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Semistate Implementation: Differentiator Example*

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Abstract. It is shown that the semistate equations can be transformed via a linear transformation into a form which is useful for physical realizations. The result is applied to the example of a semistate described differentiator which is then realized through an op-amp circuit composed of integrators.

1. Introduction

Recently there has been a large interest in the semistate theory of circuits [1]–[7]. This is because the semistate description is a natural one for circuits which is also very general and contains the state variable one as a special case when the latter exists. The semistate equations for nonlinear time-varying circuits were described in [1] in the canonical form of

$$\mathcal{A}\,dx/dt + \mathcal{B}(x,t) = \mathcal{D}u \tag{1a}$$

$$y = \mathcal{L}x \tag{1b}$$

where u = input, y = output, x = semistate, and $\mathcal{A}$, $\mathcal{D}$, $\mathcal{L}$ are constant matrices. $\mathcal{B}(.,.)$ is a nonlinear time-varying operator. In the linear case many authors have considered the solution of the semistate equations, where the equivalent standard canonical form is used in the analysis [3]–[5]. In a somewhat different approach the Drazin inverse has been used in the analysis of such systems to obtain the solutions in "one fell swoop" [5], [6]. However, except in [2], semistate theory has not been used in the design of circuits. This is in contrast to one of the distinct advantages resulting because of the

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possibility of realizing state variable designs by integrating all of the state variables via integrator circuits. Here we show that the same type of advantage should be available within semistate theory, thus opening up the possibility for design via the canonical semistate equations of a system.

In particular, in this paper we will show that through a linear transformation it is also possible to obtain an equivalent representation of the semistate equations which is useful in the physical realization of semistate described systems. The resultant equivalent representation can be used in the construction of semistate described systems by incorporating integrators on all of the semistate variables. The details of this transformation are given in Section 2. The results are then used in the realization of a differentiator example which is discussed in Section 3 and 4 with simulation details given in Section 5.

2. Transformation

We consider the canonical semistate equations of Equation **(1)**. First we premultiply by a nonsingular matrix P. Second, we change the semistate variable by a linear nonsingular transformation of matrix Q

$$x = Qx. \tag{2}$$

Finally we add dx/dt to both sides of the result. Letting I denote the identity the transformed equations can be arranged into the form

$$dx/dt = (I - P\mathcal{A}Q)dx/dt - P\mathcal{B}(Qx,t) + P\mathcal{D}u \tag{3a}$$

$$y = \mathcal{L}Qx \tag{3b}$$

Since the transformations which brought **(1)** to **(3)** can all be inverted, it is clearly possible to transform **(3)** back to **(1)**. Consequently, the relationship between the rerepresentations **(1)** and **(3)** is an equivalence relationship preserving the input and output map and under it we consider that **(1)** and **(3)** are equivalent. This equivalence contains a number of others in literature [7], **[8]** and the freedom to choose among all possible P and Q gives considerable latitude in the circuit structure allowing us to look for ones with certain desirable properties, as will be seen by the example which follows.

In the case that $\mathcal{B}(.,.)$ is a linear time-invariant operator, that is a matrix $\mathcal{B}$, then it gets transformed to $P\mathcal{B}Q$ by the above process. It should also be noted that P and Q can be found to bring $\mathcal{A}$ to the form $I \dotplus 0$ where $\dotplus$ denotes the direct sum and 0 denotes a zero matrix; in some cases this form for the transformed $\mathcal{A}$ may be desirable but sometimes not.

3. Differentiator example

In order to test out the ideas we tried them on what one might consider as a worst case example, that of a differentiator. This is a linear example for which the state variable equations do not exist and for which semistate equations are natural. Likewise, although op-amp circuit constructions can be given [**9**, p.156] they are avoided because of the eventual disturbance to the circuit of noise [**10**, p.276]. This latter will still occur with our circuit because it is an inherent property of differentiators. However, we will use only integrators in the realization thus leading to the possibility of improved performance when nonideal or imprecise components are used.

