ELECTRONIC CIRCUITS FOR CHAOS USING HYSTERESIS

CIRCUITOS ELECTRONICOS PARA GENERAR CAOS MEDIANTE HISTERESIS

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TWO MEANS OF GENERATING CHAOS BY USING HYSTERESIS IN ELECTRONIC CIRCUITS ARE DESCRIBED. ONE SYSTEM USES BINARY HYSTERESIS AND HAS THE ADVANTAGES THAT TO DATE IT IS THE ONLY ONE FOR WHICH THE SIGNALS CAN BE PROVEN CHAOTIC AND THAT IT IS ONLY OF DEGREE TWO. THE OTHER SYSTEM IS A THREE-DIMENSIONAL ONE USING BENT HYSTERESIS AND GENERATING SIGNALS VIA THE INSERTION OF DYNAMICS INTO A TWO-DIMENSIONAL DESIGN IN THE LIENARD PLANE. THE SYSTEM DESIGNS ARE PRECEEDED BY A DISCUSSION OF THE CONCEPT OF CHAOS IN THE CONTEXT OF ELECTRONIC CIRCUITS.

SE DESCRIBEN DOS FORMAS DE GENERAR CAOS UTILIZANDO CIRCUITOS ELECTRONICOS CON COMPORTAMIENTO HISTERETICO. UNO DE LOS SISTEMAS EMPLEA HISTERESIS BINARIA Y PRESENTA LAS VENTAJAS DE QUE HASTA LA FECHA ES EL UNICO PARA EL CUAL SE PUEDE DEMOSTRAR QUE SUS SEÑALES HAN SIDO DISEÑADAS PARA SER CAOTICAS, ASI COMO DE SER UN SISTEMA DE SEGUNDO GRADO UNICAMENTE. EL OTRO SISTEMA PRESENTADO ES DE ORDEN TRIDIMENSIONAL, Y SE BASA EN UNA HISTERESIS DE DOBLADURA, VINIENDO DADO SU FUNCIONAMIENTO POR LA EVOLUCION DINAMICA DE UN SISTEMA BIDIMENSIONAL EN EL PLANO DE LIENARD. LOS DISEÑOS PRESENTADOS VAN PRECEDIDOS POR UNA DISCUSION ACERCA DEL CONCEPTO DE CAOS EN EL CONTEXTO DE LOS CIRCUITOS ELECTRONICOS.

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Recently the field of chaotic systems has come into prominence due to the strange nature of its results. In particular one can obtain noise like signals from very deterministic low order systems. As a consequence it becomes possible to ascribe strange occurrances in well designed systems that were previously never thought possible to have some of the observed behaviors, such as the chaotic motion of engineered bridges or randomness in deterministic models of economic & weather systems. Perhaps more importantly it now seems possible to model such phenomena as epileptic fits and heart arrythmias by chaotic systems. This being the case, it is of interest to get analog models of chaotic systems so that nondestructive simulations of possibly very dangerous effects can be made. Thus we are led to the design of electronic circuits which exhibit chaos, which we take as the topic of this paper. Although the field of chaotic systems has really only developed during the last ten years there is already an immense literature. In contrast to the general field of chaotic systems the literature on design of electronic circuits to give chaotic phenomena is still rather limited. Nevertheless, in principle it is rather easy to create electronic circuits which have chaotic behavior; one simply obtains an electronic analog for the differential equations known to yield chaos. Although this is a straightforward matter it most often is not practical because of the difficulty of realizing the necessary nonlinearities in electronic form. However, once one sees what the principles of chaos generation are one can sometimes create the desired phenomena by using nonlinearities that are relatively easily obtained in electronic circuits. Here we discuss two cases where this latter is the case by using hysteresis as the nonlinearity. These systems have the advantage that the concepts behind their operation are relatively easily understood and the electronic circuits are simple to construct.

CHAOS

What does it mean for a system to be chaotic? There is a precise mathematical meaning which we will look at shortly, but first let us consider the more intuitive nature of chaotic signals, where in this paper we consider continuous time real valued systems. In essence we consider any signal arising from a deterministic system but that looks noise-like, or to be behaving in a strange *nonpredictable* way, to be chaotic. Thus, the signal of figure 1a is of the type that is considered chaoctic (here if one runs the signal for a long period of time the number of peaks and valleys between the maximum peaks is unpredictable). If one has a pair of such signals representing state variables for such a system, say x (t) and y (t), and one plots x versus y with time t as the running parameter, then for a chaotic system one expects a portion of the x y plane to be filled in, as in figure 1b. Likewise if one observes a chaotic signal on a spectrum analyzer one sees a continuous spectrum which often is jumping around. These observations are then helpful in recognizing chaotic signals, and indeed are often used in claiming that a system is chaotic when no other means, such as mathematical proofs, are available. And this is most often the case for continuous time systems since mathematical proofs of the chaotic nature of signals is often beyond the reach of present knowledge. However, due to experimental error or limited observation time or computer round off error, etc., one can never be sure that the system under test is really chaotic in the mathematical sense.

