A THEORETICAL EXPLANATION FOR THE BEHAVIOR OF THE RESIDUAL ERROR ENERGY IN PREDICTIVE FILTERING

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SUMMARY

It is well known that the Residual Error Energy at the output of a Linear Prediction Filter must be a positive function of the Filter order showing a monotonical decrease in its value as the order increases. This fact may be of great interest for the design of truncation criteria to establish the maximum size of Predictive Filters. Through the present paper a theoretical explanation is given to justify such behavior and to expose the most frequent patterns which can be found in real energy traces from different signals. Finally, conclusions are discussed and several applications are pointed out.

1. INTRODUCTION

A Prediction Error Filter [1] may be viewed as a Linear Operator $H_k\{\cdot\}$ that when applied to a given signal $x(n)$ yields a Residual Error Signal (RES) $\varepsilon_k(n)$:

$$\varepsilon_k(n) = H_k\{x(n)\} = \sum_{i=0}^{k} h_{ki}\, x(n-i) \qquad (1)$$

whose Energy E_k:

$$E_k = \sum_n (\varepsilon_k(n))^2 \qquad (2)$$

has been minimized with respect to the set of filter coefficients $\{h_{ki}\}$:

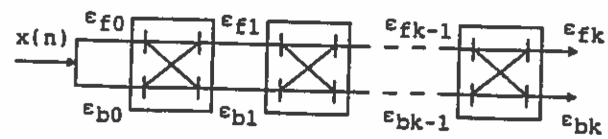
$$\frac{\partial E_k}{\partial h_{ki}} = 0, \qquad 0 \le i \le k \qquad (3)$$

This Prediction Error Filter may be implemented as a Lattice Structure (see Fig. 1) by means of Levinson's Recursion [2], in which case the Residual Error Energy (REE) may be considered minimum with respect to the so called PARCOR Coefficient (also known as Reflection Coefficient) c_k [3]:

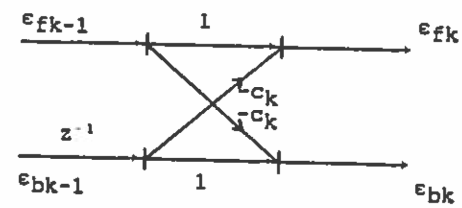
$$\frac{\partial E_{fk}}{\partial c_k} = 0 \qquad 1 \le k \le p \qquad (4)$$

with p being the order of the process $x(n)$. In (4) E_{fk} is the Forward Residual Error Energy trace (FREE), and may be defined as $E_{fk}=E_k$, because as it is well known [4], the output ε_k of the Linear Predictor in (1) may be identified with the Forward Residual Error signal $\varepsilon_k(n)=\varepsilon_{fk}(n)$. Based on these facts Itakura and Saito [5] proposed a mixed method to estimate c_k which they called PARtial CORrelation (PARCOR) coefficient as:

$$\{c_{Ik}\}^2 = \frac{\{T_{k-1}\}^2}{E_{fk-1}\cdot E_{bk-1}} \qquad (5)$$



a) General Structure.



b) Structure of a single section.

Fig. 1. Prediction-Error Lattice Filter.

which allows (2) to be rewriten as:

$$E_{fk} = E_{fk-1}\,(1-\{c_k\}^2) \qquad (6)$$

When dealing with real series, as it will be our case, E_{fk} will be positively defined as deducted from (2). We will also show that the following relation holds:

$$|c_k|^2 \le 1 \qquad (7)$$

For such purposes we will consider both $\varepsilon_{fk}(n)$ and $\varepsilon_{bk}(n)$ as a pair of vectors in a given N-dimensional vector space, for $0 \le n \le N-1$, and then in (5) it may be considered that:

$$T_{k-1} = \langle \varepsilon_{fk-1}(n), \varepsilon_{bk-1}(n-1)\rangle \qquad (8)$$

