\, T_- \qquad \text{(III. 23)}$$

Here the synthesis can be performed in the same manner as for Case A. Let

$$y_i'(p_o) = \begin{cases} (\lambda_i+1) + j(\omega_o/\sigma_o)(\lambda_i - 1) & i = 1, \ldots, r_- \\ 1 + j\dfrac{\omega_o}{\sigma_o} & i = r_-+1, \ldots, r_- + r_+ - r_o \end{cases} \qquad \text{(III. 24)}$$

Clearly $y_i'(p_o)$, for $i > r_-$, is realized by a capacitance of value $1/\sigma_o$. For $i = 1, \ldots, r_o$, Method 2 gives

$$y_i'(p_o) = \frac{1}{p\ell_i' + \dfrac{1}{pc_i'}} \qquad \text{(III. 25)}$$

with

$$\ell_i' = \frac{1}{\sigma_o[(\lambda_i+1)^2 + (\omega_o/\sigma_o)^2(\lambda_i-1)^2]}, \quad c_i' = \frac{\sigma_o[(\lambda_i+1)^2 + (\omega_o/\sigma_o)(\lambda_i-1)^2]}{\lambda_i(\sigma_o^2 + \omega_o^2)} \qquad \text{(III. 26)}$$

Here $(\omega_i')^2 = 1/(\ell_i' c_i') = 1/(\lambda_i(\sigma_o^2 + \omega_o^2))$ which is the reciprocal of Eq. (III. 13). For $i = r_o + 1, \ldots, r_-$, $y'(p_o)$ is realized by an inductance of value $\sigma_o/(\sigma_o^2 + \omega_o^2)$.

r_o

Consequently, the Method 2 realization for Case B has the same form as Fig. 8. However, the final inductances and capacitances are interchanged, subscript +'s are replaced by -'s and vice versa, and different expressions are used for the first r_o element values.

IV. SYNTHESIS OF N_P WHEN $Y_{ISS} \neq 0$, $n = 2$

We will give two methods of synthesis for 2 x 2 matrices. The first of these is quite simple but doesn't extend to n > 2. The second method is much more complicated but should extend to larger matrices. We begin by writing Eq. (II. 4) as

$$Y(p_o) = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} + \begin{bmatrix} 0 & g_{21} \\ -g_{21} & 0 \end{bmatrix} + j\begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} + j\begin{bmatrix} 0 & b_{21} \\ -b_{21} & 0 \end{bmatrix} \quad \text{(IV. 1)}$$

The real symmetric term is positive definite since $g_{11}g_{22}-(g_{12})^2 \geq (b_{21})^2 > 0$ [as $q_- \geq 0$ by assumption]. Thus we can find a real, non-singular transformation to put $Y(p_o)$ in the more usable form.

$$Y_o = \tilde{T}^{-1} Y T^{-1} = 1_2 + j[b_1 \dotplus b_2] + (g + jb)\, E \quad \text{(IV. 2)}$$

where E is as defined in Eq. (III. 2). A network N for Y then results from terminating a transformer network T by a network N_o for Y_o. Physically, it is then clear that N_o has $q_- \geq 0$, however, this can also be seen by noting that Q_- for N_o results from Q_- for N by replacing V by TV.

Method 1: Here a gyrator of unit gyration resistance is extracted from terminal pair one, as shown in Fig. 9. The new network, N_c, has

$$Y_c = \frac{1}{1+jb_1}\begin{bmatrix} 1 & g+jb \\ g+jb & (1+jb_1)(1+jb_2)+g^2-b^2 \end{bmatrix} \quad \text{(IV. 3)}$$

which has $(Y_{ISS})_c = 0$. We can realize Y_c by the method of Section III, since N_o and N_c have the same $q_- \geq 0$ (this can be proven by using Eq. (II. 3)). It should be noted that, besides giving a synthesis,

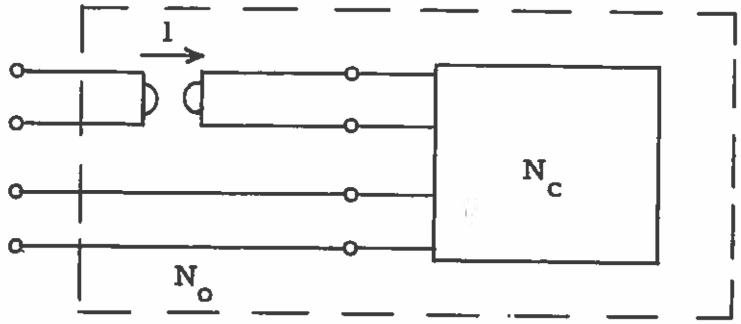


FIG. 9. --Gyrator extraction.

this method can be used to obtain the constraints on the parameters in order that a given network be passive at p_o. This is illustrated by the following examples.

