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LAPLACE TRANSFORMS – DISTRIBUTIONAL THEORY

by

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LAPLACE TRANSFORMS - DISTRIBUTIONAL THEORY

ABSTRACT

As Romans the gods of old Greece once revised,
So here we report in form generalized,
Transforms to whose name S. Laplace does apply,
Abstract at the time when L. Schwartz did espy.

The theory proceeds, distributions its start,
Come then the ideas Fourier at its heart.
By simple extension all properties come,
A multiplication included among.

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LIST OF SYMBOLS

Symbol	Definition or Explanation	First Appears
t	Independent variable (time).	5
$\varphi(t)$	Testing function.	5
$\mathfrak{D}$	Space of testing functions.	5
n	Index of a sequence, as in $\{\varphi_n\}$.	5
k	Order of derivatives taken with respect to t.	6
$T(t)$, $S(t)$	Arbitrary distributions of the variable t.	6
ϵ	Contained in, as in $\varphi \epsilon \mathfrak{D}$.	6
$\langle T(t), \varphi(t) \rangle$	Scalar product of a distribution $T(t)$ and a testing function $\varphi(t)$.	6
$\mathfrak{D}'$	Space of distributions; topological dual of $\mathfrak{D}$.	6
$h(t)$, $g(t)$	Function of t, or locally integrable function continuous or not.	7
$u(t)$, u	Unit step function.	7
$\delta(t)$, δ	Impulse occurring at $t = 0$.	7
$\mathfrak{D}'_+$	Space of distributions with support bounded on the left.	9
$(\ )'$, $(\ )''$	First and second derivatives with respect to t.	9
PF	Pseudo function (or Hadamard's finite part).	10
$S*T$	Real or complex convolution.	16
$\mathcal{S}$	Space of good functions.	32
$\mathcal{S}'$	Space of tempered (slowly increasing) distributions; topological dual of $\mathcal{S}$.	32
ψ	Good function; infinitely differentiable function of rapid decrease.	32
m	Real power of a variable, as in t^m.	33
$\not\epsilon$	Not contained in.	33
$\mathcal{F}_\omega[\ ]$	Direct Fourier transform into the ω domain.	33
$\bar{\mathcal{F}}_t[\ ]$	Inverse transform into the t domain.	33
$j = \sqrt{-1}$		33
$A \subset B$	Set A is contained in B.	41
α	Most important use: fairly good (slowly increasing) function.	45

$\mathcal{O}_M$	Space of fairly good functions.	45
$\mathcal{O}'_c$	Space of rapidly decreasing distributions.	46
$\mathcal{D}'_{L_q}$	Discussed in note-IV-20; finite sums of derivatives of L_q functions.	47
p	Complex frequency variable; $p = \sigma+j\omega$.	55
$\Gamma(T)$	Convergence strip of T (set of σ for which $e^{-\sigma t}T(t)$ is tempered).	55
σ_ℓ	Greatest lower bound of $\Gamma(T)$.	56
σ_r	Least upper bound of $\Gamma(T)$.	56
$\mathring{\Gamma}(T)$	Interior of $\Gamma(T)$.	56
$A \cap B$	Intersection of sets A and B.	56
$\mathcal{L}_p[\ ]$	Bilateral Laplace transform into the p domain.	57
p^*	Complex conjugate of p; $p^* = \sigma-j\omega$.	59
L_2	Square integrable functions.	61
L_q	q th - integrable function.	61
$\mathcal{L}_t^{-1}[\ ]$	Inverse Laplace transform into the t domain.	70
$F(p)$	Any function of p; a Laplace transform.	69

I. INTRODUCTION

For many years now some of the key notions which have been considered in system analysis and design are those of the Laplace transform and the associated frequency domain techniques. Also for many years it has been realized that a considerable number of operations performed by engineers with the Laplace transform were done on an intuitive basis. For example, one freely interchanged orders of limits and integration, worked with singularity functions and impulse responses, etc. It seems very strange to us that even though the manipulations used by engineers can be very simply justified, as well as conditions of validity given, this nowhere occurs in the engineering literature or even in complete form in the English language.

A decade ago a very simple, but still the most conceivably general, theory of Laplace transforms was presented by Laurent Schwartz, [SC3]. Schwartz's concise paper, based upon his theory of distributions, is, however, quite abstract and has never really been properly put to use. It is, therefore, the purpose of this report to present Schwartz's material in a form useful to graduate engineers and scientists.