Possible semistate equations for a differentiator are

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} dx(t)/dt + \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} x(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{4a}$$

$$y = [1 \quad 1]x(t) \tag{4b}$$

That this gives a differentiator is seen from the fact that $x_1 = -u$ and $x_2 = -x_1 - dx_1/dt = u + du/dt$ with $y = x_1 + x_2 = -u + u + \mathrm{d}u/dt = du/dt$. By adding *Idx/dt to both sides of* (**4a**) we obtain the following possible equivalent equations

$$dx/dt = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} dx/dt - \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{5a}$$

$$y = [1 \quad 1]x \tag{5b}$$

Equations (**5**) are realized in terms of integrators by the signal-flow graph of Fig. 1. However, it is clear that this realization is unstable due to a self loop gain of 1. Therefore we look for more stable realizations using the transformations of Section 2.

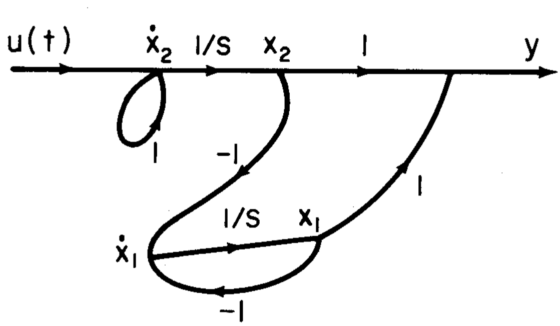

Figure 1. Signal Flow Graph for Equation (5).

4. Stable realization

In this section we wish to find P and Q for **(3)** such that the resultant realization of the differentiator starting with **(4)** is stable. Let

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \tag{6a}$$

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \tag{6b}$$

Then **(4)** becomes

$$dx/dt = \begin{bmatrix} 1 + p_{11}q_{11} & p_{11}q_{12} \\ p_{21}q_{11} & 1 + p_{21}q_{12} \end{bmatrix} dx/dt +$$

$$\begin{bmatrix} (p_{11}-p_{12})q_{11}+p_{11}q_{21} & (p_{11}-p_{12})q_{12}+p_{11}q_{22} \\ (p_{21}-p_{22})q_{11}+p_{21}q_{21} & (p_{21}-p_{22})q_{12}+p_{21}q_{21} \end{bmatrix} x + \begin{bmatrix} -p_{12} \\ -p_{22} \end{bmatrix} u \tag{7a}$$

$$y = [q_{11}+q_{21} \quad q_{12}+q_{22}]x \tag{7b}$$

For stable loops we desire, for the **(1.1)** and **(2.2)** derivative terms,

$$1 + p_{11}q_{11} < 0, \quad |1 + p_{11}q_{11}| < 1 \tag{8a}$$

$$1 + p_{21}q_{12} < 0, \quad |1 + p_{21}q_{21}| < 0 \tag{8b}$$

and, similarly, for the **(1.1)** and **(2.2)** x terms,

$$(p_{11}-p_{12})q_{12}+p_{11}q_{21} < 0, \quad |(p_{11}-p_{12})q_{12}+p_{11}q_{21}| < 1 \tag{8c}$$

$$(p_{21}-p_{22})q_{12}+p_{21}q_{21} < 0, \quad |(p_{21}-p_{22})q_{12}+p_{21}q_{21}| < ' \tag{8d}$$

Besides these conditions one would also like to guarantee that all of the eigenvalues of the matrix multiplying x on the right hand side of the equation has eigenvalues in the left hand plane; if the matrix can be made upper triangular then conditions **(8c,8d)** guarantee this. A possible choice is

$$P = \begin{bmatrix} 1 & 1/2 \\ -3/2 & 0 \end{bmatrix} \tag{9a}$$

$$Q = \begin{bmatrix} -3/2 & 1 \\ 1/2 & -1/2 \end{bmatrix} \tag{9b}$$

which gives

$$dx/dt = \begin{bmatrix} -1/2 & -3/2 \\ -3/2 & -1/2 \end{bmatrix} dx/dt + \begin{bmatrix} -1/4 & 0 \\ -1 & -3/4 \end{bmatrix} x + \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} u \quad \textbf{(10a)}$$

$$y = [\,-1 \quad -3/4]x \quad \textbf{(10b)}$$

Here the P and Q have been chosen to give the **(2.1)** zero in the multiplier of x with the inequalities of **(8)**, which are nonlinear in the P and Q jointly, being attained by iteration of trial solutions. In doing this we guarantee the desired constraint on the eigenvalues for the coefficient matrix of x in **(10)** (note, however, that the right hand coefficient of dx/dt has a left and a right half plane eigenvalue). The result is realized by the signal flow graph of Fig. 2 where it is seen that all loops are stable.