Example 1: Under what conditions is the network described by the following Y matrix passive at p_o with $\omega_o > 0$?

$$Y(p_o) = 1_2 + jbE$$

Using the connection of Fig. 9, we get

$$Y_c = \begin{bmatrix} 1 & jb \\ jb & 1-b^2 \end{bmatrix}$$

Then, using Theorem 3, we form

$$(Y_{RS})_c \pm (\sigma_o/\omega_o)(Y_{IS})_c = \begin{bmatrix} 1 & \pm \frac{\sigma_o}{\omega_o} b \\ \pm \frac{\sigma_o}{\omega_o} b & 1-b^2 \end{bmatrix}$$

which must be positive semi-definite. Thus

$$1-b^2 \geq 0 \qquad \text{(equality can only hold if } \sigma_o = 0)$$

$$(1-b^2) - (\sigma_o/\omega_o)^2 \, b^2 \geq 0$$

The condition for passivity is then

$$|b| \leq \frac{\omega_o}{|p_o|}$$

Example 2: Consider a network N described at p_o with $\sigma_o, \omega_o > 0$ by

$$Y = Y_H = [1 \dotplus b^2] + jbE$$

Here

$$Y_c = \begin{bmatrix} 1 & jb \\ jb & 0 \end{bmatrix}$$

Clearly N can't be passive at p_o even though the Hermitian part is positive semi-definite.

Method 2: Here we will synthesize Y directly by extracting a term $Y^{(1)} = (p^2+\omega_1^2)^{-1}[pC^{(1)} + D^{(1)}]$. Such a term is realizable by a lossless network using a gyrator and is studied in detail in Ref. 3 (pp. 56-113) (a less detailed but more readily available study is Ref. 4 (p. 27)). For this it is convenient to use another canonical form in conjuction with Eq. (IV. 2). In Appendix 2, it is shown that $1 \pm (\sigma_o/\omega_o)\, b_i > 0$, for $i = 1$ & 2, and thus both of $(Y_{RS})_o \pm (\sigma_o/\omega_o)(Y_{IS})_o$ are positive definite. We can then diagonalize $(Y_{RS})_o + (\sigma_o/\omega_o)(Y_{IS})_o$ to 1_2 while at the same time diagonalizing $(Y_{RS})_o - (\sigma_o/\omega_o)(Y_{IS})_o$ to $[\lambda_1 \dotplus \lambda_2]$ with $\lambda_2 \geq \lambda_1 > 0$. We can then write

$$Y_o = \tilde{T}_o Y_c T_o = \tilde{T}_o \left\{ \begin{bmatrix} 1+\lambda_1 & 0 \\ 0 & 1+\lambda_2 \end{bmatrix} + j(\omega_o/\sigma_o) \begin{bmatrix} 1-\lambda_1 & 0 \\ 0 & 1-\lambda_2 \end{bmatrix} + (g'+jb')\,E \right\} T_o \qquad \text{(IV. 4)}$$

For the case $b_1 \geq b_2$ we can explicitly find

$$T_o = (1/\sqrt{2}) \begin{bmatrix} \left(1+\frac{\sigma_o}{\omega_o} b_1\right)^{1/2} & 0 \\ 0 & \left(1+\frac{\sigma_o}{\omega_o} b_2\right)^{1/2} \end{bmatrix}$$

$$b' = 2b\left[\left(1+\frac{\sigma_o}{\omega_o} b_1\right)\left(1+\frac{\sigma_o}{\omega_o} b_2\right)\right]^{-1/2}, \quad g' = 2g\left[\left(1+\frac{\sigma_o}{\omega_o} b_1\right)\left(1+\frac{\sigma_o}{\omega_o} b_2\right)\right]^{-1/2}$$

$$\lambda_1 = \left[1-\frac{\sigma_o}{\omega_o} b_1\right] / \left[1+\frac{\sigma_o}{\omega_o} b_1\right] \tag{IV. 5}$$

$$\lambda_2 = \left[1-\frac{\sigma_o}{\omega_o} b_2\right] / \left[1+\frac{\sigma_o}{\omega_o} b_2\right]$$

If $b_1 < b_2$ we can permute the rows and columns of Y_o and then relabel to obtain the above situation.