To be sure there are several alternate theories available, but these all have disadvantages and aren't rigorous, general, simple, and as intuitively clear as that of Schwartz. Listed among these, one should mention the ideas of Korevaar, [KO4], and König & Rehberg, [KO2] and [RE1], as well as the operational calculus of Mikusiński, [MI2]. Unfortunately, even the most recent texts in English essentially ignore the advantages to be gained by the use of the very simple theory of distributions.

The structure of the report is as follows. In Section II we review in some detail the theory of distributions. Although some of this material is available in English, most often the treatment is limited to giving a proper definition of the impulse. Thus we include the important results, such as the impulse response characterization listed in paragraph 7, as well as sufficient motivation, mainly through examples, to make the general ideas seem useful for other purposes. For those unfamiliar with Schwartz's ideas we have inserted many notes which work out details or ramify points which seem to us hard to grasp at a first meeting of the subject.

The short Section III gives the most important ideas concerning the Fourier transform of "good" functions. From two properties of these functions the entire distributional theory follows by a simple extension. This extension is carried out in Section IV to the most general case of "tempered" distributions. By then multiplying with real exponentials, Section V generalizes these ideas to obtain the Laplace transform of distributions which are tempered in a region. The detailed properties of this most general bilateral Laplace transform are then studied, also in Section V. As in Section II, these last three sections contain many notes giving proofs and discussions of points which, although of interest, tend to obscure the fundamental concepts.

Results concerning the Laplace transform are extensive, and exhaustive treatise exist giving complete results for the classical theory, [DO1], [DO2], [DO3], [WI1]. Here we can not pretend to be complete, but do attempt to present the ideas we feel of most importance in their greatest generality. As a consequence many properties normally treated separately will be seen to result as special cases in this distributional theory.

Because of the many symbols and abstract spaces present in the theory, notation becomes a problem. In general we follow that of Schwartz, where possible, and suggest that the reader consult the list of symbols or the appendix on spaces when in doubt.

II. DISTRIBUTIONS

Summary: This section consists of the definition of distributions and their most important properties. It serves as an introduction to this important theory for those unfamiliar with it. For those already versed in the theory of distributions it serves to define our notation and to compile the results needed in the later chapters.

The concept of a distribution will be regarded here as an attempt to generalize that of a function. For this, in paragraph 2 we define the space $\mathfrak{D}$ of testing functions. These functions are infinitely differentiable and of compact support. Distributions are then defined as linear functionals in this space, continuous in a convenient way. The space $\mathfrak{D}'$ of distributions is the dual of $\mathfrak{D}$, and the scalar product of a distribution T and a testing function φ is denoted by $< T, \varphi >$.

In paragraph 3 we show how the usual notion of function and the concept of distribution are related in such a way as to generalize the notion of function. We then define the derivative of a distribution, with the consequence that every distribution is infinitely differentiable. Paragraph 5 consists of the notion of convergence of distributions. The importance of this concept stems from its relation to physical reality. Thus any distribution can be approximated by a sequence of testing functions and a sequence of distributions can always be differentiated term by term.

It is impossible to define the product of any two distributions. In paragraph 6 we define the product of a distribution, T, and an infinitely differentiable function α. A similar situation holds for convolution. The convolution, S*T, of two distributions S and T, obtained from $< S(t), \psi(t) >$ where $\psi(t) = < T(\tau), \varphi(t+\tau) >$ is important for the Fourier and Laplace transforms. Paragraph 7 summarizes the most important consequences of distribution theory, while in the last paragraph a physical interpretation is given.

1. Introduction

In many branches of applied science it is desirable to deal with forces of large magnitude actuated for a short time only. A convenient mathematical expression for such forces is the well known impulse function (Dirac-delta function), $\delta(t)$, commonly defined by

$$\int_{-\infty}^{\infty} \delta(t)dt = 1, \ \delta(t) = 0 \text{ if } t \neq 0 \qquad \text{(II-1a)}$$

The above definition is perfect in the sense that the physical properties are conveniently described; mathematically, however, it is highly unsatisfactory. Here we will make a brief attempt to point out the difficulties.

