Design of a Torus Knot Oscillator

Robert W. Newcomb
Microsystems Laboratory
Electrical Engineering Department
University of Maryland
College Park, MD 20742 USA
Phone: (301) 454-6869

Abstract:
A system of semistate equations for the generation of (m,n)-torus knots is set up in a form suitable for electronic circuit realization.

I. Introduction

Since almost every day a person ties a knot, there appear to be sufficient applications for the rather developed mathematical theory of knots [1]. For example, one can imagine the need for a robot in a textile factory to have its end effectors following knot trajectories in tying off finished products. Consequently, here we look at the design of electronic oscillators whose outputs are torus knot trajectories in three-dimensional space. Toward this we recall that a torus can be considered as the locus of a latitude circle (sometimes called a meridian) as it is revolved around a longitude (or, alternatively, its center around an axial) circle [2, p.5][3, p.3]. If a trajectory on the torus returns to itself after tracing along the longitude circle m times and along the axial circle n times it is called an (m,n)-torus knot [3, p.185].

Previously Parris [4] has proposed the system of equations
(here ' = d /dt)

$$x' = -my + nxz \tag{1a}$$

$$y' = mx + nyz \tag{1b}$$

$$z' = (n/2)[1 - x^2 - y^2 + z^2] \tag{1c}$$

which has for solutions the (m,n)-torus knots. Although it is possible to implement these equations in electronic hardware the resulting circuit is very difficult to make properly operational for several reasons. Among these reasons we note: 1) the necessity of using five multipliers with 2) any errors in the multipliers affecting the dynamics while 3) the gain settings determining the several n (and similarly m) need to be precisely ganged. Consequently, we look for an alternate to Eqs. (1), for which we take the clue for our treatment the idea behind Parris setting up of the above equations.

II. Torus System

Since the torus can be considered as the topological product of two circles [5, p.15], we start with two uncoupled sine-wave oscillators

$$\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} \tag{2a}$$

which generate the two circles

$$(\omega_1 X)^2 + Y^2 = R_1^2 \tag{2b}$$

$$(\omega_2 Z)^2 + W^2 = R_2^2 \tag{2c}$$

with radii R_1 and R_2 fixed by initial conditions. The (X,Y)-plane circle is traversed at a radian frequency ω_1 while the (Z,W)-plane circle is traversed at radian frequency ω_2. Equations (2) do describe a torus in the real four dimensional Euclidean space R^4 [6. p.161], and, hence, describe (ω_1,ω_2)-torus knots. However, we are interested in trajectories on a torus in real three dimensional Euclidean space R^3. To achieve this end we form a quotient system [7, p.21] for which, following Parris [4], we introduce the algebraic constraints

$$W = D - \Delta, \quad \Delta = \text{constant} \qquad (2d)$$

$$\underline{X} = X/D,\ \underline{Y} = Y/D,\ \underline{Z} = Z/D \qquad (2e)$$

To see that the quotient system yields a torus in three dimensional space we substitute (2d) into (2c), divide by D^2 and note that (2b) after division also by D^2 yields the constant terms divided by D which can be substituted back into the manipulated (2c); further rearrangement yields the result

$$z^2 + (R - [x^2 + y^2]^{1/2})^2 = r^2 \qquad (3)$$

where

$$x = \omega_1\underline{X},\quad y = \underline{Y},\quad z = (R_1\omega_2[\Delta^2-R_2^2]^{-1/2})\underline{Z} \qquad (2f)$$

$$R = \Delta R_1/[\Delta^2-R_2^2]^{1/2},\quad r = R_1R_2/[\Delta^2-R_2^2]^{1/2} \qquad (2g)$$

Equation (3) is the equation of a torus with a meridian circle of radius r revolved around an axial circle of radius R with the axis of revolution being the z axis [8, p.220]. Equations (2), therefore, are a set of (semistate [9]) equations for trajectories on a torus, these trajectories forming (ω_1,ω_2)- knots; we identify m = ω_1, n = ω_2].