As in the case where $Y_{ISS} = 0$, we can write, p. 10, eqs (III.15), (III.16), (III.17).

$$y_1(p_o) = (1+\lambda_1) + j(\omega_o/\sigma_o)(1-\lambda_1)$$

$$= \frac{\gamma_1 p_o}{p_o^2 + \omega_1^2} \tag{IV. 6}$$

where

$$\gamma_1 = \sigma_o \left[(1+\lambda_1)^2 + (\omega_o/\sigma_o)^2(1-\lambda_1)^2\right]/\lambda_1 = \frac{4\sigma_o[1+b_1^2]}{\left(1-\frac{\sigma_o}{\omega_o} b_1\right)\left(1+\frac{\sigma_o}{\omega_o} b_1\right)}$$

$$\omega_1^2 = [\sigma_o^2 + \omega_o^2]/\lambda_1 \tag{IV. 7}$$

Since $\lambda_2 \geq \lambda_1$ we can split y_2 conveniently by writing

$$\lambda_2 = \lambda_1 + \lambda_o \tag{IV. 8}$$

which gives

$$y_2(p_o) = [(1+\lambda_1) + j(\omega_o/\sigma_o)(1-\lambda_1)] + \lambda_o\left(1-j\frac{\omega_o}{\sigma_o}\right)$$

$$= y_1(p_o) + \frac{\lambda_o[\sigma_o^2 + \omega_o^2]}{\sigma_o p_o} \qquad \text{(IV. 9)}$$

Thus we can write for Y_c of Eq. (IV. 4)

$$Y_c(p_o) = \frac{\gamma_1 p_o}{p_o^2 + \omega_1^2} 1_2 + \frac{\lambda_o[\sigma_o^2 + \omega_o^2]}{\sigma_o p_o} [0 \dotplus 1] + (g' + jb')\, E \qquad \text{(IV. 10)}$$

This is the matrix we will use for synthesis, which proceeds by cancelling the imaginary skew-symmetric term.

For this consider the positive real in the extended sense, EPR, matrix

$$Y^{(1)}(p) = (p^2+\omega_1^2)^{-1}[pC^{(1)}+D^{(1)}] \qquad \text{(IV. 11)}$$

Written out in full, this is

$$Y^{(1)}(p) = \frac{\{\sigma[\sigma^2+\omega^2+\omega_1^2]-j\omega[\sigma^2+\omega^2-\omega_1^2]\}C^{(1)}+\{[\sigma^2-\omega^2+\omega_1^2]-j2\sigma\omega\}\, D^{(1)}}{[\sigma^2-\omega^2+\omega_1^2]^2 + [2\sigma\omega]^2} \qquad \text{(IV. 12)}$$

We wish to form such a matrix to cancel the jb'E term of Y_c. Thus we wish for the (1, 2) term of $D^{(1)}$

$$d_{12} = \left\{\frac{[\sigma_o^2-\omega_o^2+\omega_1^2]^2+[2\sigma_o\omega_o]^2}{-2\sigma_o\omega_o}\right\} b' \qquad \text{(IV. 13)}$$

From this $D^{(1)}$ we then wish to form a minimal $C^{(1)}$ such that

$$Y^{(2)}(p_o) = Y_c(p_o) - Y^{(1)}(p_o) \qquad \text{(IV. 14)}$$

can be realized. Now, by definition,

$$D^{(1)} = j\omega_1\,[K - \widetilde{K}] = d_{12}E \qquad \text{(IV. 15)}$$

where K is a non-negative Hermitian (residue) matrix. Further

$$C^{(1)} = K + \widetilde{K} \qquad \text{(IV. 16)}$$

Thus let

$$K = \begin{bmatrix} k_{11} & k_1 + jk_2 \\ k_1 - jk_2 & k_{22} \end{bmatrix} \tag{IV. 17}$$