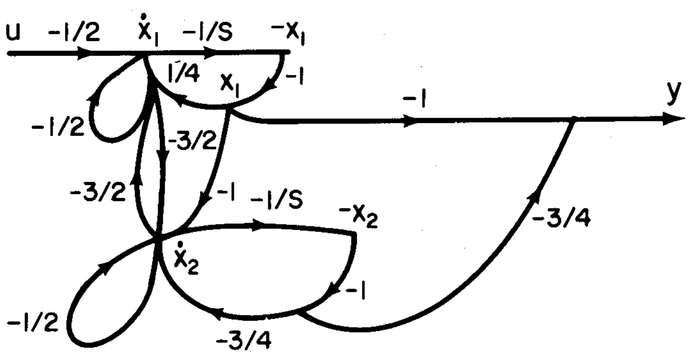

Figure 2. Signal Flow Graph for Equation (10).

5. Circuit realization and simulation results

Fig. 3 shows the op-amp circuit which is used to implement the semistate equivalent equations of **(10)**. As is seen only op-amp integrators are used to realize the dynamics, and thus, differentiators are directly avoided even though the circuit realizes differentiation. Because of the presence of resistor only loops, which, because of loading, would change the characteristics from those desired in **(10)**, some extra isolation amplifiers are included, these being op amps 4, 7, 16, and 20 in Fig. 3 where the numbering corresponds to the netlist of Fig. 4.

Simulations were done on a personal computer using the MICRO-CAP program **[11]** with the results show in Figs. 5 and 6. Fig. 4 shows the MICRO-CAP layout along with the parameter values used (including a netlist of connections). In Fig. 5 a transient response is shown with the input being a 1

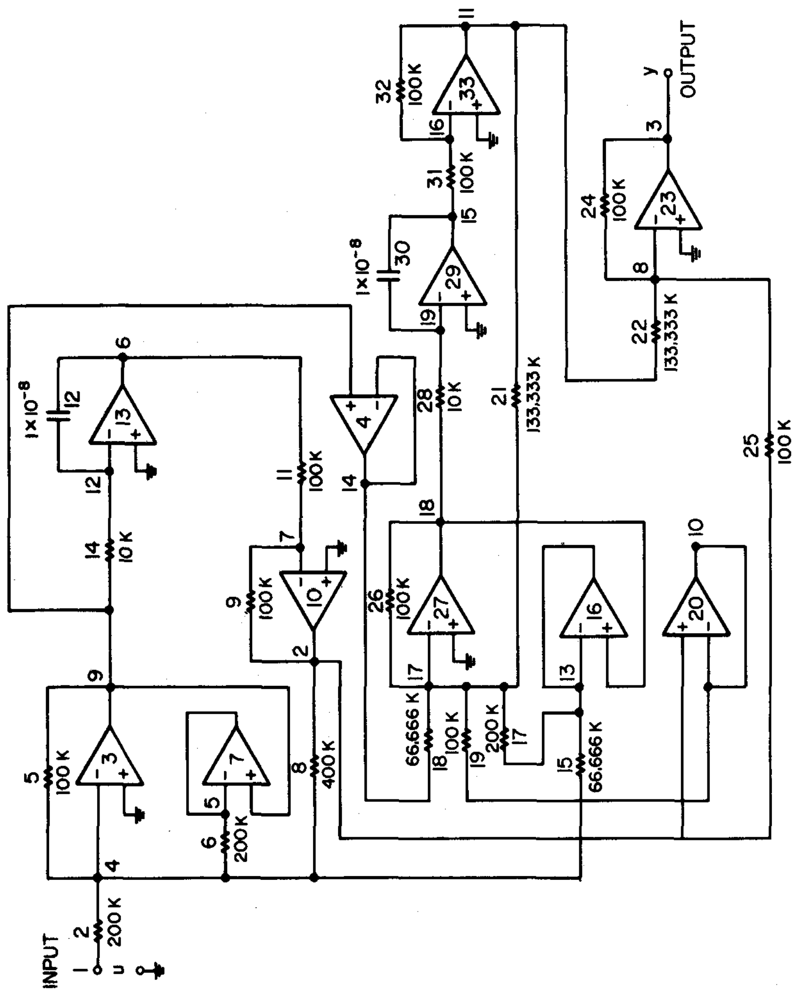

Figure 3. Op-Amp Circuit Realization of Equation (10).