A mathematical definition of chaos has been put forth in the fascinating paper of Li & Yorke.[1] Although the work of Li & Yorke was phrased for discrete time systems, their definition can be taken over to continuous time systems as follows. Let there be an open region of initial values which a system signal x can assume and let X_1 and X_2 be two of these initial values (that is, values of x at $t = 0$) and let $x_1 = x_1(.)$ and $x_2 = x_2(.)$ be the resulting signals defined on $t \geq 0$. Then we can call the system chaotic if there exist initial X_1 & X_2 such that the difference

$$d(t) = x_1(t) - x_2(t) \tag{1a}$$

has the following two properties

$$\liminf_{t \to \infty} | d(t) | = 0 \tag{1b}$$

$$\limsup_{t \to \infty} | d(t) | > 0 \tag{1c}$$

Equations (1) say that a system under this definition is chaotic if there exists two starting values such that as time goes on the two resulting signals become arbitrarily close at some instants of time while also at other instants of time the signals are separated. It should be noted that this need not happen for any starting values; indeed for some starting values there may be periodic solutions for which d(t) is identically 0.

Li and Yorke gave a criteria which guaranteed that a discrete time system would be chaotic, this being the period three implies chaos result. To phrase this criteria more precisely we convert to a discrete time system by selecting distinguished values of time, which we index, giving a sequence of times t_0, t_1, t_2, t_3, etc... Next we can consider the value of x(.) at the present indexed time t_i to be a map M(.) of x(.) at the previous indexed time t_{i-1}. Thus,

$$x(t_i) = M(x(t_{i-1})) \tag{2}$$

The discrete system map M(.), considered to be real valued, is determined by the original continuous time system and is said to be of period three if there are points p_1, p_2, and p_3 such that

$$p_2 = M(p_1) > p_1$$
$$p_3 = M(p_2) = M^2(p_1) > p_2$$
$$p_4 = M(p_3) = M^2(p_2) = M^3(p_1) = p_1 \tag{3}$$

That is, the map M is periodic of period three if there is a starting point p_1 such that after three iterations the value of the mapping is the same as the starting point, i. e. $p_4 = p_1$. The Li & Yorke period three result is that a discrete time system with a continuous map M(.) is chaotic if M is of period three. In actual fact the result is a bit nicer in that the system is chaotic if p is no larger than p in (3).

As for the concept of chaotic systems, this period three result was really only given by Li & Yorke for discrete time:

$$d(t) = M^t(X) - M^t(X) \text{ with } M^t(.) = M(M^{t-1}(.)).$$

Consequently, it is necessary to spell out how we should choose M(.) and the distinguished times to allow us to convert from x(t) to M (X), which in turn gives the relation of the chaotic discrete time map M to a chaotic continuous time system under consideration. To undertake this conversion one wishes to carry over the chaotic motion of the continuous time systems to the related discrete time map. In our case we can do this by using the local peaks of the continuous time systems, that is, choosing the distinguished times such that the discrete time map is a map of one local continuous time peak into the next. This procedure is illustrated by the binary hysteresis chaos generator discussed below.

In the following we give two kinds of hysteretic systems, one for which we make a design to guarantee that a period three map M exists and another for which we obtain chaotic kind of signals, but for which no proof yet exists to guarantee that they are actually chaotic. For the first situation we give a design using binary hysteresis in an otherwise simple linear second order system while in the second we use bent hysteresis in a rather intriguing third order system design.

