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A Binary Hysteresis Chaos Generator

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Abstract:

Motivated by studies on fibrillation of the heart a chaos generator is introduced. The generator is based on the switching of a second order system s trajectories between two planes which are determined by the two branches of binary hysteresis. Chaos is insured by designing the system so that the first return map has a period three point.

I. Introduction.

Recently the theory of chaos has received considerable interest in both the mathematical and circuits & systems communities. For example the August & September, 1983, issues of the IEEE Transactions on Circuits and Systems, [1],[2], contain a number of articles on the topic while the mathematical literature in the area is far too extensive to begin to reference here. However, there is very little available on the design of circuits to give chaotic responses. In this regard the original equations of Lorenz [3, p. 135] have been simulated by Theis [4], a synchronization circuit has been designed by Tang, Mees, & Chua [5, p. 625], and a bent hysteresis design has been presented by one of us with Sathyan [6]. Although they do work, designs based upon the Lorenz equations are really quite unwieldy since they require a number of multipliers and are of third order. The synchronization design of [5] is quite interesting, being of first order, but it uses two 555 timer chips making the overall circuit also quite extensive. The bent hysteresis circuit of [6] is quite easy to implement but is of third order.

Here, because of our interest to obtain electronic simulations that will allow for study of the heart under arrhythmic behavior, we look at a second order system, the heart being basically a second order system. In order to get chaotic behavior, which in essence should be able to represent fibrillations, we switch the parameters in the second order system at various points on the trajectory. This is accomplished by having parameters dependent upon the value of a hysteresis branch and then switching between the two branches of binary hysteresis in such a manner that the conditions of Li & Yorke [7, p. 987] are satisfied for the generation of chaos. In particular, we make the design such that the first return map has a period three point, this being a sufficient con- dition that the system will yield chaotic trajectories.

In Section II we briefly discuss the motivation for such a system, this being followed in Section III, the main portion of the paper, by the technical description of the system, its design, and its behavior. Section IV gives a brief discussion of the results which open a number of future research paths.

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II. Motivation - The Heart As A Chaotic System.

The heart as an electronically controlled pumping system is fairly well understood [8]. It is also well recognized that it is subject to various types of chaotic behavior, though these are much less well understood. Nevertheless, mathematical theories for chaotic behavior of the heart do exist, as presented by Keener [9]. Strangely, the main model presented by Keener does not involve basic dynamics of the heart [9, p. 307], whereas we believe the primary dynamics of the heart should be modeled by a dynamic system of at least second order [10]. In searching for a second order system which may give chaos we have observed that the third order system discussed by Rössler [11, p. 379] becomes a second order one with hysteresis if a limiting process is used to convert the state-variable equations to semi-state ones. It is on this basis that we have been led to the following chaos generator.

III. The Chaos Generator.

Our philosophy is to take an unstable second order system forced by constants of binary hysteresis such that these constants force the solution to return to a region through which the trajectory has previously passed when the hysteresis switches from one branch to the other. The equations from which we begin are taken in the following canonical semistate form [12]

$$\dot{X}_1 = A_{11}X_1 + A_{12}X_2 + A_{14}X_4 \qquad (1a)$$
$$\dot{X}_2 = A_{21}X_1 + A_{22}X_2 + A_{24}X_4 \qquad (1b)$$
$$\dot{X}_3 = A_{31}X_1 + A_{32}X_2 + A_{33}X_3 \qquad (1c)$$
$$0 = A_{44}X_4 + BH(X_3) \qquad (1d)$$

where the A_{ij} and B are real constants and where $H(.)$ is binary hysteresis (here H_+ and H_- are real constants)

$$H(X_3) = \begin{cases} H_+ & \text{for } X_3 \text{ on the upper branch} \\ H_- & \text{for } X_3 \text{ on the lower branch} \end{cases} \qquad (1e)$$

By a series of transformations and normalizations (see appendix) we can bring these to the form

$$\dot{x}_1 = x_2 + a_1 h(x_3) \qquad (2a)$$
$$\dot{x}_2 = -x_1 - 2\sigma_0 x_2 + a_2 h(x_3) \qquad (2b)$$
$$0 = a_{31}x_1 + a_{32}x_2 - x_3 \qquad (2c)$$

with the binary hysteresis now normalized to unit step function form

$$h(x_3) = \begin{cases} 1 & \text{for } X_L < x_3 \text{ (on upper branch)} \\ 0 & \text{for } x_3 < X_U \text{ (on lower branch)} \end{cases} \qquad (2d)$$

Here X_L and X_U are the normalized hysteresis jump points.

