Realization of the Connected Sum of Two Knots*

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Abstract:

This paper gives a technique to realize the connected sum of two knots given that semistate equations for the realization of each of the two knots are already on hand.

I. Introduction

Since knots are a part of everyday life it is of interest to have equipment that can tie knots. Toward this there is an extensive mathematical theory of knots, a basic work being [1]. What could then be done is to make robot like equipment whose end effectors essentially trace trajectories that describe knots in the mathematical sense. On investigation of this possibility one learns that it is possible to make certain basic knots and then splice these together to form more complicated knots, for example the granny knot can be considered as two trefoil knots spliced together. Mathematically the operation of splicing two knots, K_1 & K_2, can be looked upon as forming the connected sum knot, $K=K_1\#K_2$, this being illustrated in Fig. 1. In terms of equipment we know that it is possible to make electronic circuits that realize some basic knots, for example the torus knots [2], in which case it would be possible to make electronic circuits to realize more knots from these basic knots if we could get electronic circuits to realize the connected sum. We show here that for nonpathological knots this is always possible.

The basic idea, as developed more fully in section III, is to keep one knot fixed and move & stretch the other knot to give a certain kind of intersection which is made into a connected sum via a step function or a binary hysteresis interface that chops out the unnecessary overlap. Section II gives necessary background.

II. Some Knot Theory Background

For our purposes we take a knot K to be a homeomorphism of the unit circle into three dimensional Euclidean space. In other words a knot can be looked upon as a closed trajectory in (x,y,z)-space with no multiple points. The projections of a knot on any axis-plane will generally have multiple points, however. Thus, we assume that the knot is in "general position," this meaning that each multiple point in an axis-plane projection comes from a true crossing of exactly two portions of the 3-dimensional trajectory. Figure 2 should help clarify the situation. Limiting the treatement to nonpathological knots we also assume that there are only a finite number of multiple points in any projection on to an axis-plane.

In some cases it is possible to realize a knot as the output trajectory in time t of a nonforced semistate describe system of semistate s; in canonical form such a system is described by [3]

$$\mathcal{A}s' + \mathcal{B}(s) = 0 \qquad (1a)$$

$$y = \mathcal{C}s \qquad (1b)$$

where $\mathcal{A}$ & $\mathcal{C}$ are constant matrices, $\mathcal{B}(.)$ is a nonlinear but time invariant operator and $'=d/dt$; $\mathcal{A}$ and $\mathcal{C}$ are generally singular but $\mathcal{C}$ has rank three. The three vector y is the output, this being the coordinates of the knot trajectory in time

$$y(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (1c)$$

From (1b,c) we see that any scalings or linear shifting of the knot can be accomplished by linear affine transformations on the semistate s, a fact that is important to making the connected sum.

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The connected sum $K=K_1\#K_2$ of two knots K_1 & K_2 is illustrated in Fig. 1. This is to be interpreted more precisely [1, p.82][4, p.26] in the sense that there is a sphere S and an arc A on S such that

1. S intersects K in exactly two points p_1 & p_2.
2. A is an arc from p_1 to p_2.
3. ((Int S)∩K)∪A is a knot equivalent to K_1.
4. ((Ext S)∩K)∪A is a knot equivalent to K_2.

Here Int means the interior, Ext means the exterior, ∩ means intersection, ∪ means union and knot equivalence means described by the same Conway polynomial [1, p.78]. In this regard we note that a (nonpathological) knot is uniquely specified by its Conway polynomial and that the Conway polynomial for $K=K_1\#K_2$ is the product of the Conway polynomials for K_1 & K_2. Consequently, it would be worthwhile having circuits that could realize prime Conway polynomials since using the techniques of this paper we could realize any Conway polynomial, and with that any (nonpathological) knot.

III. Realization of the Connected Sum

We consider two knots, K_1 & K_2, described by (1) with semistates s_1 & s_2, and outputs $\mathcal{Y}_1$ & $\mathcal{Y}_2$, respectively. We assume that these two knots possibly have already been stretched and twisted to guarantee that there are two strands, one on each knot, which project out from the rest of the knots, the portion on K_1 being for larger x_1 value and the portion on K_2 being for smaller x_2 values (as illustrated in Fig.1). With this assumption on hand we will keep the origin for K_1 fixed and move the origin for K_2. First we locate the maximum x_1 value, x_{max1}, on K_1 and the minimum x_2 value, x_{min2}, on K_2. Then we linearly move the x_2 origin so that there is a slight overlap of x values on the two knots, that is such that $x_{max1}=x_{min2}+\Delta$ for Δ a small positive number. There will be at least two points on K_1 with $x_1=x_{max1}-(\Delta/2)$; we locate the one with the smallest y_1 value, call it p_1, and linearly shift the y_2 and z_2 origins such that the two knots intersect at p_1. We then rotate K_2 around the $x_2(=x=x_1)$ axis such that the knots passing through p_1 have another intersection and we call the intersect point p_2. By our assumption on the two strands being joined we know there is a small enough Δ such that there are no other intersections of the two knots and we choose such a small enough Δ. At p_1 and p_2 we have achieved $\mathcal{Y}1=\mathcal{Y}2$. Next we patch the two set of semistate equations together using unit step functions.