On forming $K - \tilde{K}$ we see that we wish to choose

$$k_2 = -\frac{d_{12}}{2\omega_1} \tag{IV. 18}$$

For a minimal $C^{(1)}$ let $k_1 = 0$ and det. $K = 0$. Then we require $k_{11}k_{22} = (d_{12})^2/4\omega_1^2$. Thus we will choose

$$k_{11} = k\,\frac{|d_{12}|}{2\omega_1} \qquad k > 0$$

$$k_{22} = \frac{1}{k}\,\frac{|d_{12}|}{2\omega_1} \tag{IV. 19}$$

Here k is a parameter which will be chosen later in such a manner as to insure the synthesis. This then fixes

$$K = (1/2\omega_1)\begin{bmatrix} k\,|d_{12}| & -jd_{12} \\ jd_{12} & (1/k)\,|d_{12}| \end{bmatrix} \tag{IV. 20}$$

The minimal $C^{(1)}$ is then given by

$$C^{(1)} = (|d_{12}|/\omega_1)\,[k + (1/k)] \tag{IV. 21}$$

We now choose k such that $\gamma_1 - c_{11} = 0$ (γ_1 is given in Eq. (IV.6)). Then

$$k = \frac{\gamma_1\omega_1}{|d_{12}|} = \frac{1}{|b|}\,\frac{[1+b_1^2]}{\left(1+\frac{\sigma_o^2}{\omega_o^2}\right)^{1/2}}\left[\left(1-\frac{\sigma_o}{\omega_o}b_1\right)\left(1+\frac{\sigma_o}{\omega_o}b_2\right)\right]^{1/2} \tag{IV. 22}$$

We have

$$\alpha \triangleq \gamma_1 - c_{22} = \gamma_1\,[1 - (1/k^2)] \tag{IV. 23}$$

k can be seen to satisfy $k^2 \geq 1$ in the following manner, and as a consequence $\alpha \geq 0$. We have

$$k^2 = \frac{1}{b^2} \frac{[1+b_1^2]^2}{\left[1+\frac{\sigma_o^2}{\omega_o^2}\right]^2} \left[\left(1-\frac{\sigma_o}{\omega_o} b_1\right)\left(1+\frac{\sigma_o}{\omega_o} b_2\right)\right]$$

$$\geq \frac{1}{b^2} \frac{[1+b_1^2]^2}{\left[1+\frac{\sigma_o^2}{\omega_o^2}\right]^2} \cdot b^2\left[1+\frac{\sigma_o^2}{\omega_o^2}\right] \geq 1 \qquad \text{(IV. 24)}$$

where use has been made of Eq. (A_2. 4) of the second appendix in getting the first inequality.

We have finally obtained

$$Y^{(2)}(p_o) = \left\{\frac{\alpha p_o}{p_o^2+\omega_1^2} + \frac{\lambda_o[\sigma_o^2+\omega_o^2]}{\sigma_o p_o}\right\}[0 \dotplus 1] + \frac{[\sigma_o^2-\omega_o^2+\omega_1^2]\, D^{(1)}}{[\sigma_o^2-\omega_o^2+\omega_1^2]^2+[2\sigma_o\omega_o]^2} \qquad \text{(IV. 25)}$$

Each term of $Y^{(2)}(p_o)$ can easily be realized; the last by a gyrator and the other two by an inductance and capacitance and an inductance, respectively. Since $Y^{(1)}$ is itself realizable we have realized the given Y.

Several points are worth noting. Method 1 uses one gyrator while Method 2 may require two. The proof that Method 2 works is essentially given by Method 1, since the inequality of Eq. (IV. 24) rests upon Eq. (IV. 3). This appears to cause trouble for generalizing to n x n matrices. The choice of K for Eq. (IV. 20) appears to be somewhat arbitrary; perhaps a better choice exists.

CONCLUSIONS

In Sections III and IV we have given methods for synthesizing a finite passive network N_P such that a given N with $q_-(p_o) \geq 0$ has the same admittance matrix as N_P. For the n-port we have shown how to synthesize N_P if $Y_{ISS} = 0$ and, in fact, three different ways of doing this were given. In the case that $Y_{ISS} = 0$, an extremely easy test for $q_- \geq 0$ was also given which avoids the absolute values needed in Eq. (II.1). Since a one-port always has $Y_{ISS} = 0$, the only unsolved case is $Y_{ISS} \neq 0$ for $n > 2$. A solution to this problem appears to be given by extending the second method of Section IV; however, the proof seems to be complicated.