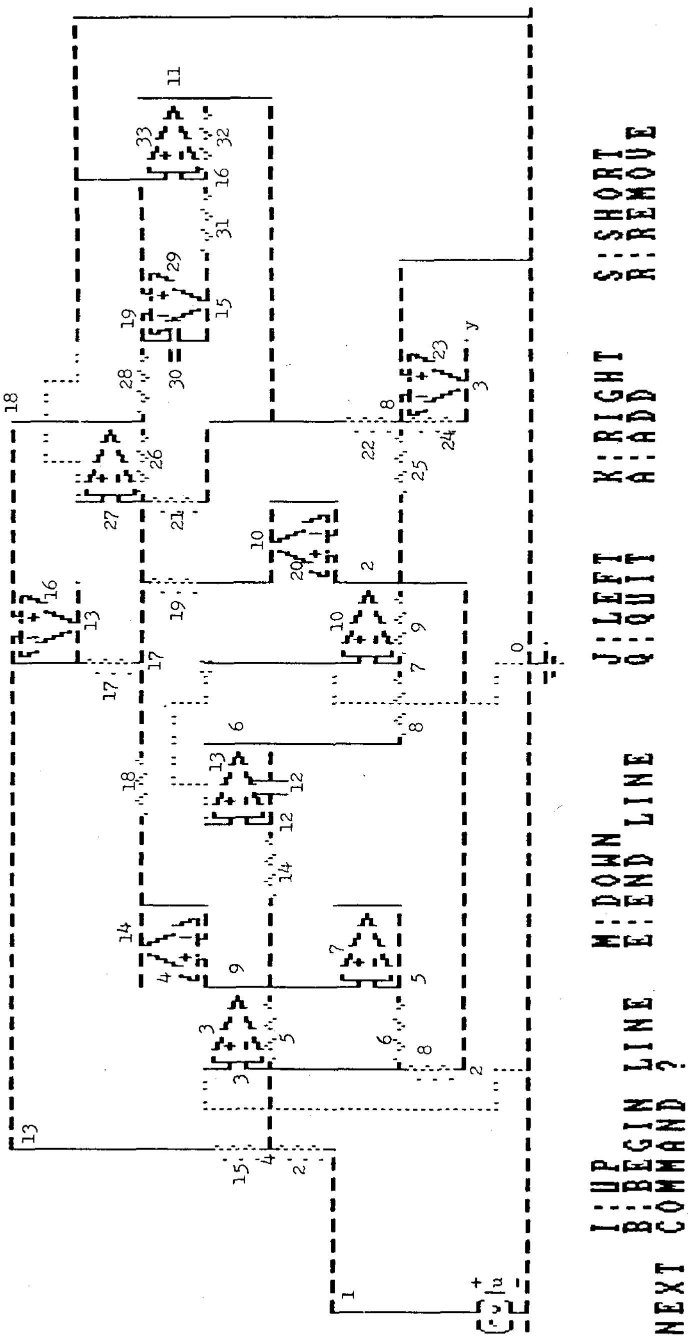

Figure 4a. MICRO-CAP Layout of Circuit of Fig. 3.

OP-AMPS
TYPE 0

0	INPUT RESISTANCE	1000000
1	OPEN LOOP GAIN	10000
2	OUTPUT RESISTANCE	20
3	INPUT OFFSET VOLTAGE	.00001
4	MAXIMUM OUTPUT VOLTAGE	10
5	OUTPUT CAPACITANCE	1E-12
6		0

SINUSOIDAL SOURCES
TYPE 3

0	FREQUENCY	1000
1	AMPLITUDE/2	.05
2	D.C. VOLTAGE LEVEL	0
3	PHASE ANGLE (RADIANS)	0
4	SOURCE RESISTANCE	.1
5		0
6		0

Figure 4b. Op-Amp and Input Source Characteristics.

