In the second case, the following equation results:

$$S_{\bar{z}}^{V_2/V_1} = \frac{V_n V_2^{(n)}}{V_1 V_2}, \qquad (5)$$

where

V_2 is the voltage between the two nodes considered when the driving voltage V_1 operates in mesh 1, and $V_2^{(n)}$ is the voltage between the two nodes under consideration when the driving voltage V_1 operates in series with the impedance $\bar{z}$, so as to cause the current of mesh n flow in the negative direction.

Eq. (5) can be derived from (4) by assuming there is an impedance branch Z_L between the two nodes in question and then letting $Z_L \to \infty$.

As a consequence, there results

$$S_{\bar{z}}^{V_2/V_1} = \lim_{z_L \to \infty} \frac{V_n' I_n^{(2)\prime}}{V_1 I_2'}$$

$$= \lim_{z_L \to \infty} \frac{V_n' I_2^{(n)\prime} Z_L}{V_1 I_2' Z_L} = \lim_{z_L \to \infty} \frac{V_n' V_2^{(n)\prime}}{V_1 V_2'}$$

where the primes are used to indicate the new voltages and the currents that result when the network under consideration is terminated in Z_L.

GIUSEPPE MARTINELLI
Istituto Superiore P. T.
Fondazione "Ugo Bordoni"
Viale Trastevere, 189
Rome, Italy

Topological Analysis with Ideal Transformers*

Perhaps because of the considerable importance of topological methods in the synthesis of transformerless circuits, ideal transformers are somewhat ignored when circuit analysis using linear graph theory is discussed. However, since ideal transformers do appear frequently in the theoretical models of many practical electronic systems, it is useful to incorporate such transformers into general methods of analysis. The customary methods of analysis require the existence of a branch impedance or branch admittance matrix,[1] neither of which exist for the ideal transformer. Consequently, in order to take advantage of the generality of topological methods, while still allowing the presence of ideal transformers, one must modify the usual treatment.

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[1] E. A. Guillemin, "Introductory Circuit Theory," John Wiley and Sons, Inc., New York, N. Y., pp. 491–501; 1953.

The modification introduced by Koenig and Blackwell[2] uses the $[A] \times [v] = [B] \times [i]$ concepts of recent interest.[3] This method requires mixing the voltage and current variables, and, as a consequence, the formulation and manipulation of the describing equations is not quite as straightforward as in the customary theory. By the artifact of augmenting the node-branch incidence matrix with the turns ratio matrix, Onodera[4] has also modified the standard theory. Here we use a different artifact to incorporate ideal transformers. This consists of replacing an ideal transformer by its equivalent in terms of cascade gyrators. Such a replacement has the disadvantage of introducing excess nodes and loops, but the equations are simply formulated and manipulated. Further the number of equations to be solved in the gyrator replacement method is no more (and may be less) than in the Koenig-Blackwell method, since the latter has one equation for each nonsource tree voltage and link current.[5] Of course none of the methods mentioned can "guarantee" a solution since circuits exist for which there is no unique solution, that is, for which no analysis method will yield independent describing equations.

Under the usual assumptions of a circuit consisting of a finite number of linear, time-invariant inductors, capacitors, resistors, gyrators and independent sources, we can write in the notation of Guillemin[1] (on the loop basis)

$$[\beta] \times [D] \times [\beta]_t \times [i] = [e_l] \qquad (1a)$$

$$[D(p)] = p[l] + [r] + p^{-1}[s] \qquad (1b)$$

where $[D]$ is the branch impedance matrix, $[\beta]$ is the tie-set matrix with $[\beta]_t$ its transpose, $[i]$ is the matrix of link currents, $[e_l]$ is the matrix of Thévenin equivalent voltage sources and p is the complex frequency variable. The branch inductance $[l]$, resistance $[r]$, and elastance $[s]$, matrices are in many cases diagonal. However, if coupled coils are present, $[l]$ will have off diagonal terms, as will be the case with $[r]$, if gyrators are met.

If ideal transformers are present, we still propose to use (1). This is accomplished in general in two stages. First, if multiple-winding transformers are present these are replaced by two-winding transformers. Then the two-winding transformers are replaced by the cascade gyrator equivalent of Fig. 1, where $\gamma_2 \neq 0$ can be chosen arbitrarily with

$$\gamma_1 = T\gamma_2. \qquad (2)$$

[2] H. E. Koenig and W. A. Blackwell, "Electromechanical System Theory," McGraw-Hill Book Co., Inc., New York, N. Y.; 1961. See p. 147.

[3] H. J. Carlin and D. C. Youla, "Network synthesis with negative resistors," *Proc. of the Brooklyn Polytechnic Institute of Brooklyn, N. Y.*, pp. 27–67; 1960.