Clearly, a dual synthesis holds for those N which have impedance matrices. For the two-ports which have no Z or Y the methods of a companion paper can be used to obtain a Z or Y while preserving q_-, (Ref. 5).

APPENDIX 1: SIMULTANEOUS DIAGONALIZATION OF TWO SEMI-DEFINITE MATRICES

Theorem A. 1: If A and B are real, symmetric, positive semi-definite matrices of rank r_a and r_b, respectively, with $r_b \geq r_a$, then there exists a real, non-singular matrix T such that

$$A = \tilde{T}\,[1_{r_a} \dotplus 0_{n-r_a}]\,T$$

$$B = \tilde{T}[\Lambda_{r_a} \dotplus 1_{r_b-r_o} \dotplus 0_{n-r_a-r_b+r_o}]\,T$$

where

$$\Lambda_{r_a} = \text{diag.}\,[\lambda_1, \ldots, \lambda_{r_a}]$$

with $\lambda_i > 0$ for $i = 1, \ldots, r_o$ (if r_o), $\lambda_i = 0$ for $r_o + 1, \ldots, r_a$.

Proof: We first find a real, non-singular T_o such that

$$A = \tilde{T}_o[1_{r_a} \dotplus 0_{n-r_a}]\,T_o$$

$$B = \tilde{T}_o\,B_o\,T_o$$

If any of the last $n - r_a$ diagonal elements of B_o are zero, the entire row and column of B_o are zero, since B_o is semi-definite. For the last $n - r_a$ diagonal elements of B_o which are non-zero we can reduce the remaining non-diagonal elements in these rows and columns to zero. We must do this by adding the diagonal element to the off-diagonal element. This won't destroy the form of $1_{r_a} \dotplus 0_{n-r_a}$ and we can then write

$$A = \tilde{T}_o\,\tilde{T}_1\,[1_{r_a} \dotplus 0_{n-r_a}]\,T_1\,T_o$$

$$B = \tilde{T}_o\,\tilde{T}_1\,[B_{r_a} \dotplus 1_{r_b-r_o} \dotplus 0_{n-r_a-r_b+r_o}]\,T_1\,T_o$$

since $r_b \geq r_a$ by assumption. Here $r_o \geq 0$ is defined as the rank of B_{r_a}. We can now diagonalize B_{r_a} by an orthogonal transformation T_2'. Let $T_2 = T_2' \dotplus 1_{n-r_a}$ and then let $T = T_2\,T_1\,T_o$ to give

$$A = \tilde{T}[1_{r_a} \dotplus 0_{n-r_a}]\, T$$

$$B = \tilde{T}[\Lambda_{r_a} \dotplus 1_{r_b-r_o} \dotplus 0_{n-r_a-r_b-r_o}]\, T$$

as required. Q. E. D.

By observing that neither the sign of 1_{r_a} nor that of $\Lambda_{r_a} \dotplus 1_{r_b-r_o}$ enter into the proof, we see that any two semi-definite matrices can be simultaneously diagonalized. Further, if A and B are allowed to be complex but Hermitian, similar results are obtained.

APPENDIX 2: CONDITIONS ON THE PARAMETERS WHEN $Y_{ISS} \neq 0$, $n = 2$

Consider Y_c of Eq. (IV. 3). We require $(Y_{RS})_c \pm (\sigma_o/\omega_o)(Y_{IS})_c$ to be positive semi-definite, by Theorem 1 (clearly, $\omega_o > 0$ since we are assuming $Y_{ISS} \neq 0$). Thus

$$\begin{bmatrix} 1 \mp (\sigma_o/\omega_o)\, b_1 & (g+bb_1) \pm (\sigma_o/\omega_o)(b-gb_1) \\ (g+bb_1) \pm (\sigma_o/\omega_o)(b-gb_1) & (1-b^2+b_1^2+g^2) \pm (\sigma_o/\omega_o)(b_2+b_1^2 b_2+b^2 b_1-g^2 b_1) \end{bmatrix} \qquad (A_2.\ 1)$$

must be positive semi-definite. Since $b \neq 0$, we clearly require

$$1 \pm (\sigma_o/\omega_o)\, b_1 > 0 \qquad (A_2.\ 2)$$

$$1 \pm (\sigma_o/\omega_o)\, b_2 > 0 \qquad (A_2.\ 3)$$

where the second of this is found by noting that we could have extracted the gyrator from terminal pair two instead of one to obtain a Y_c.

