A Continuation-Type Method for Solving Semistate Equations

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Abstract:

A continuation-type method is proposed for solving nonlinear time-varying semistate equations. Details of the technique are outlined and proposals made for further study.

I. Introduction

Because semistate equations arise naturally in a number of physical situations, as for example in the analysis of linear [1] and nonlinear [2] electrical circuits, there is presently considerable interest in solution methods. A summary of a number of the methods is given by Petzold and Gear [3] while others are presented by Campbell and coauthors [4][5][6]. Here we introduce another technique which stems from continuation method ideas [7]. Although we do not find applications in the literature of the continuation method to differential equations, its success in solving nonlinear non-dynamic equations seems to make it a natural technique to apply to semistate equations where there are non-state-variables present.

II. The Method

We begin with the canonical semistate equations [2]

$$\mathcal{A}\dot{x} + \mathcal{B}(x,t) = \mathcal{D}u \tag{1a}$$

$$y = \mathcal{F}x \tag{1b}$$

where $\mathcal{A}, \mathcal{D}, \mathcal{F}$ are constant generally singular matrices and u, x, y are the semistate, and output vectors. By straightforward transformations we can bring the nxn matrix $\mathcal{A}$ to be the direct sum of the mxm identity and the zero matrix. Partitioning the resulting semistate, again called x, accordingly into an m-vector x_1 and an (n-m)- vector x_2 we can rewrite (1a) as

$$\dot{x}_1 + \mathcal{B}_1(x,t) = \mathcal{D}_1 u \tag{2a}$$

$$\mathcal{B}_2(x,t) = \mathcal{D}_2 u \tag{2b}$$

Our desire is to solve equation (2) where we will assume: i) $n > m$ so that (2b) is actually present and ii) a unique solution exists for the semistate x(t) for $t > 0$

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given $x_1(0)$ and $u(t)$ for $t > 0$.

To proceed toward the continuation method we introduce a parameter λ and constant matrices B_{11}, B_{21} and B_{22}, the latter assumed nonsingular, and write

$$\dot{x}_1 + \lambda[\mathcal{B}_1(x,t) - B_{11}x_1] + B_{11}x_1 = \mathcal{D}_1 u \tag{3a}$$

$$\lambda[\mathcal{B}_2(x,t) - B_{21}x_1 - B_{22}x_2] + B_{22}x_2 = \mathcal{D}_2 u - B_{21}x_1 \tag{3b}$$

For $\lambda = 1$ equations (3) reduce to (2) while for $\lambda=0$ these are easily solved being linear, time-invariant and having (3a) uncoupled to (3b). We will thus wish to extend the solution $x(t,\lambda)$ of (3) from $x(t,0)$ to the desired solution $x(t,1) = x(t)$.

At $\lambda=0$ we solve (3a), $\dot{x}_1 + B_{11}x_1 = \mathcal{D}_1 u$, subject to the given initial conditions, $x_1(0)$. Then using this solution, $x_1(t,0)$, (3b) is solved by

$$x_2(t,0) = B_{22}^{-1}\,[\mathcal{D}_2 u - B_{21}x_1(t,0)] \tag{4}$$

Next we solve (3b) for $x_2(t,\lambda)$ via the continuation method (see below) over a small increment Δ in λ, $\lambda \in [0,\Delta]$, for x_1 fixed at $x_1(t,0)$. This gives a value for $x_2(t,\Delta)$ which we then use in (3a) with $\lambda = \Delta$ to solve for x_1 by any standard ODE technique [8], thus giving a value for $x_1(t, \Delta)$. This cycle is repeated, the next time giving $x_2(t,2\Delta)$ by the continuation method on (3b) over $\lambda \in [\Delta, 2\Delta]$ for x_1 chosen as the previously found $x_1(t, \Delta)$; $x_1(t, 2\Delta)$ is then found from (3a) with $\lambda = 2\Delta$ and $x_2 = x_2(t, 2\Delta)$, etc.. Choosing $\Delta = 1/N$, for N a large enough integer, this cycle is repeated N times such that the final iteration has $\lambda = \lambda_N = N \cdot \Delta = 1$ in which case the solution approximates that of (2) giving the desired semistate.