$\langle\cdot,\cdot\rangle$ being defined as the scalar or inner product between two given vectors in the space considered. Then (5) may be regarded as:

$$\{c_k\}^2 = \frac{\langle \varepsilon_{fk-1}, \varepsilon_{bk-1}\rangle^2}{\|\varepsilon_{fk-1}\|^2\cdot\|\varepsilon_{bk-1}\|^2} = \cos^2\theta_k \qquad (9)$$

with $\|\cdot\|$ the norm of a vector in the above mentioned space, and θ_k being the trigonometric angle between ε_{fk-1} and ε_{bk-1}. As such, when dealing with real vectors we would expect that $0 \le \cos^2\theta_k \le 1$. Now having in mind (6) and (9) it will be easy to see that E_{fk} must be a monotonically decreasing series with k increasing.

2. GENERAL MODELS

The profile for E_{fk} when plotted against k show certain patterns depending on the nature of the signal $x(n)$ being considered. This signal may be modeled by a Production System, as the ones shown in Fig. 2 a, b, c and d. Several main factors will then influence the mentioned profiles, these being:

- The nature of $x(n)$, which may be purely autoregressive (AR), or mixed autoregressive moving average (ARMA).

- The kind of signal being used as input to the production model considered, which may be Line Power Spectra (LPS) or White Power Spectra (WPS) [6].

- The arrangement shown by PARCOR coefficients with k for the signal under study. The cases which will be considered here, are the Almost Flat (AF), the Peak Type (PT), the Regularly Alternating (RA) and the Shifted Hiperbolic (SH).

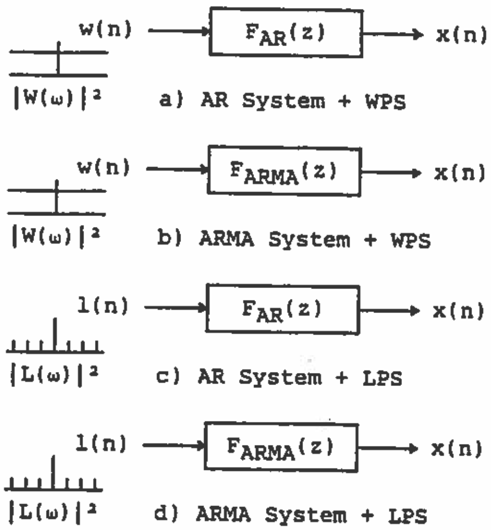


Fig. 2. Different models for $x(n)$.

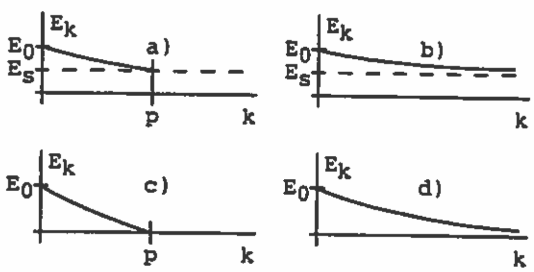


Fig. 3. Different tendencies for E_k.

According with [6], the FREE in a Prediction Error Filter may evolve in one of the following ways depending on the Production Model:

a) As a Decreasing Series starting at E_0 and ending at a steady value E_s for a given $k=p$ (see Fig. 3.a). In this case $c_{Ik}=0$ for $k>p$, $x(n)$ being modeled as in Fig. 2.a. A special case of a) takes place when $E_s=0$ (see Fig. 3.c), which means that the modeling system is of the kind in Fig. 2.c. As it is well known an AR process may be characterized by a transfer function such as:

$$F_{AR}(z) = \frac{A_0}{\sum_{i=0}^{p} a_{pi}\, z^{-i}} \qquad (10)$$

$\{a_{pi}\}$ being the set of filter coefficients, and p the order of the model. If such a system is fed at its input with a WPS signal $w(n)$, the z-transform of the signal at the output should be characterized by an all-pole power spectrum. When such a signal is fed at the input of a Prediction Error Filter as the one in (1), it will yield a FREE trace at the output $\varepsilon_{fk}(n)$ with a z-transform given by:

$$\varepsilon_{fk}(z) = S_{\varepsilon fk}(z)\cdot W(z) \qquad (11)$$

in which case $S_{\varepsilon fk}(z)$ will be:

$$S_{\varepsilon fk}(z) = H_k(z)\cdot F_{AR}(z) = A_0 \frac{\sum_{i=0}^{k} h_{ki}\, z^{-i}}{\sum_{i=0}^{p} a_{pi}\, z^{-i}} \qquad (12)$$