Equations (2) are a set of semistate equations which have trajectories in the (x_1,x_2)-plane which change when x_3 passes through either X_L or X_U. One can consider the situation as that of two second order dynamical systems, one forced by $h(.) = 1$ and the other forced by $h(.) = 0$, patched together through the nondynamic variable x_3, where in our case we have chosen the determining parameter, σ_0, to be the same for both dynamical systems (so that the overall system remains second order rather than fourth). We choose the damping factor, σ_0, negative so that the system will be self-starting and give chaos as desired. With negative damping, in each of the hysteresis planes [that is the two (x_1,x_2)-planes, one for $h=1$, and one for $h=0$] we obtain outgoing spirals with the two spiral centers being at different points. The nondynamical equation (2c) allows us to control the means of switching between the two hysteresis planes and for convenience in this paper we simply choose $x_3 = x_1$. Based on other factors to be discussed we choose the parameters for our numerical example as

$$\sigma_0 = -0.2,\ a_1 = -1,\ a_2 = -1.349966731,$$
$$a_{31} = -1,\ a_{32} = 0,\ X_L = 0,\ X_U = 0.3 \qquad (3)$$

Figure 1a) shows the nature of the semistate trajectory planes while Fig. 1b) shows a top view (that is, looking toward the origin down the positive x_3 axis) of the trajectory sets in the two hysteresis planes superimposed to show how they intersect each other; the actual trajectory (for given initial conditions) will switch between these two sets of trajectories when the spiral from the upper center intersects the line $x_1 = X_L$ or the one from the lower center intersects $x_1 = X_U$. As an example, considering Fig. 1a) we can consider a trajectory which passes through the point labelled 1 on the lower plane spiral; this continues in the lower plane until the point 2 where it jumps up, due to the hysteresis (it having hit the jump line $x_1 = X_U$), from the lower plane spiral to the upper plane one at point 3. The trajectory continues on the upper plane spiral from point 3 until point 4 where it jumps down to the lower plane at point 5, the upper trajectory having arrived at $x_1 = X_L$; the trajectory continues in this manner to points 6 through 13, which in the case of a periodic trajectory, as shown, will equal a point passed through, point 1 in this case. In the case of a chaotic trajectory of course no point will be met twice on any one trajectory. Figure 2 shows the points of intersection of trajectories on the nonnegative x_2 axis (that is, the intersection with the $x_1 = X_L$ line) versus the intersection point on the x_2 axis when the line $x_1 = X_L$ was crossed the time before. Figure 2 is thus a "first return" map for the system. Also on Fig. 2 we have noted a period three point. Since, by the strange looking choice of a_2 above, we have guaranteed continuity of this first return map, the theorem of Li & Yorke [7, p. 987] applies to show that this system has chaotic behavior in the first return map and, hence, in the real time trajectories.

IV. System Realization

Figure 3 shows a signal flow graph for a practical set of equations of the form of Eqs. (1). As seen, two integrators are required to realize Fig. 3, and, hence, the realization is second order. Since the binary hysteresis, H(.), is readily realized by an operational amplifier circuit [12], as are the other components of the signal flow graph, this chaos generator can readily be realized by an active-RC circuit.

V. Discussion

By inserting binary hysteresis into an unstable second order dynamical system we have been able to change it into one which creates chaotic signals. This occurs by feeding back an outgoing spiral on one hysteresis plane to the inside of a similar outgoing spiral on the other hysteresis plane. We have guaranteed the existence of chaos by applying a theorem of Li & Yorke to the first return map, where we have taken the first return as the crossing of trajectories on the nonnegative x_2 axis. In applying the theorem of Li & Yorke it is necessary to have a continuous first return map. However, in general the first return map of the systems under discussion, found as we describe, are not continuous, there usually being a discontinuity at the first cusp (near $x_2 = 0.2$ in our example) of the first return map (similar to Fig. 2). The presence of a discontinuity at that point does not seem to affect the presence or absence of chaos, but this seems still to be an open question (though see [5] where a somewhat different discontinuity also occurs). So we have chosen parameters to obtain the continuity used in the theorem cited.

By superimposing two linear systems by using hysteresis which assumes only one of two possible values, we have been able to explicitely solve the resulting semistate equations, the solutions for $x_1(t)$ and $x_2(t)$ being exponential in time for traversal of each of the two hysteresis branches. Space limitations do not allow complete presentation of these solutions here, but they were used to calculate the curves of Figs. 1b) & 2, these calculations being simple enough to have been carried out in total on an HP 15C vest pocket calculator.

