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Trajectory Calculations and Chaos Existence in a Binary Hysteresis Chaos Generator

Abstract:

The method for calculating trajectories is given for a binary hysteresis chaos generator. From this a set of parameters is given for which it is proven the system exhibits chaos.

I. Introduction

Previously we have exhibited [1] a real continuous-time second order system designed to generate chaotic signals. An advantage of this system, as we show here, is that it is possible to prove that it generates chaos. This is in contrast to other continuous-time systems, such as the Lorenz equations [2, p.135] (these being the basic ones from which continuous-time chaos theory has evolved) and the system of Brockett [3][4]; see especially the discussion in [3, p.936] and in [5] on the point that proofs do not as yet exist to confirm that these other systems actually generate chaos. Along the way to proving that our system gives chaos we show how the trajectories can be calculated. As the calculations are simple and straightforward they are easily carried out on programmable pocket calculators, for example all of our calculations were performed on an HP15C.

II. The System

In normalized form the system under discussion is

$$x_1 = x_2 - h(x_1) \qquad (1a)$$
$$x_2 = -x_1 - 2\sigma_o x_2 + \underline{a}h(x_1) \qquad (1b)$$

Here σ_o (the damping factor) and $\underline{a}$ are real design parameters and $h(.)$ is binary hysteresis of unit step form

$$h(x_1) = \begin{cases} 1 \text{ for } X_L < x_1 \text{ (on upper hysteresis branch)} \\ 0 \text{ for } x_1 < X_U \text{ (on lower hysteresis branch)} \end{cases} \qquad (1c)$$

where X_L and X_U are the real jump positions for the hysteresis. To achieve the results desired we choose

$$-1 < \sigma_o < 0 \quad \text{and} \quad \underline{a} < 0 \qquad (1d)$$

By a trajectory of the system (1) we mean a solution vector $\underline{x}(t)$ of components $x_1(t)$ and $x_2(t)$ for $t \geq 0$ and we write $\underline{x}=(x_1,x_2)$.

III. Solution Technique

Equation (1) takes on one of two forms depending upon whether the solution is on the lower, h=0, or upper, h=1, hysteresis branch

$$x_1 = x_2 \qquad \text{lower} \quad (2a)$$
$$x_2 = -x_1 - 2\sigma_o x_2 \qquad \text{branch} \quad (2b)$$

or

$$x_1 = x_2 - 1 \qquad \text{upper} \quad (3a)$$
$$x_2 = -x_1 - 2\sigma_o x_2 + \underline{a} \qquad \text{branch} \quad (3b)$$

To reduce these to a common form we choose in the first case

$$x = x_1 \quad \text{which is} \quad x_1 = x \qquad \text{lower} \quad (4a)$$
$$y = x_2 \quad \text{which is} \quad x_2 = y \qquad \text{branch} \quad (4b)$$

and in the second case

$$x = x_1 + 2\sigma_o - \underline{a} \quad \text{which is} \quad x_1 = x - 2\sigma_o + \underline{a} \qquad \text{upper} \quad (5a)$$
$$y = x_2 - 1 \quad \text{which is} \quad x_2 = y + 1 \qquad \text{branch} \quad (5b)$$

With these choices for x and y in both cases the differential equations of (2) and (3) are reduced to

$$x = y \qquad (6a)$$
$$y = -x - 2\sigma_o y \qquad (6b)$$

Therefore, we can always solve (6) and transform back to (1) via the variable changes of (4) and (5), piecing together the trajectories that jump between (2) and (3) at the hysteresis jump points by using the appropriate initial conditions in (6). Although there are jumps in the right sides of (1) the solution vector $\underline{x}(t)$, that is the trajectories, are continuous, as is clear from standard systems theory for second order systems.

Equations (6), being linear and second order, are easily solved, again by standard means. For calculational purposes we put the solution in the form

$$x(t) = K(\exp[-\sigma_o t])\cos[\omega_1 t + \varnothing] \qquad (7a)$$
$$y(t) = K(\exp[-\sigma_o t])\cos[\omega_1 + \varnothing + \angle s_1] \qquad (7b)$$

where

$$\omega_1 = (1 - \sigma_o^2)^{1/2} \qquad (7c)$$
$$\angle s_1 = \arctan[\omega_1/(-\sigma_o)] = \text{angle of } -\sigma_o + j\omega_1 \qquad (7d)$$

and K and Ø are determined from initial conditions

$$x(0) = X \quad \text{and} \quad y(0) = Y \qquad (7e)$$

according to whether x(0) is nonzero or not by:

If $x(0) = X \neq 0$: | If $x(0) = X = 0$:

$$K = [\text{sign}X][X^2 + 2\sigma_o XY + Y^2]^{1/2}/\omega_1 \quad \text{or} \quad K = -Y/\omega_1 \qquad (7f)$$
$$\varnothing = \arcsin[(Y + \sigma_o X)/(-\omega_1 K)] \qquad \varnothing = \pi/2 \qquad (7g)$$

Given initial conditions $x_1(0)$ & $x_2(0)$ as well as the hysteresis branch on which the trajectory is, we evaluate the initial $x(0)$ & $y(0)$ from (4) or (5), obtain the parameters K & Ø from (7f,g) and then determine the solution via (7a,b). The system trajectory remains on this solution until a hysteresis jump occurs. If before the jump the solution is on the upper [lower] branch, h=1 [h=0], then this jump is to the lower [upper] branch and takes place at a time t_J for which $x(t_J) = X_L + 2\sigma_o - \underline{a}$ $[= X_U]$; we can determine t_J and evaluate the value $Y(t_J)$ of $y(t_J)$, for which we have $y(t_J) = x_2(t_J) - 1$ $[= x_2(t_J)]$. We then reset initial conditions for solving on the jumped to lower [upper] branch by using continuity of the solution, this giving $x(0) = X_L$ [$=X_U+2\sigma_o - \underline{a}$] and $y(0) = Y(t_J)$ [$=Y(t_J)-1$]. We note that the solutions to (6) are continuous functions of its initial conditions (as is seen by observing the formulas for K and π above, or by appealing to the literature [6, p.465]). To plot the trajctories it is, therefore, important to know $Y(t_J)$; this can be found by a number of different calculational means, one being to solve (7a) for t_J knowing the value of $x(t_J)$ and another being to obtain the equation [7, p.29] for the spiral relating x to y and solve this for y knowing $x(t_J)$ (in either case a transcendental equation is to be solved, this being easily carried out by the SOLVE routine of the HP15C). Again an important item is that the trajectory vector point $\underline{x}(t_J)$ is a continuous function of the initial conditions from which it arises via the system equations (1).

We note that the solutions of the x-y system, (6), and, hence, of the subsystems (2) and (3), are spirals emanating from an unstable focus [8, p.127] and that the lower hysteresis branch spiral has its center at

$$x_1 = x_2 = 0 \qquad (8a)$$

while the upper branch spiral is centered at

$$x_1 = \underline{a} - 2\sigma_o, \qquad x_2 = 1 \qquad (8b)$$

It is also to be noted that in a practical electronic circuit construction of the system using symmetric binary hysteresis, if initially set with zero capacitor charge, the start up point in the $\underline{x}$-plane will be half way along the line joining the centers of the two spiral centers [1], that is at

$$x_1(0) = (\underline{a}-2\sigma_o)/2, \qquad x_2(0) = 1/2 \qquad (8c)$$

[this because the system (1) is normalized to bring the negative value of the circuit's hysteresis to 0].

Figure 1 shows overlayed on top of each other some $\underline{x}$-plane solution paths for equations (2) and (3) for the design parameters given. We can trace a trajectory starting at t=0 at a given point, say that marked I and given by (8c). Since the point I has its x_1 value less that X_L it is on the lower hysteresis branch and the solution spiral, labelled A, then satisfies eqs. (2) until x_1 has increased to the value X_U at which time a switch to the upper hysteresis branch, and with it to eqs. (3), occurs and the path to be followed by the trajectory is that labelled B in Fig. 1. When path B has its x_1 value equal to X_L another switch occurs, back to the lower hysteresis branch and to eqs. (2), and the new path, labelled C continues to satisfy (2) until its x_1 value becomes X_U, this process continues on and on.

IV. The Existence of Chaos

We apply the theorem of Li and Yorke [9, p.987] which states that a continuous discrete time map having a period three solution exhibits chaos. To convert our continuous-time system to one in discrete time we take as our discrete time map M(.), called the return map, the transformation which takes one intersection on the positive x_2 portion of the line $x_2 = X_L$ into the next intersection on this same line. Figure 2 shows a map of this kind. The major problem to be faced in applying Li & Yorke's theorem is to insure that this map is continuous. We show that with the choices

$$\sigma_o = -0.2, \; X_L = 0, \; X_U = 0.3 \tag{9a}$$

there exists a value $\underline{a}_o$ of the design parameter $\underline{a}$ which gives continuity of the map M(.). To the accuracy of the HP15C this value is