As it is well known [4], when the set $\{h_{ki}\}$ is chosen according with the minimization process implied in (4), $\{h_{ki}\}$ shows a convergence tendency to $\{a_{pi}\}$ as $k \to p$. This value of k is then called the "optimal size of the predictor". In order to detect when we are approaching to this optimal value for k, we will make use of the energy trace E_{fk}. First we will assume that the energy of $x(n)=\varepsilon_{f0}(n)$ is $E_{f0}>0$. Then from (6) we will find that:

$$E_{fk} = E_{f0} \cdot \prod_{i=1}^{k} (1-c_i^2) \qquad (13)$$

and having in mind (7) it will be easy to see that E_{fk} will decrease as far as $c_i \neq 0$, (see Fig. 3.a). As k approaches p, the Residual Error series will more resemble a WPS signal:

$$\varepsilon_{fk}(n) \to A_0\cdot w(n) \qquad (14)$$

and from this fact it may be shown [6, pg. 417] that $T_{k-1}=0$ which in turn renders $c_k=0$ and then from (13):

$$E_{fk} = E_s; \quad k > p \qquad (15)$$

showing that the optimal size of the filter must be that value of k which renders E_{fk} to a steady value, and thus serving as a good marker to estimate filter orders.

b) As a Decreasing Series starting at E_0 and tending but never reaching a steady value E_s (see Fig. 4.b). In this case $\{c_k\}^2>0$ for $0\leq k<\infty$, and $x(n)$ may be seen as the output of the system in Fig. 2.b. A special case of b) takes place when $E_s=0$ (see Fig. 3.d), which means that the modeling system is of the kind in Fig. 2.d. In a similar way we may express the transfer function of an ARMA system as:

$$F_{ARMA}(z) = A_0 \frac{\sum_{i=1}^{q} b_{qi}\, z^{-i}}{\sum_{i=0}^{p} a_{pi}\, z^{-i}} \qquad (16)$$

$\{api\}$ and $\{bqi\}$ being the set of coefficients for the AR and MA parts, and p and q the orders of the model. In this case $X(z)$ will be characterized by a pole-zero power spectrum. When feeding this signal to the input of a Prediction Error Filter as that in (1), we will get a FREE trace at the output $\varepsilon_{fk(n)}$ with z-transform given by:

$$\varepsilon_{fk}(z) = S_{\varepsilon fk}(z)\cdot W(z) \qquad (17)$$

having defined:

$$S_{\varepsilon fk}(z) = H_k(z)\cdot F_{ARMA}(z) =$$

$$= A_0 \frac{\sum_{i=1}^{q} b_{qi}\, z^{-i}}{\sum_{i=0}^{p} a_{pi}\, z^{-i}} \sum_{i=0}^{k} h_{ki}\, z^{-i} \qquad (18)$$

In contrast with the AR case, when we adjust the set $\{h_{ki}\}$ according with (4) there is not a finite value of k which renders E_{fk} to a steady value. Rather than that, it may be shown [6, pg. 418] that for a general ARMA process $E_{fk} \to Es$ only with $k \to \infty$, as it may be seen in Fig. 3.b, and the kind of Prediction Error Filter which would result in this case, $H_\infty(n)$ is known as the "equivalent Wiener Filter". A further discussion on the convergence properties of this approach to model ARMA processes may be found in [7]. A slowly but non stopping decrease in E_{fk} should then be interpreted as an evidence of the MA or ARMA nature of $x(n)$ [8].