APPENDIX: Normalization of Semistate Equations

By a similarity transformation on the two vector of components X_1 and X_2, and substitution of (1d) into (1c), we can bring Eqs. (1) to a form pertinent to Fig. 3 (for which we assume $\omega_n^2 = A_{11}A_{22} - A_{12}A_{21}$ and $-2\sigma_n\omega_n = A_{11} + A_{22}$ are both nonzero in order to have a meaningful second order system). Assuming that Eqs. (1) have first been brought to this form, we have

$$\dot{X}_1 = \omega_0 X_2 + a_1\omega_0 H(X_3) \qquad (A1a)$$
$$\dot{X}_2 = -\omega_0 X_1 - 2\sigma_0\omega_0 X_2 + a_2\omega_0 H(X_3) \qquad (A1b)$$

where H takes the value H_+ on the upper branch and H_- on the lower branch. Dividing by ω_0 and setting $\dot{} = d/d\omega_0 t$ gives

$$\dot{x}_1 = x_2 + a_1 H(x_3) \qquad (A2a)$$
$$\dot{x}_2 = -x_1 - 2\sigma_0 x_2 + a_2 H(x_3) \qquad (A2b)$$

Next let

$$\underline{x}_2 = x_2 + a_1 H_- \qquad (A3a)$$

which will lead to

$$\dot{x}_1 = \underline{x}_2 + a_1(H - H_-) \qquad (A3b)$$

in which we have obtained a lower zero in the hysteresis term of the X_1 equation. We then choose X_1 itself to give a lower zero in the hysteresis term of the X_2' equation: $X_2' = -X_1 - 2\sigma_0(X_2 - a_1H_-) + a_2H$ or

$$\underline{X}_1 = X_1 - 2\sigma_0 a_1 H_- + a_2 H_- \qquad \text{(A3c)}$$

we then have

$$\dot{\underline{X}}_2 = -\underline{X}_1 - 2\sigma_0\underline{X}_2 + a_2[H - H_-] \qquad \text{(A3d)}$$

In order to have the upper hysteresis value be unity we divide our present variables by $H_+ - H_-$:

$$x_1 = \underline{X}_1/[H_+-H_-], \quad x_2 = \underline{X}_2/[H_+-H_-] \qquad \text{(A4)}$$

in which case Eqs. (2) result. We note that now in both the upper and lower hysteresis planes we have equations of the form

$$\dot{u} = v \qquad \text{(A5a)}$$
$$\dot{v} = -u - 2\sigma_0 v \qquad \text{(A5b)}$$

under the transformations

	lower plane		upper plane
$u =$	x_1	or	$x_1 - 2\sigma_0 a_1 - a_2$
$v =$	x_2	or	$x_2 + a_1$

These show that in the (x_1, x_2)-plane the lower spiral is centered at (0,0) and the upper spiral at $(2\sigma_0 a_1 + a_2, -a_1)$. The initial value $(X_1, X_2) = (0,0)$ for the system of Fig. 3 becomes for the normalized system of Eqs. (2) $(x1, x2) = (a1, -2\sigma_0 a_1 + a_2)H_-/(H_+ - H_-)$.

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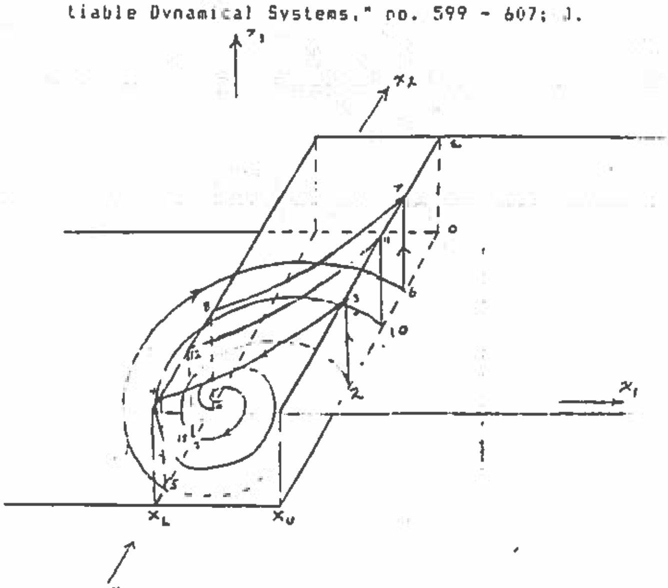

Figure 1a)

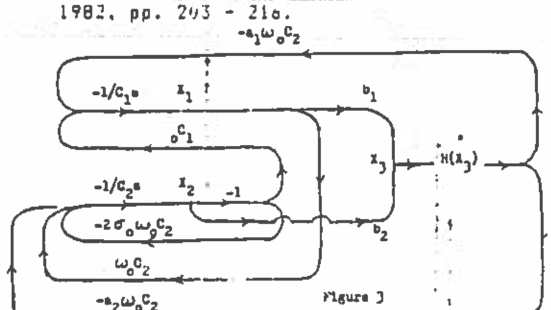

Figure 3

Signal-Flow Graph for Practical Op-Amp Realization

$$\dot{x}_1 = \omega_0 x_2 + a_1\omega_0 H(x_3)$$
$$\dot{x}_2 = -\omega_0 x_1 - 2\sigma_0\omega_0 x_2 + a_2\omega_0 H(x_3)$$
$$x_3 = b_1 x_1 + b_2 x_2$$