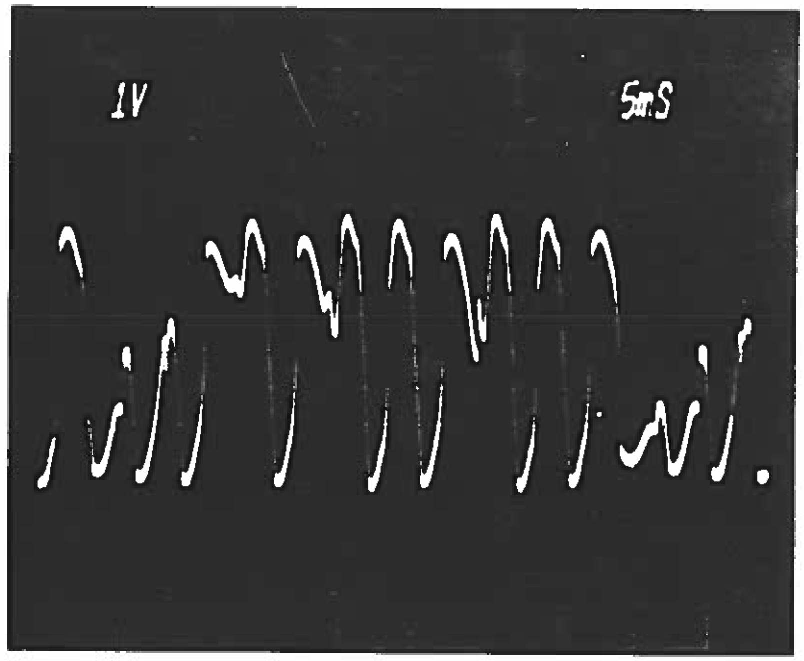
BINARY HISTERESIS CHAOS GENERATOR

In this section we discuss the binary hysteresis chaos generator presented in (2). The importance of this circuit is twofold in that to date it is the only one for which the trajectories have been proven to be chaotic and, secondly, it is of only degree two (thus, using only two dynamic elements, capacitors here, whereas other continuous time circuits require three or more).

A signal-flow graph is given in figure 2a for our system from which we proceed to develop the idea behing the circuit's operation as well as the circuit itself. In figure 2a there are two integrators, designated by the Laplace transform integration symbol 1/s, and a binary hysteresis path designated by h(x). One observes from the signal-flow graph that in the absence of the hysteresis the system is a negatively damped second orden linear one in which case it would be very unstable. The presence of the hysteresis changes the system characteristics in such a way that the system becomes stable but in a very strange way. To understand this we introduce the mathematical characterizations of the system components. Thus, by binary hysteresis we mean the hysteresis given in figure 2b and described by

$$h(x) = \begin{cases} 1 \text{ if } x_L < x \text{ upper branch} & (4a) \\ 0 \text{ if } x < x_U \text{ lower branch} & (4b) \end{cases}$$

In reality binary hysteresis is a kind of switch where switching is different depending upon the



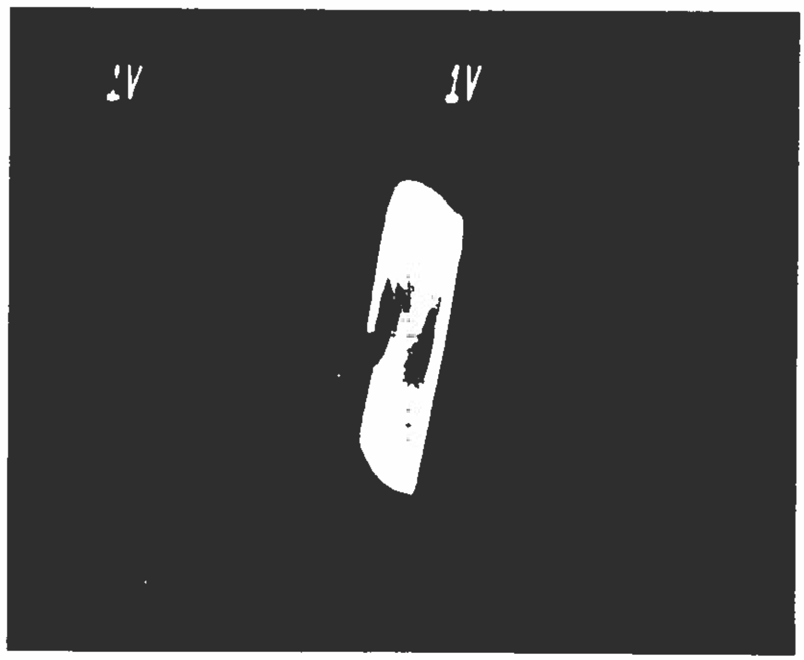


FIGURE 1

Chaotic kinds of signals (from[3])

FIGURE 1a Versus time. FIGURE 1b Phase plane plot.