KHz sinewave of 0.05v amplitude applied to the circuit with zero initial conditions. As can be seen the circuit rapidly adjusts to performing its differentiation function. However, if nonzero inconsistent (or too large consistent) initial conditions are present then a very noisy, or overly large, initial response may occur which in many cases causes the op-amps to saturate. In Fig. 6 is shown a Bode plot from which it is seen that the circuit differentiates over a large range, from about 1 Hz to 10 KHz with the op-amp poles taking over at higher frequencies to cause nonideal behavior. In the case of a differentiator this nonideal behavior may actually be desired since the noisy behavior of the differentiator becomes limited due to the failure to differentiate high frequency signals.

NETLIST
NEWCAPAM CIRCUIT

REF	COMPONENT	CONNECTIONS				PARAMETER
		IN		OUT		OR
NO.	NAME	−	+	−	+	TYPE
1	VSIN	0	1	0	0	3...P0=1000 P1=.05 P2=0 P4=.1 P5=0 P6=0
2	RESISTOR	1	4	0	0	200000
3	OPAMP	4	0	0	9	0... RI=1000000 A0=10000 R0=20 VOFF=.0001 VMAX=10 C0=1E-12
4	OPAMP	14	9	0	14	0... RI=1000000 A0=10000 R0=20 VOFF=.0001 VMAX=10 C0=1E-12
5	RESISTOR	9	4	0	0	100000
6	RESISTOR	4	5	0	0	200000
7	OPAMP	5	9	0	5	0... RI=1000000 A0=10000 R0=20 VOFF=.0001 VMAX=10 C0=IE-12
8	RESISTOR	4	2	0	0	400000
9	RESISTOR	2	7	0	0	100000
10	OPAMP	7	0	0	2	0... RI=1000000 A0=10000 R0=20 VOFF=.0001 VMAX=10 C0=IE-12
11	RESISTOR	7	6	0	0	100000
12	CAPACITOR	6	12	0	0	IE-8
13	OPAMP	12	0	0	6	0... RI=1000000 A0=10000 R0=20 VOFF=.0001 VMAX=10 C0=IE-12
14	RESISTOR	12	9	0	0	10000
15	RESISTOR	4	13	0	0	66666
16	OPAMP	13	18	0	13	0... RI=1000000 A0=10000 R0=20 VOFF=.0001 VMAX=10 C0=IE-12
17	RESISTOR	13	17	0	0	200000
18	RESISTOR	17	14	0	0	6666
19	RESISTOR	17	10	0	0	100000
20	OPAMP	10	2	0	10	0... RI=1000000 A0=10000 R0=20 VOFF=.0001 VMAX=10 C0=IE-12
21	RESISTOR	17	11	0	0	133333
22	RESISTOR	11	8	0	0	133333
23	OPAMP	8	0	3	3	0... RI=1000000 A0=10000 R0=20 VOFF=.0001 VMAX=10 C0=IE-12
24	RESISTOR	3	8	0	0	100000
25	RESISTOR	8	2	0	0	100000
26	RESISTOR	17	18	0	0	100000
27	OPAMP	17	0	0	18	0... RI=1000000 A0=10000 R0=20 VOFF=.0001 VMAX=10 C0=IE-12
28	RESISTOR	18	19	0	0	10000
29	OPAMP	10	0	0	15	0... RI=1000000 A0=10000 R0=20 VOFF=.0001 VMAX=10 C0=IE-12
30	CAPACITOR	15	19	0	0	IE-8
31	RESISTOR	15	16	0	0	100000
32	RESISTOR	16	11	0	0	100000
33	OPAMP	16	0	0	11	0... RI=1000000 A0=10000 R0=20 VOFF=.0001 VMAX=10 C0=IE-12

Figure 4c. Netlist of Circuit Connections and Components.