For the continuation method we wish to solve the nondynamic equation

$$C(x_2,t,\lambda) = 0 \tag{5a}$$

where C is the "continuation" function

$$C(x_2,t,\lambda) = \lambda[\mathcal{B}_2(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},t) - B_{21}x_1 - B_{22}x_2] + B_{22}x_2 - \mathcal{D}_2 u + B_{21}x_1 \tag{5b}$$

Equation (5a) is to be solved at the ith step over $[i-1]\Delta < \lambda < i\Delta$ subject to $x(t,[i-1]\Delta)$ given while x_1 is held fixed at $x_1(t,[i-1]\Delta)$; this is to be done for $i = 1, \cdots, N = 1/\Delta$ with $x_1(0)$ given [$x_1(t,[i-1]\Delta)$ being found from (3a) as described above]. To proceed with the continuation method we form

$$\frac{dC}{d\lambda} = 0 = \frac{\partial C}{\partial x_2} \cdot \frac{\partial x_2}{\partial \lambda} + \frac{\partial C}{\partial t} \cdot \frac{\partial t}{\partial \lambda} + \frac{\partial C}{\partial \lambda} \tag{6a}$$

or, since t & λ are taken to be independent,

$$\frac{\partial x_2}{\partial \lambda} = -\left[\frac{\partial C}{\partial x_2}\right]^{-1}\frac{\partial C}{\partial \lambda} = -\left[\lambda\frac{\partial \mathcal{B}_2}{\partial x_2} + (1-\lambda)B_{22}\right]^{-1}\left[\mathcal{B}_2 - B_{21}x_1 - B_{22}x_2\right] \quad (6b)$$

Equation (6b) is the key differential equation for x_2 in λ. As for the ODE for x_1, (6b) can be solved for x_2 by any standard ODE technique [8] over the intervals $[i-1]\Delta < \lambda < i\Delta$ with given $x_1(t,[i-1]\Delta)$ and initial $x_2(t,[i-1]\Delta)$ to yield $x_2(t,x)$ over $[i-1]\Delta < \lambda < i\Delta$ from which $x_2(t,i\Delta)$ is found.

III. Discussion

By introducing linear terms which dominate at $\lambda=0$ but disappear at $\lambda=1$ we have given a continuation type of method for solving semistate equations. This method solves the nondynamic part separately from the dynamic portion, in contrast to another method previously given [2] where both parts are solved together. Because of the coupling between the dynamic, (2a), and the nondynamic, (2b), portions of the semistate equations the two portions are solved iteratively over small intervals with the continuation method itself applying to the nondynamic portion. In so doing we have assumed that the semistate equations have a unique solution, for example there being no hysteresis in the nondynamic part though the overall system could exhibit hysteresis, this being one important area to which semistate theory applies [9].

The ideas outlined here raise many points for further study. Of most interest is the accuracy of the solution and how this depends upon the continuation step size. Closely related to this is the efficiency of the calculations when compared to other semistate solution techniques. The answers to these and other questions undoubtedly have some relation to the choice of the B_{jk} made. At first glance it appears that choosing B_{22} to be the (n-m) x (n-m) identity matrix with all other B_{jk} zero is convenient. However, there may be some merit in choosing different B_{jk} at different iteration steps, possibly along with different size Δ's. Rigorization of the theory and conditions of validity are also needed.

Various variations of the technique can be given, an obvious one is to solve (3b) by other means, one being,

$$x_2(t,i\Delta) = B_{22}^{-1}\{\mathcal{B}_2 u(t) - B_{21}x(t,[i-1]\Delta) - i\Delta[\mathcal{B}_2(x(t,[i-1]\Delta\,) - B_{21}\,x_1(t,[i-1]\Delta) - B_{22}x_2(t,[i-1]\Delta)]\} \quad (7)$$

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