[4] R. Onodera, "Topological synthesis of transfer-admittance matrices," IRE TRANS. ON CIRCUIT THEORY, vol. CT-7, pp. 112–120; June, 1960.

[5] Koenig and Blackwell, p. 151.

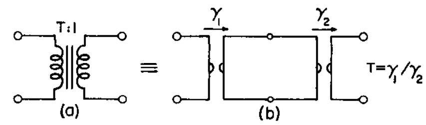

Fig. 1—Gyrator equivalent of ideal transformer.

The new graph can be solved by the use of (1) and the extraneous "internal gyrator" voltages and currents ignored to get the voltages and currents of the original network. It should be observed that this method increases the order of the matrices involved in the solution of (1), as well as the number of separate parts in the network. However, since coupled coils can be realized by inductor-loaded transformers, such a method guarantees that we can always assume, if so desired, that $[r]$ is the only nondiagonal matrix in (1b). In fact, since inductors are equivalent to capacitor-loaded gyrators, we can also assume $[l] = [0]$ if so desired. Further, by using equivalent circuits in terms of the elements mentioned above (1) for transistors and tubes, such elements can immediately be covered by (1), with perhaps an excessive number of equations, however. In many cases, this latter inconvenience is of course not necessary, since the describing equations of multiterminal components can sometimes be directly incorporated in $[D]$.

As a simple illustrative example, consider the circuit of Fig. 2(a), which is replaced by the equivalent, Fig. 2(b), which in turn has the graph of Fig. 2(c) with the tree in heavy lines. Then

$$[i]_t = [i_4, i_5, i_6, i_7]$$

$$[e_l]_t = [e_s, 0, 0, 0,]$$

$$[\beta] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[D] = \begin{bmatrix} 0 & T & 0 & 0 & 0 & 0 & 0 \\ -T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & pl & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Eq. (1a) is then

$$\begin{bmatrix} e_s \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -T \\ 0 & r + pl & -r & T \\ 0 & -r & r & -1 \\ T & -T & 1 & 0 \end{bmatrix} \begin{bmatrix} i_4 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix}.$$

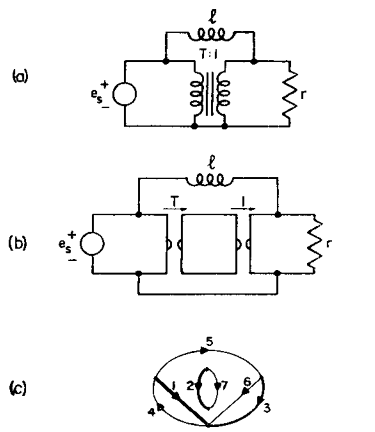

Fig. 2—Example circuit and derived graph.

In this, one need not solve for i_7 since it is not present in the original circuit. For comparison the method of Koenig and Blackwell uses the graph of Fig. 2(c) with branches 2 and 7 absent. Choosing branches 4 and 6 for the tree, this formulation, after some manipulation, yields[5]

$$\begin{bmatrix} T^{-1} \\ 0 \\ 1 \\ 0 \end{bmatrix} [e_s] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & T^{-1} & -T^{-1} \\ 1 & 0 & 0 & pl \\ 1 & 0 & -r & 0 \end{bmatrix} \begin{bmatrix} v_6 \\ i_1 \\ i_3 \\ i_5 \end{bmatrix}.$$

R. W. NEWCOMB
Dept. of Elec. Engrg.
Stanford University
Stanford, Calif.

Some Observations on the Errors in the Estimation of the Impulse Response*

A linear, time-invariant system with input $x(t)$, output $z(t)$ and impulse response $h(\tau)$ is observed over a period $[0, T]$. The output is disturbed by additive noise $n(t)$ so that it is only possible to observe

$$y(t) = \int_0^\infty x(t-\tau)h(\tau)\,d\tau + n(t), \qquad 0 \le t \le T. \tag{1}$$

If the various signals can be observed and manipulated in continuous time, the estimate $\hat{g}(\tau)$ that minimizes

$$\mathcal{E} = \int_b^T \left[y(t) - \int_0^b x(t-\tau)g(\tau)\,d\tau \right]^2 dt, \qquad 0 < b < T \tag{2}$$

is the solution of the integral equation[1]

$$\int_0^b \hat{g}(\tau')\,d\tau' \int_b^T x(t-\tau')x(t-\tau)\,dt = \int_b^T x(t-\tau)y(t)\,dt. \tag{3}$$

The time b has to be chosen finite and in fact $< T$ to make it possible to obtain an estimate at all $0 \ge \tau \ge b$. If the signals are only observed at times separated by intervals of length Δ the operation of the system can be approximated by

$$y_m = \Delta \sum_{k=0}^{K} h_k x_{m-k} + n_m \qquad m = K, K+1, \cdots, N, \tag{4}$$

where $y_m = y(m\Delta)$, $h_k = h(k\Delta)$, etc.; $b = K\Delta$; $T = N\Delta$.