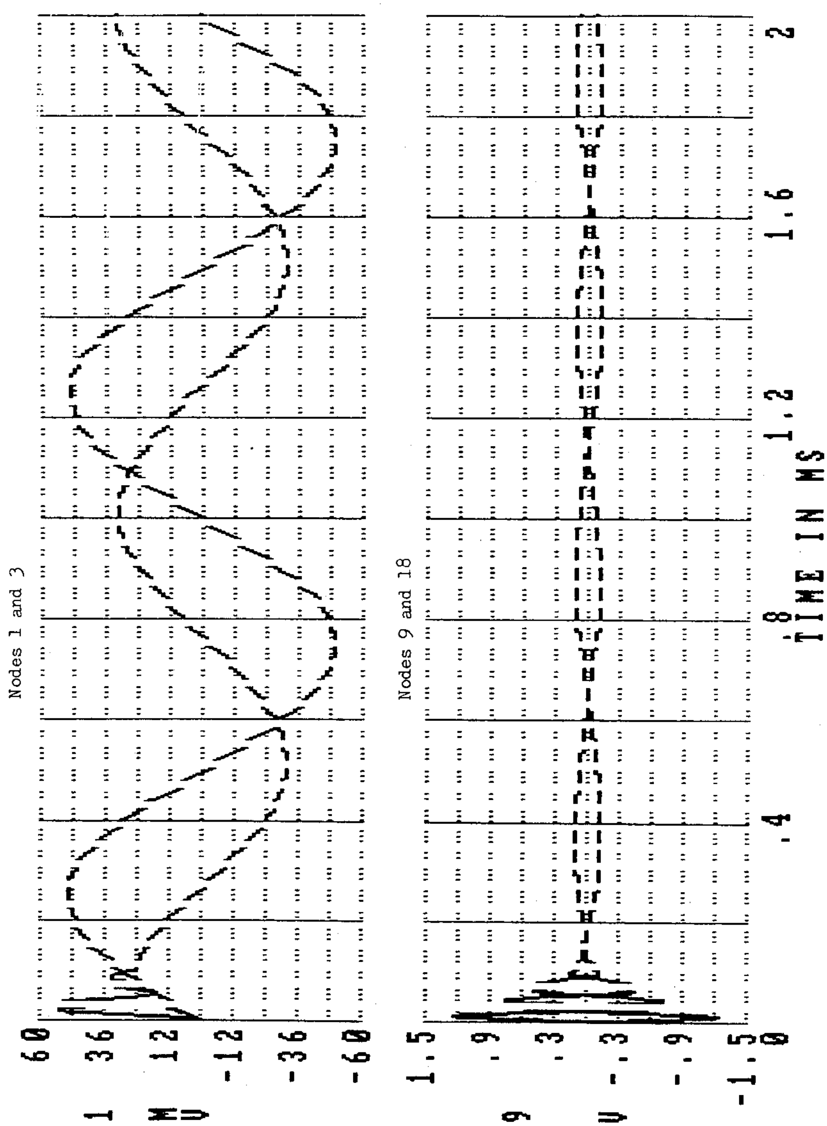

Figure 5. Node Voltage Waveform Plots of Input, Node 1, and Output, Node 3.