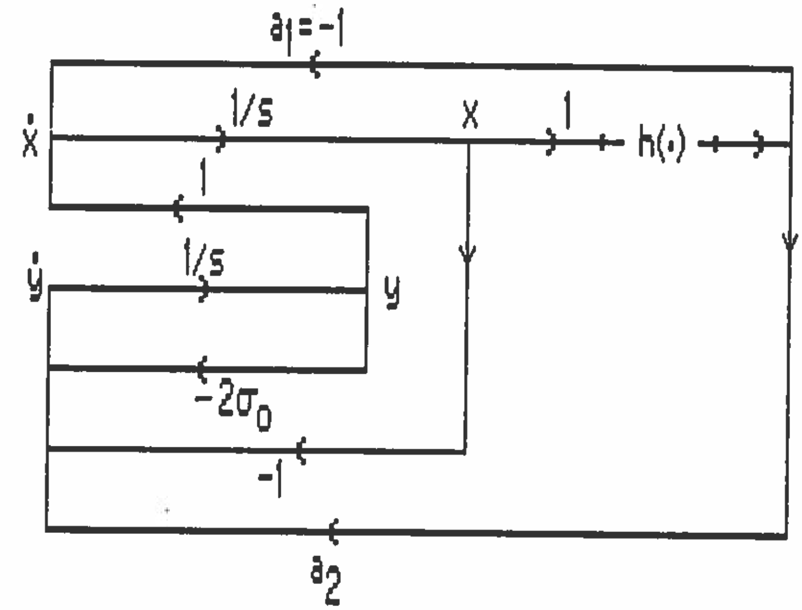


FIGURE 2

Binary hysteresis chaotic system. FIGURE 2a Signal-flow graph.

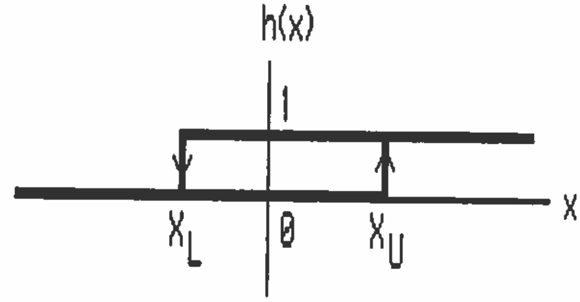


FIGURE 2b Binary hysteresis.

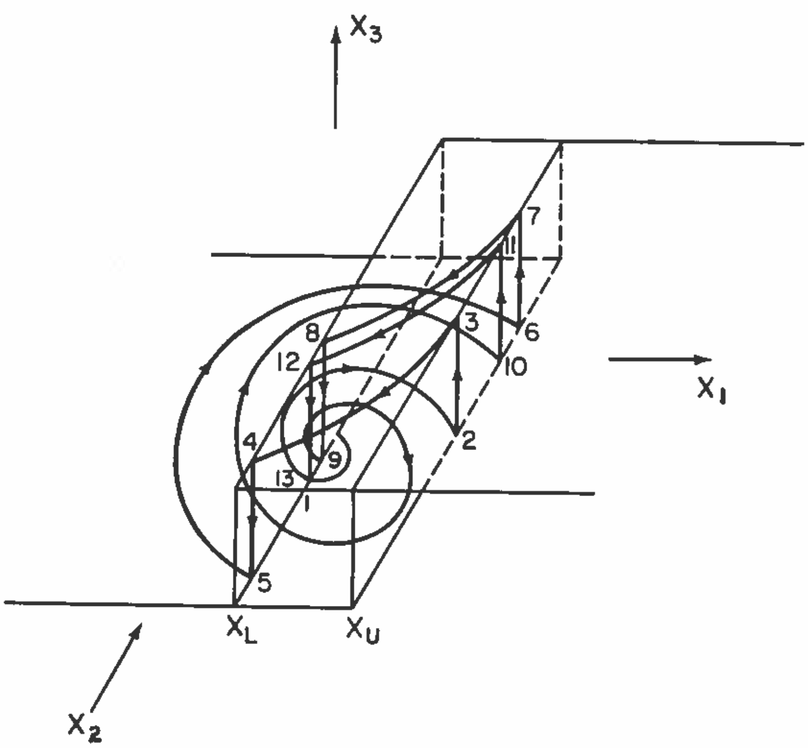


FIGURE 3

Binary hysteresis chaotic trajectories.

immediate past history (the history being something which is not quite reflected in equations and which hence requires further properties to completely characterize). Figure 2a represents the system graphically while mathematically it leads to the following differential equation description.