Let 1(.) denote the unit step function, x_p be the (common) value of x at p_1 & p_2, and subscripts 1 & 2 on system quantities refer to knots K_1 and K_2, respectively. Then we define for $K=K_1\#K_2$ the full system's semistate as

$$s = \begin{bmatrix} s_1 1(-(x-x_p)) \\ s_2 1(x-x_p) \end{bmatrix} \qquad (2a)$$

With this we have the first components of s being zero when the trajectory is on K_2 and the second components zero when on K_1. Consequently

$$\mathcal{Y} = \begin{bmatrix} \mathcal{C}_1, & \mathcal{C}_2 \end{bmatrix} s \qquad (2b)$$

and

$$s' = \begin{bmatrix} s_1' 1(-(x-x_p)) - s_1\delta(x-x_p)x' \\ s_2' 1(x-x_p) + s_2\delta(x-x_p)x' \end{bmatrix} \qquad (2c)$$

where δ is the unit impulse distribution. Using

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{bmatrix}, \qquad \mathcal{B}_o(s) = \begin{bmatrix} \mathcal{B}_1(s_1) \\ \mathcal{B}_2(s_2) \end{bmatrix} \qquad (2d)$$

we obtain for the connected sum

$$\mathcal{A}s' + \left\{ \begin{bmatrix} 1(-(x-x_p)) & 0 \\ 0 & 1(x-x_p) \end{bmatrix} \mathcal{B}_o(s) + \mathcal{A} \begin{bmatrix} s_1 x' \\ -s_2 x' \end{bmatrix} \delta(x-x_p) \right\} = 0 \qquad (3a)$$

$$\mathcal{Y} = \begin{bmatrix} \mathcal{C}_1, & \mathcal{C}_2 \end{bmatrix} s \qquad (3b)$$

Equations (3) are the complete semistate equations for the connected sum where $\mathcal{B}(s)$ is the term inside the braces {}. In order to form $\mathcal{B}$ it is necessary to evaluate s_1x' and s_2x' at the connection points p_2 and p_1 (both of x coordinate x_p), respectively (note the reversal of subscripts). These quantities are known from the original semistate systems for K_1 and K_2 and through the impulse serve to reset the pertinent portion of s from 0 to the proper initial value at the connection points to continue smoothly on the new connected sum knot. It is to be noted that x' is evaluated just at $x=x_p$ and, consequently, is really a constant; the value on the jumped to portion of the knot is to be used since it is this value that is needed to identify the connected sum with the original knot on the jumped to portion of the knot (and similarly for the s value used at $x=x_p$). As with either of the original knots, it is necessary to start the semistate equations on the knot trajectory, but this can be insured by choosing an initial semistate to be on a trajectory of one of the two subknots.

IV. Discussion

Here we have presented a theory for realizing the connected sum of two knots assuming that each of the two knots has been separately realized. As yet there are many knots that we do not know how to realize, but in the special case of torus knots realizations do exist [2]. And in this case of torus knots a different theory for realization of connected sums can be given [5]. By factoring the Conway polynomial corresponding to a given knot into irreducible polynomials it is possible to reduce the construction of an arbitrary knot of the kind treated here into the simplest knots. These latter it seems could be realized using van der Pol oscillators and the ideas developed in [6]. In any event at this point there are a number of open problems in the field toward which it is of first interest to obtain numerical examples of the technique given here and then to compare in the case of torus knots with the theory of [5].

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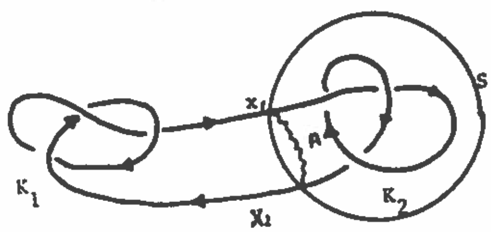

Figure (1) Connected Sum of Two Knots

Figure (2) A Knot and Its Projection
a) In General Position
b) Not in General Position

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