$$\underline{a}_o = -1.349966731 \tag{9b}$$

Using this value gives the map of Fig. 2 where the period three solution is also traced. To show that a continuous return map M(.) is possible we focus attention in Fig. 1 on the lower plane spiral S, called the "separating spiral, which is tangent to the jump line $x_1 = X_U$, the point of tangency being $\underline{x}=(X_U,0)$. Through this tangency point goes an upper plane solution T which differs for different parameters $\underline{a}$ but which is continuous in $\underline{a}$. Now T intersects the jump line $x_1 = X_L$ at some point of x_2-value b with b varying continuously in $\underline{a}$. If c (which is indepanedent of $\underline{a}$) is the x_2 intersection value of the lower plane spiral S with the last mentioned jump line $x_1 = X_L$, then a little thought will convince one that the return map M(.) is continuous if b=c. For the numbers of (9a) we have c = -0.396754584; for $\underline{a}=\underline{a}_1=-1.3$ we find $b=b_1=-0.385451946$ while for $\underline{a}=\underline{a}_2=-1.4$ we find $b=b_2=-0.407978164$. By the continuity of b as a function of $\underline{a}$, there is, therefore, an $\underline{a}=\underline{a}_o$, as in (9b), between -1.3 and -1.4, since $b_2=b(\underline{a}_2) < c=b(\underline{a}_o) < b(\underline{a}_1)=b_1$, such that M(.) exhibits chaos. As M(.) has chaos then so does our system (1), since this M(.) has a period three point.

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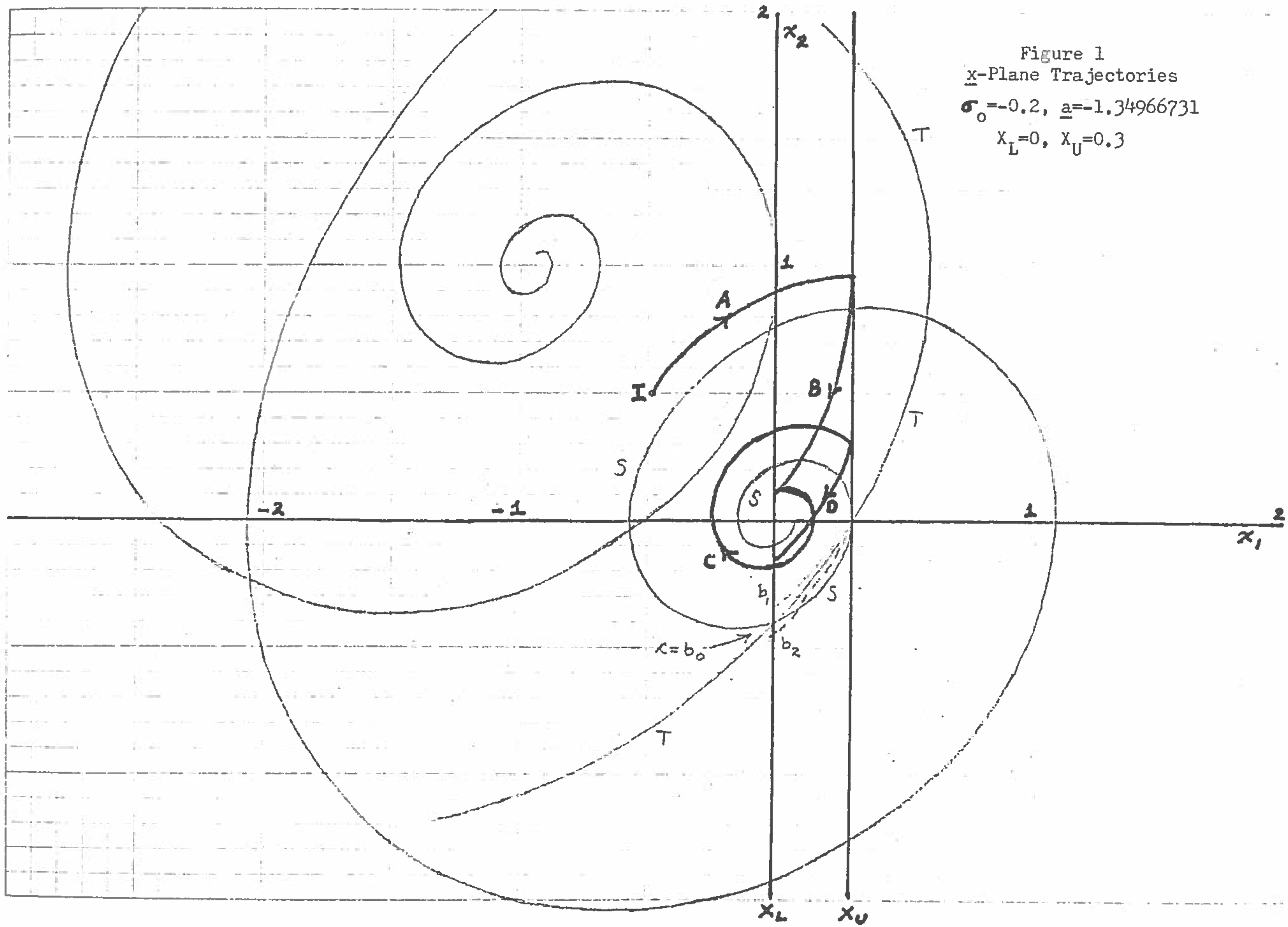
Figure 1
x-Plane Trajectories
$\sigma_o = -0.2$, $\underline{a} = -1.34966731$
$X_L = 0$, $X_U = 0.3$