III. Circuit Design Considerations

Figure 1 shows a signal-flow graph suitable for electronic realization of the semistate equations (2). The radii, R_1 & R_2, of the circles of which the torus is the direct product are set by the initial conditions on the four integrators, as per (2b,c). Since it is most convenient to have two of these initial conditions to be zero we can choose

$$X(0) = 0,\ Y(0) = R_1,\ Z(0) = 0,\ W(0) = R_2 \qquad (4)$$

Further, it is also very convenient to normalize ω_1 to 1 and choose the time scale via the integrator capacitors (effectively denormalizing ω_1); this places a requirement to make all integrator capacitor ratios unity while also placing the adjustment of m/n ratios upon variation of the one gain constant, ω_2^2, which itself can be fixed as the ratio of resistors.

A major problem in the electronic circuit construction of Fig. 1 is the presence of dividers. These dividers can be realized by the use of multipliers in feedback circuits [11, p.453] with the significant points about realizing (2) rather than (1) being that the multipliers do not appear in the dynamics of the system with at most three multipliers being needed for (2) rather than the five needed for (1). If all of the dividers have the same gain k this merely acts to scale the torus. However for oscilloscope display only two of the three variables, x,y,z, would be seen and for that only two dividers are necessary. Indeed to observe the knots it is probably best to monitor the projections of the trajectories on the (x,y)-plane and this is easily done by feeding the x and y outputs to the horizontal and vertical inputs of an oscilloscope (the oscilloscope traces being unscaled when ω_1 = 1). It should be observed that if the output scalings to go from $\underline{X},\underline{Y},\underline{Z}$ to x,y,z are omitted the torus becomes squashed and has ellipsoidal meridians rather than circular ones in whcich case the knots are essentially unchanged.

In realizing the divisions it is important to keep the denominators sufficiently removed from zero. Fortunately this system allows for this since min D = $\Delta > 0$, as seen from (2c) expressed in terms of D. Therefore, it appears that we can choose Δ large enough so that the dividers will properly operate. A convenient choice appears to be $\Delta^2 = R_1^2 + R_2^2$, since the coefficient on z becomes ω_2 (see also the appendix), but then one needs to feed the dividers with a quantity, Δ, which varies with the initial conditions chosen; this can of course be a major nuisance in practical operation of the circuit.

A significant problem in getting a circuit realization to operate is the means of setting the m/n ratio. If this is rational then true knots occur but if it is irrational then a portion of the projection on the (x,y)-plane becomes filled up, since the trajectories are then dense on the torus; in essence for irrational m/n ratios the responses must look somewhat chaotic. The practical problem is that there appears to be as yet no apriori method to set resistors for achieving rational m/n; practically any randomly picked resistor ratios will be irrational. But note that by adjusting the resistor setting ω_2^2 the (x,y)-plane projections of the trajectories will vary from filling regions to being knots and this could be used to determine when resistor ratios are rational!

IV. Discussion

In the above we have rephrased the equations for torus knots to be in a form that can possibly be implemented using modern day electronic circuits, in particular analog integrated circuits. Key to this is the moving of the multipliers out of the dynamical part and into the algebraic part of the semistate equations describing the system. To go along with this we have replaced the several instances of occurance of the knot parameters m and n by one occurance of each (and in reality by choosing $m = 1$ with just one occurance of n in the form of n^2); in practice, though, this is a little deceptive since m and n are only defined to within the common factor of a time scaling and this time scaling depends upon the multipliers of s in the multipliers. These s multipliers are in practice set by capacitors while $m = \omega_1$ and $n = \omega_2$ are set by resistors; it being much easier to control capacitor ratios than resistor ratios the proposal to use (2) rather than (1) as a basis for constructing a torus knot oscillator is still advantageous on this point. However, the serious problem of actually setting the ratio of m/n to be rational (rather than irrational) remains in any conceivable implementation; this appears now to be the real challange associated with implementing torus knot oscillators for which we await actual experimental investigations.

It is worth noting that Birman and Williams [11] have shown that the Lorenz equations [12, p.135] also realize knots and some of very fascinating variety. However, although the Lorenz equations are easier to realize [13] than those of (1) of Parris, [14] still they possess similar problems and our experience is that they are already very difficult to make work practically. Thus, it appears that it may be worth looking at the Lorenz or similar types of equations in terms of the ideas of this paper.