If one insists on viewing δ as an ordinary function, the above definition is meaningless, and the fact that many correct results are obtained with it is due to the reason that only some properties of δ are significant. These are assumed independently of the definition of δ and no consistent mathematical derivation of these properties can be made from (II-1a). Among the most important properties are the "sifting property"

$$\int_{-\infty}^{\infty} \delta(t)\varphi(t)dt = \varphi(0) \qquad \text{(II-1b)}$$

for any function φ continuous at the origin, and the corresponding property (obtained by formal integration by parts, taking $\varphi(\pm\infty) = 0$),

$$\int_{-\infty}^{\infty} \delta^{(k)}(t)\varphi(t)dt = (-1)^k \varphi^{(k)}(0) \qquad \text{(II-1c)}$$

In the usual mathematical sense, the sifting property can be obtained by the use of the Stieltjes integral, which in fact shows that δ can't be a function by the customary meaning.[1] For the derivatives appearing in (II-1c) no similar representation exists.

Here the difficulties are solved by generalizing the notion of function. The traditional concept of a mathematical function is widened, and a theory is made such that improper functions, such as impulsive derivatives, acquire a meaning.

The task of the theory of distributions is quite clear:[2] to generalize the notion of function, arriving at the notion of distributions, in a manner to include such useful (and pathological) entities as the impulse function; then it is necessary to establish the rules of a calculus for distributions so that the usual results of Fourier analysis and Laplace transform theory remain valid. These calculation rules turn out to be extremely simple, and, since they coincide with the methods used by engineers, they serve in themselves as ample justification for a study of distributions. Of such interest are the facts that every distribution can be infinitely differentiated with differentiation and series summation being freely interchanged. Further, every normal function is a distribution and functional and distributional operations coincide when the former can be carried out. Thus, although the original goal was to give a rigorous theory for improper functions, much more is actually achieved.

As a necessary prerequisite to the Laplace transform material, a simple summary of the basic ideas and results along these lines will be given in this chapter.

2. Definition of Distributions

Let R denote the real line whose variable is (time) t and let φ be a complex valued function whose domain of definition is R. By the support of φ is meant the closure[3] of the set of t contained in R for which $\varphi(t) \neq 0$.

The theory is based upon the set $\mathcal{D}$ of functions φ which are infinitely (continuously) differentiable and of compact support. Such functions will often be referred to as testing functions, since they serve to test a distribution for its properties. A typical testing function is shown in Fig. II-1, if infinite differentiability is assumed.[4]

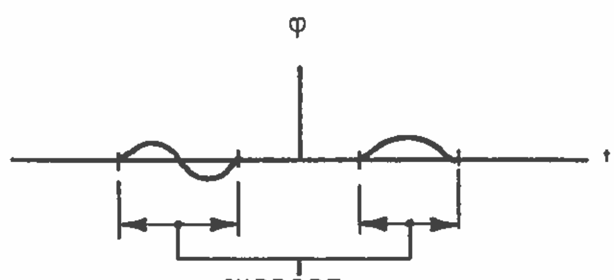


FIG. II-1. TYPICAL TESTING FUNCTION.

A matter of theoretical concern is the introduction of a notion of convergence[5] in $\mathcal{D}$. This is done to set a "topology", which in turn allows the properties to be rigorously obtained in the framework of modern mathematics.

We say that a sequence of testing functions, $\{\varphi_n\}$, converges (in $\mathcal{D}$) to zero as $n \to \infty$ if: a) all φ_n have their support in the same bounded set of R, and b) for every integer $k \geq 0$, the sequence of k th derivatives, $\{\varphi_n^{(k)}\}$, converges uniformly to zero (in the customary sense).

Definition II-1:

A <u>distribution</u> T(t) is a linear functional[6] defined over $\mathfrak{D}$ which is continuous in terms of the above notion of convergence in $\mathfrak{D}$.

A distribution, or generalized function, T(t) is then a process of assigning to every testing function $\varphi \epsilon \mathfrak{D}$ a complex number which we shall denote[7] by $< T(t), \varphi(t) >$. We comment that in this expression t acts as a dummy variable, since $< T, \varphi >$ is a fixed number for fixed T and φ.