$$dx/dt = y + a\,h\,(x) \qquad (5a)$$

$$dy/dt = -x - 2\sigma y + a\,h\,(x) \qquad (5b)$$

Introducing $z = h(x)$ we can plot the trajectories of eqs. in three dimensional space as indicated in figure 3. From figure 3 and these equations the philosophy of operation is seen to be as follows. The system is of degree two and on each branch of the hysteresis looks to be a linear system. Since there are two branches of the hysteresis, it being binary, we call these two linear systems S and S, respectively, with S denoting operation on the upper hysteresis branch (where h = 1) and S on the lower branch. The branch of hysteresis upon which the full system S is operating depends upon the value of the output of the first integrator, x. If x is above an *upper* threshold value x_U the system looks like S while if it is below the lower threshold value x_L it looks like S. As seen by eqs. both S and S are the same except for a shift in the origin of the unstable focus; when the trajectories of S and S are overlaid they comprise sets of intersecting outward winding spirals. We now trace a trajectory and create a map M that will have a period three point. We choose a starting point p of the lower plane, $z = h = 0$ (which defines S), on the line $x=X_L$ and with its y value nonnegative; the trajectory spirals away from the origin and eventually hits the hysteresis jump line, $x = x_U$, at which time the trajectory jumps to the upper hysteresis plane, $z = h = 1$ (which defines S_1). At the time of jumping it switches to a spiral trajectory in the upper plane and eventually runs into the lower hysteresis jump line (since the spiral is unstable and must increase in size); the origin for trajectories of S_1 has been shifted via the parameters a_1 and a_2 in eqs.[5] such that when the trajectory hits the lower jump line, $x = x_L$, and jumps to the lower plane the third time it will again be at the point p_1 when it crosses $x=x_L$ with y nonnegative. If we take M to be the map of net values of y when crossing the line $x=x_L$ for the same sign or y (we call this a *same side return map*), then M will be of period three. By constructing it to be continuous the theorem of Li & Yorke applies to show that M is a chaotic map. Thinking about the meaning of this M we convince ourselves that if this M is chaotic as a discrete time map then so is the continuous time system since the local peak and valley values of the continuous time system trajectories are related (monotonically), respectively, to the positive and negative values of M. The real problem is then to choose the system parameters a_1 and a_2 such that M has a period three point and at the same time is continuous. For $\sigma = -0{,}2$ and $a_1 = -1$ the value $a_2 = -1.349966731$ results to yield these desired properties. For these values a map M is shown in figure 4 for which a period three point is $y = p_1 = 0{,}069253575+$ (with $M(p_1) = p_2 = 0{,}24871608+$, $M(P_2) = p_3 = 0{,}4916892+$ and $M(p_3) = p_4 = p_1$). It is worth commenting that M is not quite the first return map for the trajectory's traversal of the line $x = x_L$ since we require also that y return to the same side of the axis; we can of course construct a first return map by counting every crossing of the line $x = x_L$, not just when y returns to a value of the same sign, but that map intermixes minima and maxima of the continuous time system and, therefore, is not so directly of interest. To summarize, the system of (5) with the parameter values given yields a period three same side return map and with that exhibits chaos in the behavior of the maxima (and minima) values of the continuous time signal x (.).

Equations (5) are ideal for realization in terms of operational amplifiers, resistors, and capacitors, and, hence available for integrated circuit constructions. A circuit has been built in lumped form and results from it confirm the available theory upon this binary hysteresis chaos generator.[3]

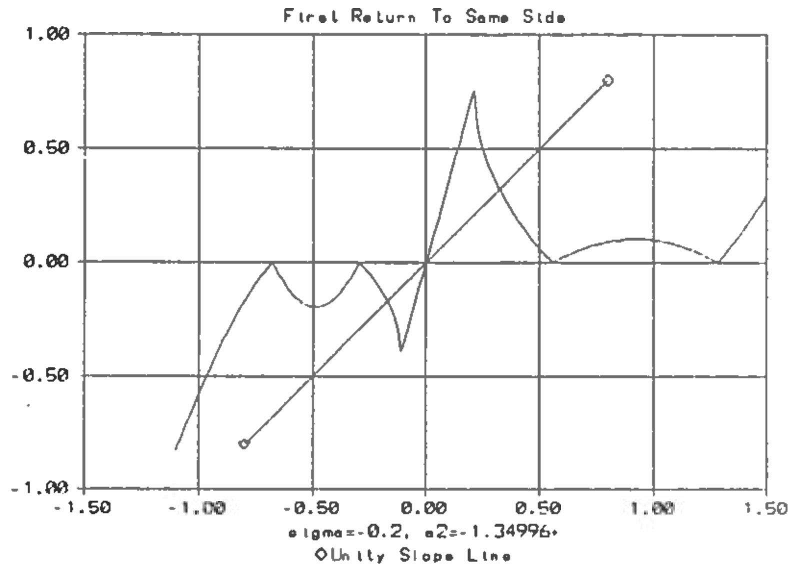


FIGURE 4

Map M.