Appendix

Here we show the general nature of equations of the type of (1) that result from (2) within the framework of this paper. Considering (2) we differentiate x, y, and z, to get

$$x' = \omega_1(DX' - XD')/D^2 \qquad \text{(A1a)}$$

$$y' = (DY' - YD')/D^2 \qquad \text{(A1b)}$$

$$z' = \{R_1\omega_2/[\Delta^2-R_2^2]^{1/2}\}(DZ' - ZD')/D^2 \qquad \text{(A1c)}$$

By (2d), (2a) and (2f) we note

$$D' = W' = -\omega_2^2 Z = -\{\omega_2[\Delta^2-R_2^2]^{1/2}\}(zD) \qquad \text{(A2a)}$$

while from (2c) and (2b) we find

$$D\Delta = (1/2)[(\Delta^2 - R_2^2) + \omega_2^2 Z^2 + D^2] \qquad \text{(A2b)}$$

$$\Delta^2 - R_2^2 = ((\Delta^2 - R_2^2)/R_1^2)[\omega_1^2 X^2 + Y^2] \qquad \text{(A2c)}$$

Using (A2a) in (A1a,b) as well as (A2a) and (A2b,c) in (A1c) finally gives

$$x' = \omega_1 y + \{\omega_2[\Delta^2-R_2^2]^{1/2}/R_1\}xz \qquad \text{(A3a)}$$

$$y' = -\omega_1 x + \{\omega_2[\Delta^2-R_2^2]^{1/2}/R_1\}yz \qquad \text{(A3b)}$$

$$z' = \{\omega_2 R_1/2[\Delta^2-R_2^2]^{1/2}\}[1 + ([\Delta^2-R_2^2]/R_1^2)(z^2 - x^2 - y^2)] \qquad \text{(A3c)}$$

If we set

$$\Delta^2 = R_1^2 + R_2^2 \qquad \text{(A4)}$$

then, on replacing x by -x and identifying $m = \omega_1$ and $n = \omega_2$, the equations (A3) become identified with (1), those of Parris.

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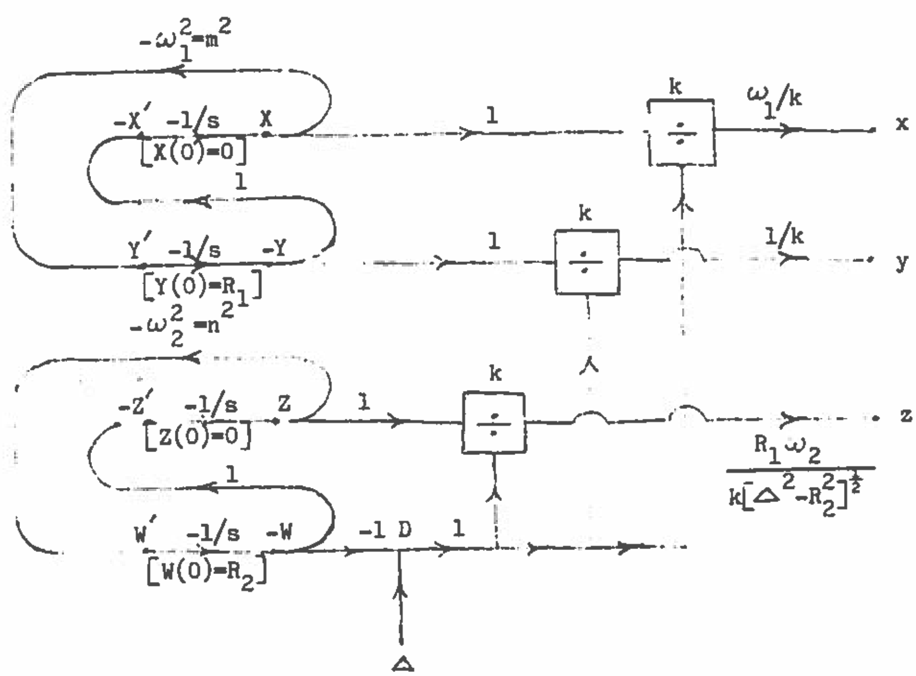

Figure 1

Signal-Flow Graph for Torus Knot Oscillator

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