Equation $(A_2.\ 1)$ must be positive semi-definite for every value of g and especially g = 0. Using this value of g we find for the determinant

$$-b^2\left[\left(1 \mp \frac{\sigma_o}{\omega_o} b_1\right)^2 + \left(b_1 \pm \frac{\sigma_o}{\omega_o}\right)^2\right] + \left(1 \mp \frac{\sigma_o}{\omega_o} b_1\right)\left(1 \pm \frac{\sigma_o}{\omega_o} b_2\right)\left(1+b_1^2\right) \geq 0$$

or

$$\left(1 \mp \frac{\sigma_o}{\omega_o} b_1\right)\left(1 \pm \frac{\sigma_o}{\omega_o} b_2\right) \geq b^2 \left(1 + \frac{\sigma_o^2}{\omega_o^2}\right) \qquad (A_2.\ 4)$$

which is the condition needed for Eq. (IV. 24). Using the two signs in separate equations which are then added we can obtain the interesting result

$$1 - \frac{\sigma_o^2}{\omega_o^2} b_1 b_2 \geq b^2 \left(1 + \frac{\sigma_o^2}{\omega_o^2}\right)$$

APPENDIX 3: THE NON-REALIZABILITY OF THE COMPLEX TRANSFORMER FOR Re $p > 0$

A device described by

$$V_2 = NV_1 \qquad\qquad I_1 = -\tilde{N}^* I_2 \tag{A_3. 1}$$

with N, an n x m complex constant matrix, is called a complex transformer. Using the notation of Fig. 1 with $\tilde{T}$ replaced by $\tilde{N}^*$ we see that if such a device is loaded at the series terminals by an n-port of admittance matrix Y_2, the input admittance is $Y_1 = \tilde{N}^* Y_2 N$. Further, we have $\tilde{V}_2^* I_2 + \tilde{V}_1^* I_1 = 0$. Such a device thus has some of the useful properties of a transformer network, and, in fact, can be realized at a single real frequency $p_o = j\omega_o$, (Ref. 6, p. 611).

Because of the nice transforming property, the problem arises as to whether such a device can be realized at a complex frequency p_o with $\sigma_o > 0$. To see that it can't, we first note that it is sufficient to consider N to be 1 x 1, since, through the use of elementary complex transformations, any N can be realized by an interconnection of two-winding transformers. Such a device is described by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1/n & 0 \\ 0 & n^* \end{bmatrix} \tag{A_3. 2}$$

In order to work with an admittance matrix we connect a gyrator of unit gyration resistance in cascade with port one of the complex transformer, see Fig. 9. The resultant transmission matrix is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_o = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/n & 0 \\ 0 & n^* \end{bmatrix} = \begin{bmatrix} 0 & n^* \\ 1/n & 0 \end{bmatrix} \tag{A_3. 3}$$

Converting this to an admittance matrix yields, after letting $(1/n) = a + jb$

$$Y_o = \begin{bmatrix} 0 & 1/n \\ -1/n^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & a+jb \\ -a+jb & 0 \end{bmatrix} \tag{A_3. 4}$$

Since Y_o has a zero Hermitian part, it has $q_- < 0$ for $\omega_o \neq 0$ and $q_- = 0$ if $\omega = 0$. These values for q_- are identical to those for the complex transformer itself since the cascade gyrator doesn't alter q_-. Consequently, the complex transformer isn't passive if $\omega_o \neq 0$ and isn't real for real p if $\omega = 0$; it can't be realized by a finite passive network. This is also expected by an inspection of Q_- where $\tilde{V}_2 Y_2 V_2 = \tilde{V}_1 \tilde{N} Y_2 N V_1 \neq \tilde{\bar{V}}_1 \tilde{N}^* Y_2 N V_1$.

It should be noted also that if $\sigma_o = 0$, $q_- = 0$ and the method of Section III can be used to realize Eq. (A_3. 3) and with it any complex N. Application of such a device can be found in Refs. 6 and 7.

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