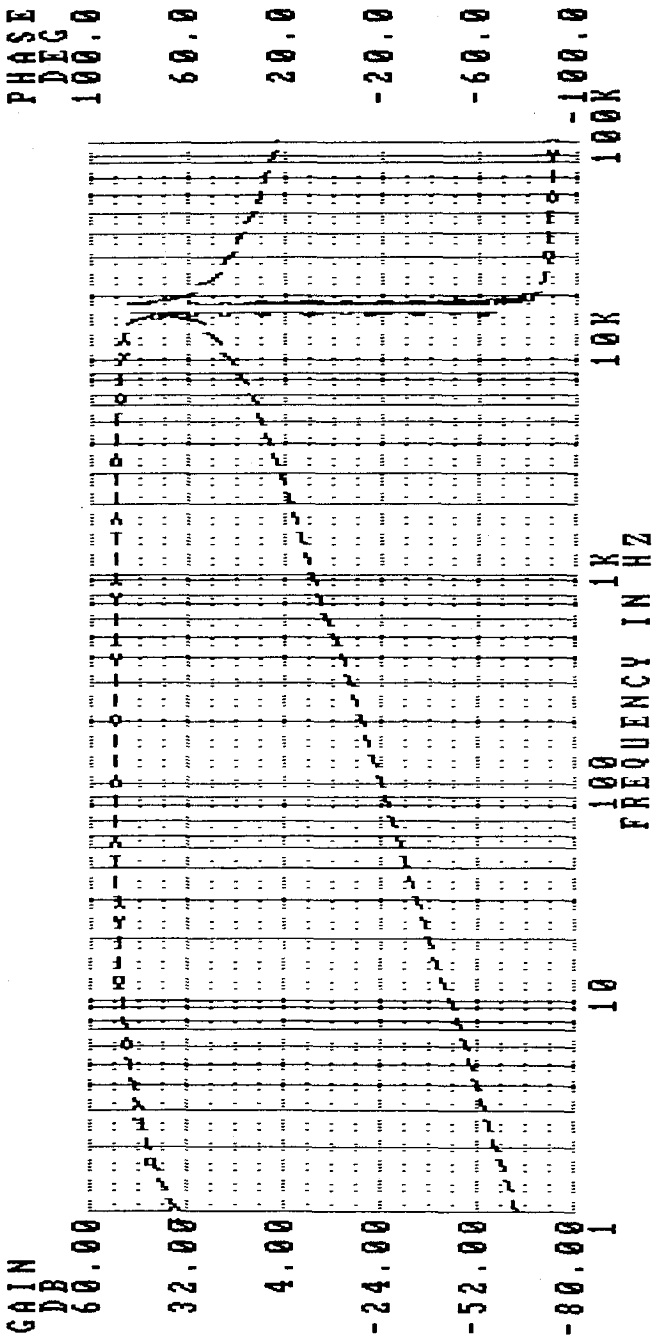

Figure 6. Bode plot (Gain and Phase)

6. Discussion

By the use of transformations which bring semistate equations into an equivalent form we have shown here that interesting and useful circuits can be realized in physically constructable form from the canonical semistate equations. The results rest upon an idea in [7] and [8] that introduces derivatives on all of the semistate variables which can then be undone via integrators. However, it is often necessary, as in the example presented here, to transform the equations out of the form suggested by Saidhmed to guarantee that the circuits are internally stable when constructed with physical devices.

We have found in practice that any set of linear transformations performed on the canonical semistate equations, which lead to an equation having dx/dt on one side of the equation, can be brought to the form given in **(3)**. That is, such linear transformations can be reduced to finding just two matrices $P$and Q. For example, one may premultiply **(3a)** by a matrix M, add dx/dt to the result, and then invert $I + M$, if nonsingular, to again obtain dx/dt on the left; but the result would have already been obtainable through some P and Q via **(3a)** directly. Similarly, the transformation used in [7] can be found in this manner. This is a handy result since it means that one need only work with the canonical form of **(3)** in searching for equations that give convenient structures. However, to find the P and the Q may not be the easiest thing analytically since P and Q enter in a jointly nonlinear fashion in **(3)**. In any event it would be worth finding a proof that **(3)** is the general form for all equations equivalent to the canonical semistate equations when dx/dt is on the left side.

Since the differentiator is a polynomic system, an alternate set of semistate equations to those of **(4)** can be obtained using the technique mentioned in [**12**, p.123]. However, such a method uses a semistate of dimension 4, twice the size of that used in our example. It does point out though that an equivalence theory which works between semistates of different dimensions for the same input-output map is worth investigating.

The results are clearly useful for design where one may desire structures of a particular form. As we have seen, desirable internal (unobservable) poles may be obtained via the transformations considered here. However, as commented to one of us by Professor F. Lewis, a least squares solution of the transformed equations may minimize a different norm than a least squares solution of the original canonical equations [**13**]. Finally it is worth pointing out that Campbell has shown [**5**, p. 149] how the time-variation can easily be transformed out of the canonical semistate equations given in **(1)** above while Hayton *et al.*, have discussed portions of the equivalence transformations used here in [**14**]. We also note the possibility of considering different dimensions for the semistate a study of which should prove quite interesting.

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