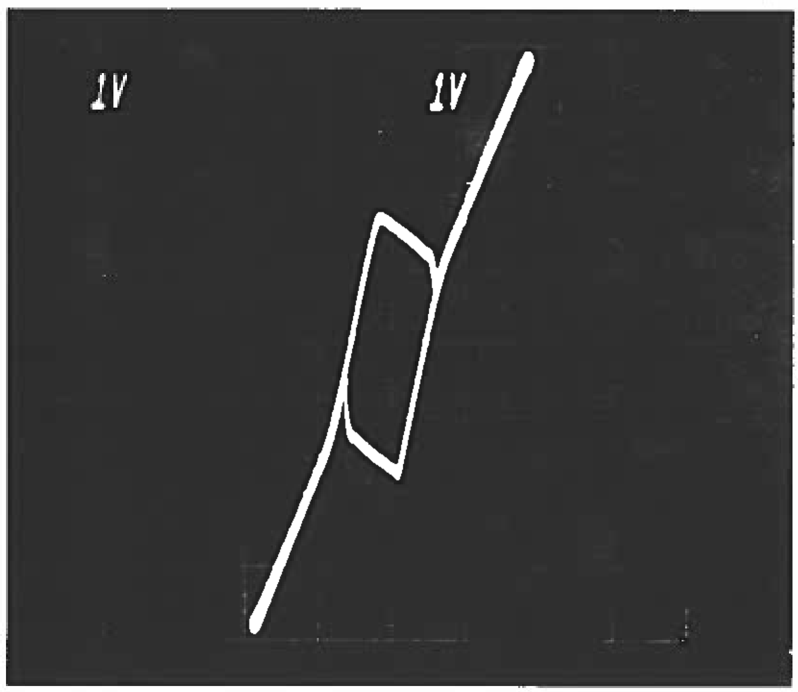


FIGURE 5

Measured bent hysteresis (from [3]).

BENT HYSTERESIS CHAOS GENERATOR

Here we take a second orden Van der Pol oscillator and introduce bent hysteresis as the nonlinearity. By introducing a third dynamic variable internal to the hysteresis generator one finds that chaotic kinds of signals result.[4] The system is designed to bring out the main properties of a circuit discussed by Shinriki, et. al.[5], which was observed to give random types of signals.

Figure 5 shows the bent hysteresis around which the system is designed. To understand how the system is conceived we consider this bent hysteresis as the nonlinearity in the Lienard plane of a second order system. There turn out to be three stable limit cycles, one around each of the smaller end portions in figure 5 and one that surrounds the rectange-like middle portion of figure 5 (really, the boundary portion of figure 1b). One way to look at what occurs is to consider the limit cycle C around the upper down-turning branch B, of the hysteresis. The *radius* of C depends upon the slope of B; if we change the right tip of the branch B, while keeping the left end fixed, the limit cycle changes. By choosing the slope of B, appropriately we can get the limit cycle C to pass through the right tip of B. Any further change in the right tip of B that causes the limit cycle C to expand also causes it to disappear. Consequently, if we choose this limiting values of B *then any oscillation in the slope of B* will cause C to go into and out of existence; when it disappears the limit cycle jumps to other limit cycles. We in essence make this to happen by introducing another dynamic variable into the system, this going into the hysteresis itself to force effective changes in the hysteresis. We will consequently have a three-dimensional state space but will look at it via the projection upon the original two-dimensional Liemard plane. A result of this projection is that trajectories seen in the Lienard plane may cross even though there can be no intersection in the full three dimensional state space.

We begin the mathematical treatment with the signal-flow graph of figure 6a where b (.) represents the bent hysteresis. The differential equations describing the signal-flow graph of figure 6a are

$$dx/dt = y - b(x) \qquad (6a)$$

$$dy/dt = - x \qquad (6b)$$

In the x—y plane we have the trajectory governed by the slope condition

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-x}{y-b(x)} \qquad (7)$$

which allows the Liemard construction for the trajectories by plotting the curve b(x) in the x-y plane (that is, one chooses a point $p = (x_p\ y_p)$ through which one wishes the trajectory to pass and creates the direction of the trajectory via eq. (7) by dropping a vertical to intersect the curve b(x) at the point (x_p, y_b), where $y_b = b(X_p)$. and then a horizontal through the intersect point to cut the y axis at $y = y_b$; the trajectory passes through the chosen point p in the direction of a circle drawn through p with center at the cut point $(0, y_b)$ on the y axis.