The linearity and continuity clauses in the definition of a distribution mean that for all finite complex constants α and β and all $\alpha_n \epsilon \mathfrak{D}$:

1) $< T, \alpha\varphi_1 + \beta\varphi_2 > = \alpha < T, \varphi_1 > + \beta < T, \varphi_2 >$ (linearity)

2) $< T, \varphi_n >$ converges to 0 if φ_n converges to 0 (in $\mathfrak{D}$) (continuity)

One can regard $\mathfrak{D}$ as a vector space, and the set of all linear functionals, continuous or not, defined over it actually forms the dual vector space[8] of $\mathfrak{D}$. Only the sub-space which consists of all continuous linear functions has interest for us, since only then can we freely transpose properties from φ to T in the scalar product $< T, \varphi >$. This space of continuous linear functions, which is the topological dual[9] of $\mathfrak{D}$, will be designated by $\mathfrak{D}'$.

As we now see, distributions can be endowed with the important properties of functions.

3. <u>Generalized Functions</u>

We shall attempt to show how the usual notion of function and the concept of distribution are related in such a way as to generalize the notion of function while incorporating "improvements".

If g(t) is a locally integrable function[10] it defines a distribution by the expression

$$< T(t),\ \varphi(t) > = \int_{-\infty}^{\infty} g(t)\varphi(t)dt$$

For conciseness, by function we will always mean a "locally integrable function".

The crucial step in the entire theory lies in calling this T, g itself. This identification is essential; it allows g to be considered a distribution as well as a function.

Definition II-2:

If g(t) is a function then, for all $\varphi \in \mathcal{D}$,

$$< g(t),\ \varphi(t) > = \int_{-\infty}^{\infty} g(t)\varphi(t)dt \qquad \text{(II-2a)}$$

We may think of distributions as generalized functions, and can then symbolically write

$$< T(t),\ \varphi(t) > = \int_{-\infty}^{\infty} T(t)\varphi(t)dt \qquad \text{(II-2b)}$$

Example II-1:

The unit step function, u(t),

$$u(t) = \begin{cases} 1 & t < 0 \\ 0 & t > 0 \end{cases} \qquad \text{(II-3)}$$

is a function which can be regarded as a distribution with

$$< u,\ \varphi > = \int_{0}^{\infty} \varphi(t)dt$$

Example II-2:

All square integrable functions are distributions, since they are locally integrable.[11] In this case all the functions which were correctly handled by transform theories without the notion of distributions are included in the present theory.

Example II-3:

The impulse function can be adequately defined as a distribution δ by

$$< \delta(t),\ \varphi(t) > = \varphi(0) = \int_{-\infty}^{\infty} \delta(t)\varphi(t)dt \qquad \text{(II-4)}$$

On the very right of this the symbolic notation of (II-2b) is used and thus the sifting property of (II-1b) acquires precise meaning. It is perhaps better not to call δ the impulse function, since, as remarked earlier, it can not be a function. A convenient graphic representation of δ is the well-known symbol of Fig. II-2.

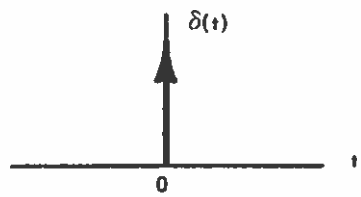


FIG. II-2. SYMBOL FOR δ.

A consequence of definition II-2 is that every function can be considered as a distribution. Equations (II-2) motivate various definitions for operations on distributions, since when a distribution coincides with a function we naturally wish distributional operations to coincide with functional ones. For instance we have, combined in pairs, four operations of summation, scalar multiplication, time shifting and time scaling.

Definition II-3:

If T_1, T_2, $T \in \mathfrak{D}'$ and α (complex), β (complex), a (real) $\neq 0$, b (real) are finite constants, then for all $\varphi \in \mathfrak{D}$ we define

$$< \alpha T_1 + \beta T_2, \varphi > = \alpha < T_1, \varphi > + \beta < T_2, \varphi > \qquad \text{(II-5a)}$$

$$< T(at+b), \varphi(t) > = < \frac{1}{|a|} T(t), \varphi(\frac{t-b}{a}) > \qquad \text{(II-5b)}$$

Note that these are simple extensions of what (II-2) gives in the functional case.[12]

Example II-4:

By (II-5b) we easily find[13]

$$\delta(at+b) = \frac{1}{|a|} \delta(t+\frac{b}{a}) \qquad \text{(II-6)}$$

The notation T(t) may mislead the reader. In general, distributions do not have definite "values at points t," as the impulse shows at t = 0. However, we do say that T(t) is zero for $t \in \Omega$, where Ω is a subset of R, if for every testing function, φ_Ω, with support contained in Ω we have

$$< T(t), \varphi_{\Omega}(t) > = 0$$

The closure of the set of all t for which T is zero is called the support of T. Thus u(t) has $t \geq 0$ for its support while the support of $\delta(at+b)$ is $t = -b/a$. Of considerable interest for physical systems is the space $\mathfrak{D}'_+$ of distributions with support bounded on the left; that is, $T \epsilon \mathfrak{D}'_+$ if T has support in $t > a$ for some finite real a.

If the distribution T is defined by a function g, as in (II-2a), then T will vanish in the interval (a, b) in the sense of our definition if and only if g vanishes, in the functional sense, almost everywhere on (a, b).[14]

At this point it is worth noting that if T has bounded support then for t outside the support of T the values of the testing function φ are unimportant. Thus, in some cases we can relax the requirement that the φ be testing functions, that is have bounded support. For the Laplace transform it is sometimes convenient to do this by using $\varphi(t) = \exp[-pt]$.

4. Derivatives of Distributions

Let us investigate a convenient way to define derivatives of distributions. That it is necessary to define the derivative of a distribution is clear if we recall that δ has been called a distribution and in applications we often encounter entities such as δ', δ'', etc.

Going to (II-2a), let us set g equal to $h' = dh/dt$. If h' is also a function, we have upon integration by parts[15]

$$< h', \varphi > = - < h, \varphi' >$$

This result shows that there is only one logical way to define the distributional derivative.

Definition II-4:

If $T \epsilon \mathfrak{D}'$ then for all $\varphi \epsilon \mathfrak{D}$ the k th derivative is defined by

$$< T^{(k)}, \varphi > = (-1)^k < T, \varphi^{(k)} > , \ k = 0, 1, 2, \ldots \qquad \text{(II-7)}$$

Clearly any distribution in $\mathfrak{D}'$ is infinitely differentiable (as a distribution) by this definition, since φ is by hypothesis.

Furthermore, our objective has been reached, that of making the distributional derivative coincide with the functional derivative, whenever this latter exists.[16]

Example II-5:

Let us group examples (II-1) and (II-3) where u and δ were defined. We have, since $\varphi(\infty) = 0$, by (II-7),

$$< u', \varphi > = < u, -\varphi' > = -\int_0^\infty \varphi'(t)dt = \varphi(0) = < \delta, \varphi >$$

or

$$\delta(t) = u'(t) = du/dt \qquad \text{(II-8a)}$$

A simple expression takes care of all derivaties of δ, by (II-4) and (II-7), is

$$< \delta^{(k)}(t), \varphi(t) > = (-1)^k \varphi^{(k)}(0) \qquad \text{(II-8b)}$$

which rigorously gives (II-1c). For instance

$$< \delta'(t), \varphi(t) > = -\varphi'(0)$$

$$< \delta''(t), \varphi(t) > = \varphi''(0)$$

and so on. Combining this with the scale change of (II-5b) gives[17]

$$\delta'(at+b) = \frac{1}{a|a|}\, \delta'(t+\frac{b}{a}) \qquad \text{(II-8c)}$$

This example illustrates the facts that:

1). δ can be considered without lack of rigour as the derivative of the step function (in the distributional sense).

2). Since every locally integrable function generates a distribution, every such function possesses derivatives (distributionally again) of arbitrary order, but these derivatives need not be functions. Here, the derivative of a function, u, was found to be a distribution, δ, which is not a function. For a function with a jump discontinuity having right and left derivates at the jump, the distributional derivative yields an impulse at the jump.[18]

Example II-6:

The mapping $1/t$ does not define a function under our definition, since it is not integrable in the neighborhood of the origin. However, we can define a distribution, denoted by $PF(\frac{1}{t})$, which coincides with 1/t when $t \neq 0$, by[19]

$$< PF(\frac{1}{t}), \varphi(t) > = \lim_{\epsilon\to 0} [\int_{-\infty}^{-\epsilon} \frac{\varphi(t)}{t}dt + \int_{\epsilon}^{\infty} \frac{\varphi(t)}{t}dt] \qquad \text{(II-9a)}$$