3. TYPICAL ENERGY PROFILES

The aim of the paper being discussed is to explore the patterns that equation (13) will yield when plotted vs. k, depending on the particular set of Reflection Coefficients $\{c_k\}$ for a given signal $x(n)$. For such we will consider that k is the discrete digitizing index of a certain arbitrary continuous variable ζ, which in some contexts such as Transmission Line Theory could be regarded as a spatial coordinate. Then, we introduce:

$$\zeta = k\cdot\Delta\zeta \qquad 0 \le \zeta \le L \qquad (19)$$

With this in mind, we will now rewrite (6):

$$\frac{\partial E_f(\zeta)}{\partial \zeta} = -\, c^2(\zeta)\cdot E_f(\zeta) \qquad (20)$$

for which we have taken:

$$\{c_k\}^2 = c^2(\zeta)\cdot\Delta\zeta \qquad (21)$$

$c(\zeta)$ being the "Reflectivity Function" or "PARCOR Function" of the system under study, whether it be a Transmission Line or not. Equation (20) may be integrated to yield:

$$E_f(\zeta) = E_f(0)\, \exp\left(-\int_0^{\zeta} c^2(\eta)\, d\eta\right) \qquad (22)$$

There are some particular cases for $c^2(\zeta)$ which we will examine in this section. First we will introduce the Almost Flat (AF) profile, in which the Squared PARCOR Function $c^2(\zeta)$ may be considered approximately constant for all $\zeta \ge 0$ (exponential line):

$$c^2(\zeta) = \alpha^2; \quad \zeta \ge 0 \qquad (23)$$

In this case, as it can be seen in Fig. 4.a and b $E_f(\zeta)$ behaves as a decaying exponential:

$$E_f(\zeta) = E_f(0)\, e^{-\alpha^2\zeta} \qquad (24)$$

The next case to be studied is the Peak Type (PT) profile, in which the Squared PARCOR Function is composed by different AF sections, each one of them with a different constant $\alpha_1, \alpha_2, \ldots \alpha_m$. as it can be seen in Fig. 4.c and d. The Energy trace will show in this case concave curvature. In the third place we will introduce the general form of the Regularly Alternating (RA) profile, in which the Squared PARCOR Function may be represented by linear sections with different slopes, as in Fig. 4.e and f. For each one of these sections we would have:

$$c^2(\zeta) = \alpha\zeta + \beta \qquad (25)$$

In this case the Energy profile will show both convex and concave cuvatures, and may be expressed as:

$$E(\zeta) = E(0)\, \exp\left(-\frac{\alpha\zeta^2}{2} - \beta\zeta\right) \qquad (26)$$

Finally we have the Shifted Hyperbolic (SH) profile (see Fig. 4.g and h), in which the Squared PARCOR Function may be represented as:

$$c^2(\zeta) = \frac{1}{\beta - \zeta} \qquad 0 \le \zeta \le \beta \qquad (27)$$

and this yields:

$$E(\zeta) = E(0)\, \frac{\beta - \zeta}{\beta} \qquad 0 \le \zeta \le \beta \qquad (28)$$

This Energy profile has a straight line shape, serving as a model for AR systems with LPS inputs as those in Fig. 2.c and 3.c.