The bent hysteresis is constructed electronically by placing (piecewise linear) nonlinear positive and negative resistors in series. However, as mentioned above, to create the chaos we introduce dynamics into this series connection by placing a capacitor in parallel with the negative resistor. If g+ and g— represent the nonlinear resistors on a conductance basis (current as a function of voltage) and z represents the capacitor voltage we have the signal-flow graph of figure 6b for which we can write

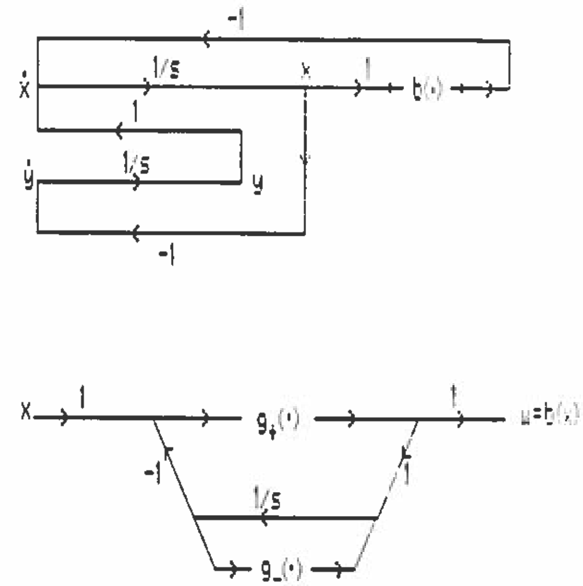


FIGURE 6

Signal-flow Graphs.

FIGURE 6a Bent hysteresis chaos system.

FIGURE 6b Bent hysteresis.

$$dz/dt = g_+\ (x-z) + g_-\ (z) \qquad (8a)$$

We also let w represent the current in g_+ thus,

$$w = g_+\ (x - z) \qquad (8b)$$

Finally we replace the static binary hysteresis b(x) in fig. 6a & eq. (6a) by

$$b(x) \rightarrow w \qquad (8c)$$

A circuit to do all of this is given in figure 7 and is discussed in; figure 1 gives typical chaotic waveforms from this circuit.

DISCUSSION

Using two types of hysteresis, binary and bent, we have presented two types of chaos generators. However, only for the first generator are we able to establish without a doubt that the system really generates chaotic waveforms. From this we see that one of the open problems of the field is to find