Differentiating we define $PF(\frac{1}{t^{\ell}})$, $\ell > 1$, by[20]

$$PF(\frac{1}{t^{\ell}}) = \frac{-1}{\ell-1} \frac{d}{dt} PF(\frac{1}{t^{\ell-1}}) \qquad \text{(II-9b)}$$

The function ln $|t|$ has its derivative functionally equal to 1/t for $t \neq 0$. This is thus still true distributionally and we have[21]

$$\frac{d}{dt} \ln |t| = PF(\frac{1}{t}) \qquad \text{(II-9c)}$$

Concerning integration, every distribution has an infinite number of primitives (indefinite integrals), two of which differ by a constant.[22] For instance two primitives of $\delta(t)$ are $u(t)$ and $-u(-t)$ with $u(t) = 1-u(-t)$.

5. Convergence of Distributions

Of special interest to engineers is the concept of distributional convergence, since in the laboratory impulses are approximated by very narrow pulses. That is, impulse responses are normally calculated in the limit where the pulse width becomes significantly less than important time constants.

Definition II-5:

A sequence of distributions $\{T_n\}$, with $T_n \in \mathfrak{D}'$, converges to $T \in \mathfrak{D}'$, written

$$T = \lim_{n\to\infty} T_n \qquad \text{(II-10a)}$$

if, for all $\varphi \in \mathfrak{D}$,

$$\lim_{n\to\infty} < T-T_n, \varphi > = 0 \qquad \text{(II-10b)}$$

One aspect of the above definition that may cause trouble is that two notions of limit are involved. In (II-10a) the limit is, by definition, in the distributional sense, while in (II-10b) it is taken in the customary sense; observe that $< T_n-T, \varphi >$ is merely a sequence of complex numbers. Of course for a series $T = \sum_{i=1}^{\infty} S_i$, $S_i \in \mathfrak{D}'$, we take

$$T_n = \sum_{i=1}^{n} S_i$$ in the standard manner.

Example II-7:

We have that distributionally[23]

$$\lim_{n\to\infty} T_n(t) = 2\delta(t)$$

with T_n as in Fig. II-3(a) or (b).

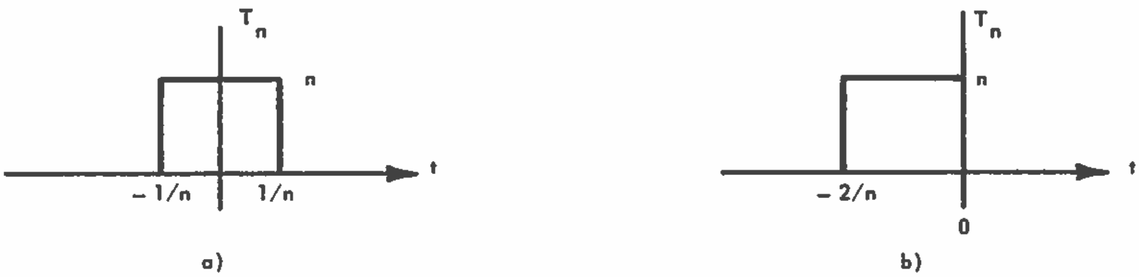


FIG. II-3. SEQUENCES OF DISTRIBUTION CONVERGING TO 2δ.

Example II-8:

$$T(t) = \sum_{k=0}^{\infty} \delta^{(k)}(t-k)$$

defines a distribution, since

$$< \sum_{k=0}^{n} \delta^{(k)}(t-k), \ \varphi(t) > = \sum_{k=0}^{n} (-1)^k \varphi^{(k)}(k)$$

and this latter is a convergent series for any fixed φ, since only a finite number of terms are non-zero, the $\varphi^{(k)}$ having compact support.

Example II-9:

$$T_o(t) = \sum_{k=0}^{\infty} \delta^{(k)}(t)$$

is not a distribution since

$$< \sum_{k=0}^{n} \delta^{(k)}(t), \ \varphi(t) > = \sum_{k=0}^{n} \varphi^{(k)}(0)$$

will in general not converge.

A fundamental and important property of distributions is the fact that a sequence, or a series, can always be differentiated term by term (with the usual rules of calculus, this process is legitimate only when both the original and the derivate series are uniformly convergent).