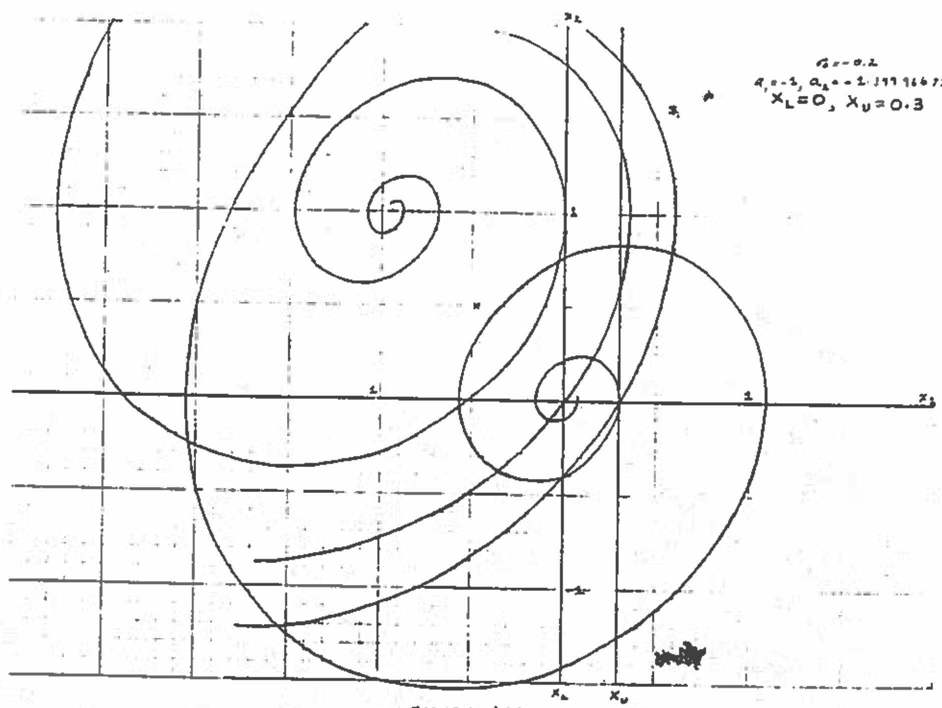

FIGURE 1b)

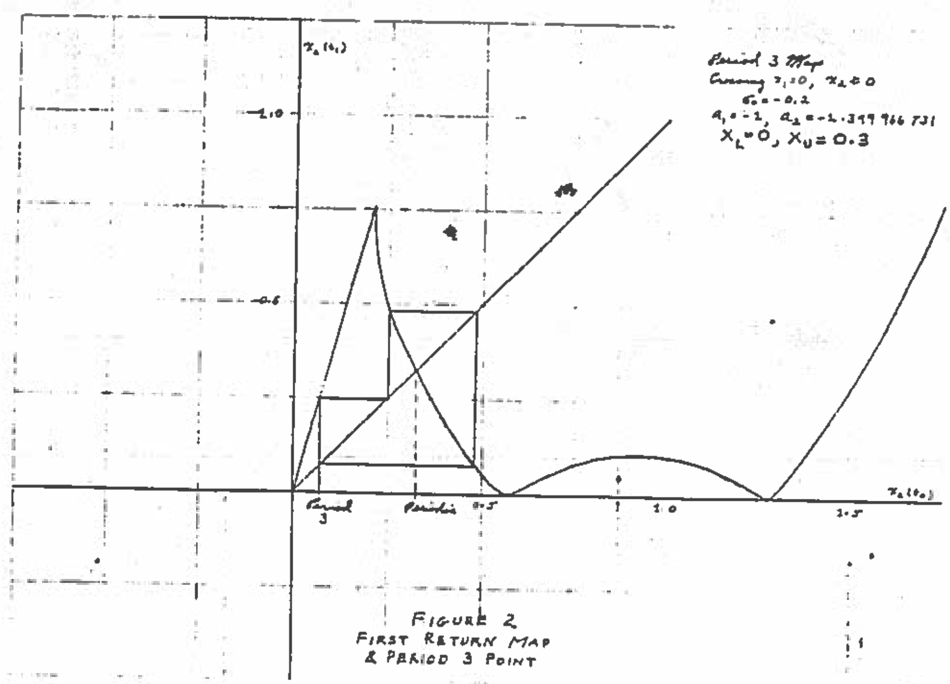

FIGURE 2
FIRST RETURN MAP
& PERIOD 3 POINT

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