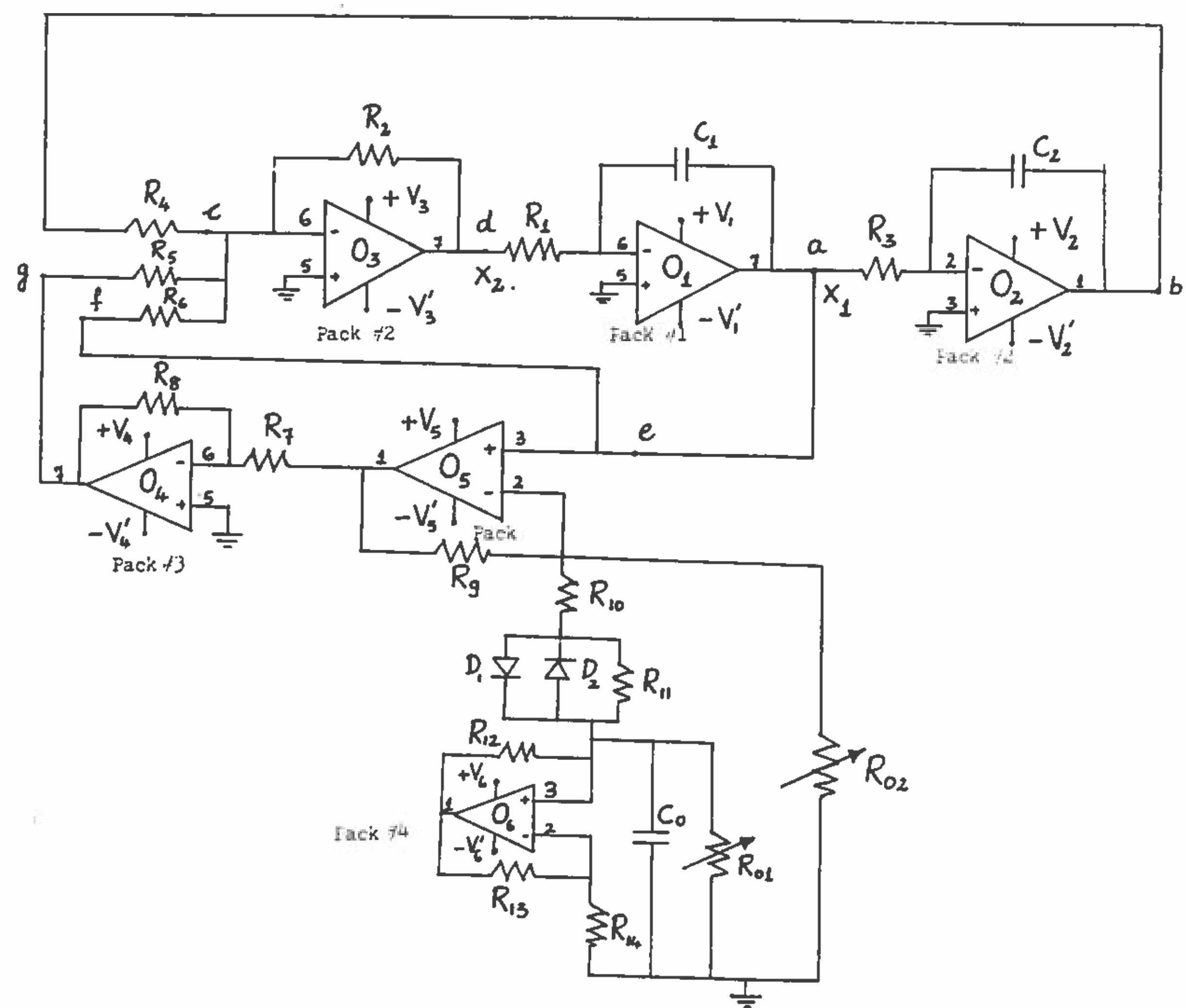


FIGURE 7

Circuit to realize bent hysteresis chaos system *(taken form Newcomb & Sathyan, IEEE trans. on Circuits and Systems, Janyary 1983).*

techniques for establishing whether a system really generates chaos or not. In being able to establish the existence of chaos for the binary hysteresis system we relied upon conversion to a discrete system map M. It does appear that such a conversion of a continuous time system to a discrete map should be possible for any continuous time chaos generator. Therefore, it would seem profitable to get other criteria for such maps that are similar to the period three implies chaos property used for the binary hysteresis case in order to be able to verify chaos in other systems. As also illustrated by the two chaos generators exhibited here, there are various kinds of chaotic signals. This being the case it would appear to be worthwhile to obtain design techniques for the various classes such that chaotic signals with specified properties could be designed on order.

Construction of electronic circuits to yield chaos has been shown here to be a simple matter. However, control of the characteristics is a horse of a different color. Thus, one would like to obtain electronic circuit chaos generators for which signal properties are easily controlled. Toward such a goal it seems that more in the nature of computer controlled experimentation could be developed. For example, in the binary hysteresis case it is relatively easy to change the circuit parameters and observe the results on an oscilloscope which could sample the signals and read them as digitized data into a PC for further analysis and consideration in constructing design charts. To have the data calculated from the differential equations generally takes much more time and suffers from the fact that to make good models of the active electronic circuit components requires more extensive capabilities than available in present day PCs. As seen from the circuit diagrams, the active electronic components used in the constructed circuits were operational amplifiers. It would also be profitable to obtain chaos generating circuits involving only a few transistors, resistors and capacitors. And it would also be profitable to obtain good mathematical models for phenomena such as heart arrythmias such that electronic circuits could be designed to simulate the phenomena.

The definition of chaos for continuous time systems does not seem to have been well considered in the mathematical or engineering literature to date. Thus, the definition used here must be considered to be a tentative one. In this regard it is of interest to know that there are other definitions than the Li-Yorke one used here of chaos for discrete time systems. For example, chaotic signals appear to be probabilistic; therefore, some sort of probability measure should exist. One can use the existence of an *absolutely continuous invariant measure* for this purpose, defining chaos to be the presence of such a measure. Doing so Saito[7] has given results closely related to ours on the binary hysteresis circuit, where he uses a degree three circuit based upon the parallel connection of two negative resistors to obtain chaos which can be looked upon as being generated via sloping hysteresis when one of the inductances in the circuit tends to zero.

Besides the use of hysteresis to generate chaos in electronic circuits there are a number of other means of realizing electronic chaos generators. The interested reader may wish to consult references[8–14] to gain entrance into this aspect of chaotic systems theory.

As a final comment it is worth noting that Plant[15] has given a model of vertebrate hearts that uses hysteresis in a second order system. Consequently, there does appear to be some interest in merging the ideas on the binary hysteresis chaotic system outlined here and the modeling of hearts for arrythmias.

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