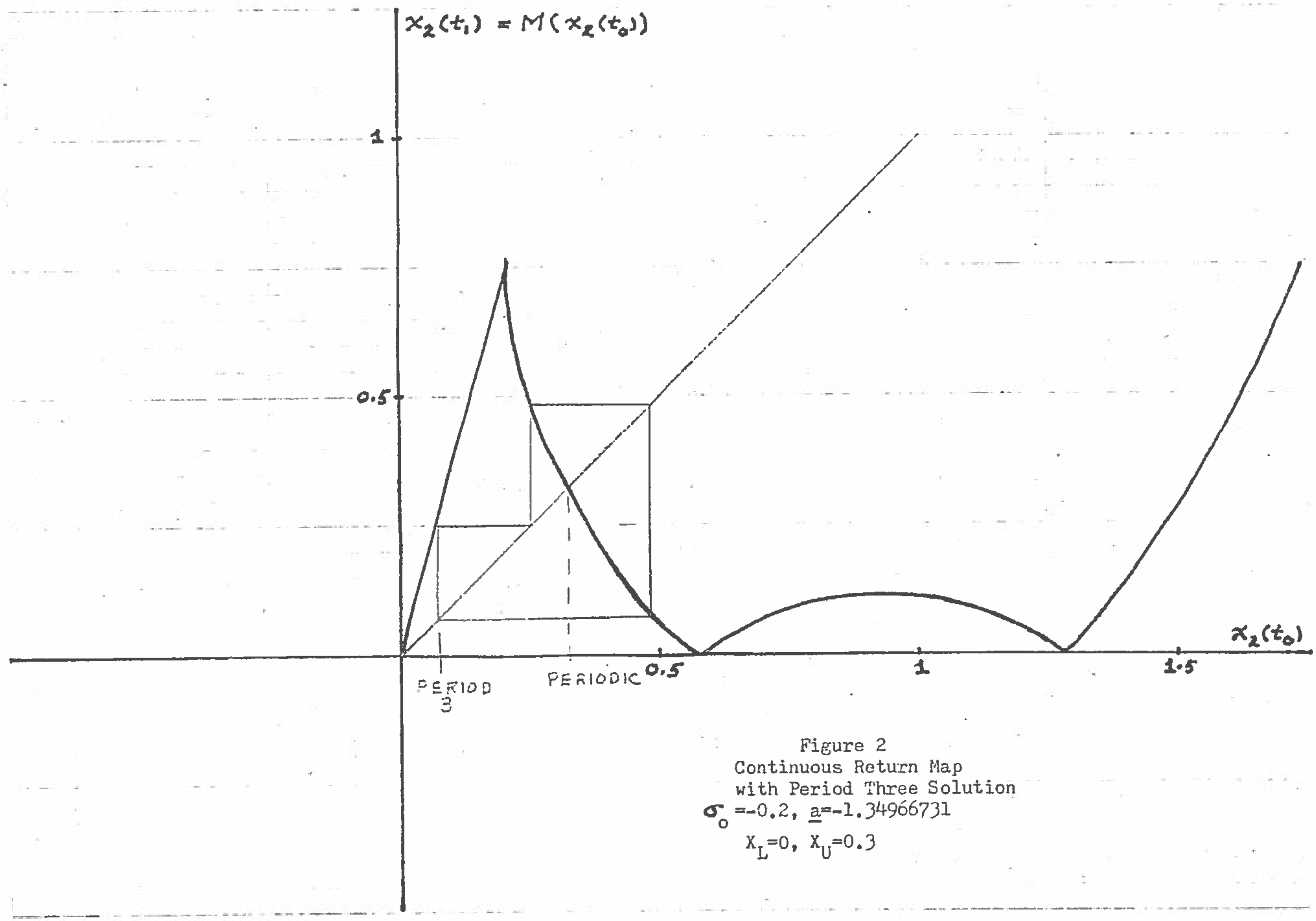


Figure 2
Continuous Return Map
with Period Three Solution
$\sigma_0 = -0.2$, $\underline{a} = -1.34966731$
$X_L = 0$, $X_U = 0.3$