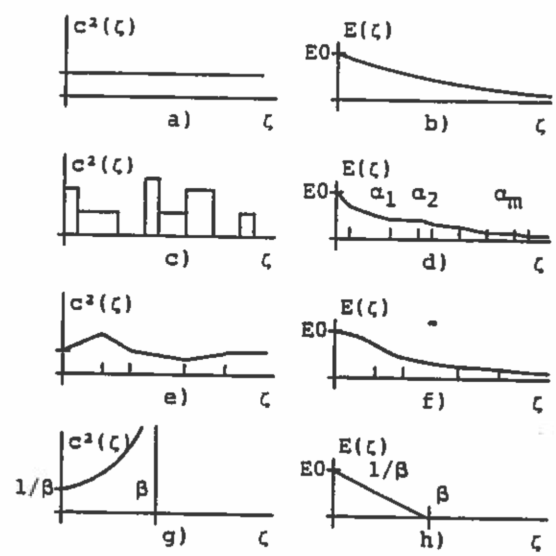
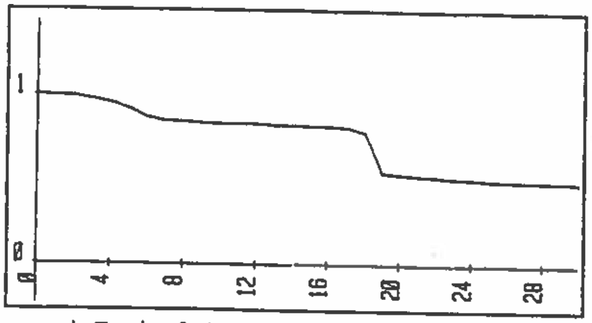
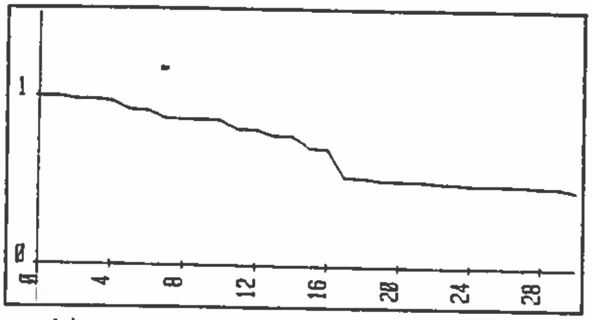


Fig. 4. Profiles for $c^2(\zeta)$ and $E(\zeta)$.

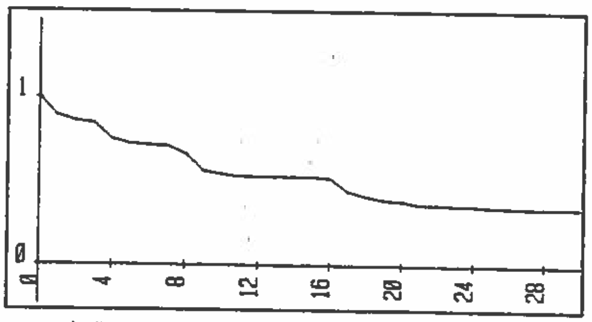
In Fig. 5.a, b and c several real FREE traces are shown corresponding to ECG (Electro Cardio Graphic) signals. As it can be seen, the first two traces are related with those in Fig. 4.e and f, and the last trace is related with Fig. 4.c and d.



a) Typical ECG FREE trace.



b) ECG FREE trace for Right Branch Blockade.



c) ECG FREE trace for Acute Infarction.

Fig. 5. Real FREE traces for several ECG signals.

4. DISCUSSION

The study of the Residual Error Energy traces may be of great interest in several fields. Among them we would mention the determination of the nature and order of a given process x(n). For example, the trace in Fig. 5.a could be produced by an AR system of k=7. The pattern in Fig. 5.b do not show marked correlation lags, and as such would require a Wiener Predictor or a MA representation [9]. In Fig. 5.c correlation lags may be easily observed, and this could be used to invert the residual error traces for their synthesis by an ARMA model [8]. Another field of application may be the classification of different kinds of traces. For such an idea to be practical a normalized trace must be produced, and a measure coefficient must be established on the similarity between two given normalized traces. A possible measure is based on the squared error between the two traces and their correlation coefficient could also be used. Both methods are of special interest, the first one measures the distance between the two traces, and the second one gives their linear dependence. Finally some other applications would include the design of criteria to develop new methods for the inversion of transmission line-like systems [10], which would find many application fields, such as non-invasive measurement techniques [11].

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