The least-squares estimate $\hat{g}_k$ of the discretized impulse response is the solution of the set of linear equations[1]

$$\Delta \sum_{k'=0}^{K} \hat{g}_{k'} \sum_{m=K}^{N} x_{m-k'}x_{m-k} = \sum_{m=K}^{N} x_{m-k}y_m, \quad k = 0, 1, \cdots, K. \tag{5}$$

By using these equations for obtaining an estimate $\hat{g}_k$ of the impulse response $h(k\Delta)$ a number of errors are introduced which will be briefly discussed in the following.

In studying the various types of errors we will examine the estimate in the continuous-time form [as obtained from the integral equation (3)] and then make some conclusions about the discrete-time solution. To see how this is done, let us compare the solutions of the equations

$$\int_0^b R(\tau, \tau')g(\tau')\,d\tau' = Q(\tau), \qquad 0 \le \tau \le b \tag{6}$$

$$\Delta \sum_{k'=0}^{K} R(k\Delta, k'\Delta)g_{k'} = Q(k\Delta), \qquad k = 0, 1, \cdots, K. \tag{7}$$

The right-hand sides of the two equations are the same for $\tau = k\Delta$, $k = 0, 1, \cdots K$. Therefore, if a solution $g(\tau)$ of the integral equation is given, we have to find the solution g_k which is equal to $g(\tau)$ in a distribution sense, *i.e.*, which makes the left-hand sides (approximately) equal for $\tau = k\Delta$. It is not difficult to check that for small Δ the following equivalences hold:

$$\begin{aligned} g(\tau) &= \text{"smooth"} \\ g_k &\sim g(k\Delta) \\ g(\tau) &= \delta(\tau - a) \\ g_k &\sim \frac{1}{\Delta}\,\delta_{k,a/\Delta} \\ g(\tau) &= \delta'(\tau - a) \\ g_k &\sim \frac{1}{\Delta^2}\,(\delta_{k+1,a/\Delta} - \delta_{k,a/\Delta}). \end{aligned} \tag{8}$$

Continuing like this, it is seen that a nth-order delta function will correspond to a linear combination of Kronecker deltas with a common factor proportional to $1/\Delta^{n+1}$.

To separate the statistical and deterministic errors substitute (1) into (3) and write

$$\hat{g}(\tau) = \bar{g}(\tau) + \tilde{g}(\tau), \tag{9}$$

where $\bar{g}$ and $\tilde{g}$ are the solutions of

$$\int_0^b \bar{g}(\tau')\,d\tau' \int_b^T x(t-\tau)x(t-\tau')\,dt = \int_0^b h(\tau')\,d\tau' \int_b^T x(t-\tau)x(t-\tau')\,dt \tag{10}$$

$$\int_0^b \tilde{g}(\tau')\,d\tau' \int_b^T x(t-\tau)x(t-\tau')\,dt = \int_b^T x(t-\tau)n(t)\,dt. \qquad 0 \le \tau \le b. \tag{11}$$

It will be observed that $\bar{g}$ is the deterministic and $\tilde{g}$ the stochastic part of the estimate $\hat{g}$.

In the following it will be assumed that with sufficient accuracy

$$\int_b^T x(t-\tau)x(t-\tau')\,dt = T'R_x(\tau - \tau'), \tag{12}$$

where $T' = T - b$, and in addition to this that $R_x(\tau)$ has a Fourier transform which can be approximated by a rational function of frequency: $N(j\omega)/D(j\omega)$.

TRUNCATION ERRORS[2]

If in (10) b were ∞, the solution would simply be $\bar{g}(\tau) = h(\tau)$ and no error would be made due to truncation. Equations of the type

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[1] M. J. Levin, "Optimum estimation of impulsive response in the presence of noise," IRE TRANS. ON CIRCUIT THEORY, vol. 7, pp. 50–56; March, 1960.

[2] A more extensive discussion of the three types of error that are considered is found in H. Kwakernaak, "Analysis of Errors in the Estimation of the Impulsive Response," University of California, Berkeley, Calif., ERL Rept., Ser. No. 60, No. 424; January 2, 1962. This report is based on a thesis submitted to the Graduate Division of the University of California, Berkeley, in partial fulfillment of the requirements for the M.S. degree.