Theorem II-1:

If distributionally

$$T = \lim_{n\to\infty} T_n$$

then

$$T^{(k)} = \lim_{n\to\infty} T_n^{(k)},\ k = 0, 1, 2, \ldots$$

In terms of series this takes the form, if $T = \sum_{i=1}^{\infty} T_i$ then $T^{(k)} = \sum_{i=1}^{\infty} T_i^{(k)}$. The proof is immediate, since

$$< T^{(k)} - T_n^{(k)}, \varphi > = (-1)^k < T - T_n, \varphi^{(k)} >$$

but $\varphi^{(k)}$ is still in $\mathfrak{D}$ and thus the right side of this has a zero limit by hypothesis.

Example II-10:

The periodic waveform of Fig. II-4(a) has the Fourier series representation[24]

$$g(t) = \frac{4}{\pi}\sum_{i=1}^{\infty} \frac{\sin(2i-1)t}{2i-1}$$

Since locally the jumps can be represented by step functions, we have (distributionally)

$$g'(t) = \sum_{i=-\infty}^{\infty} 2(-1)^i \delta(t-i) = \frac{4}{\pi}\sum_{i=1}^{\infty} \cos(2i-1)t$$

as shown in Fig. II-4(b). Of course the series can be differentiated again.

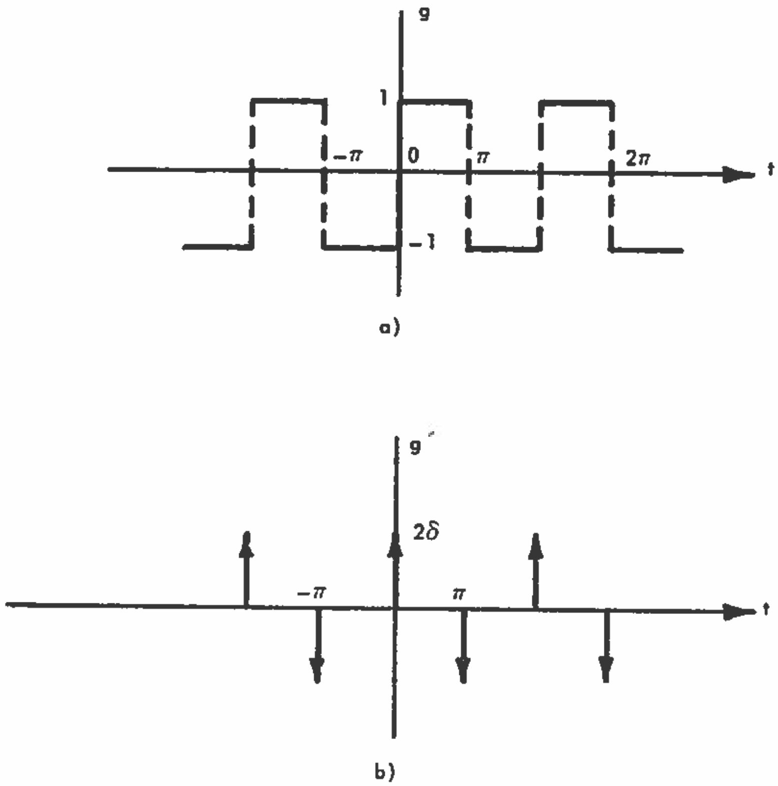


FIG. II-4. A PERIODIC WAVE a) AND ITS DERIVATIVE b).

6. Multiplication and Convolution

In general, multiplication can not be defined for any two distributions. That this is true can be seen if we look at the product of two functions g and h, their product is not necessarily another function, a distribution is not necessarily defined.[25] If δ^2 were defined it would physically represent an infinite point charge, since $q\delta(t)$ is the charge density of a point charge of charge q. But a charge could become infinite in many ways, showing that δ^2 could have no useful meaning. However, if $\alpha(t)$ is locally more regular than $T(t)$ is irregular then αT can be defined. This will be the case if $\alpha\varphi$ acts as a testing function for T, which in fact is always the case if α is an infinitely differentiable function, that is $< T, \alpha\varphi >$ uniquely defines a distribution.

Definition II-6:

If $T \epsilon \mathcal{D}'$ and α is an infinitely differentiable function then αT is defined by

$$< \alpha(t) \cdot T(t), \varphi(t) > = < T(t), \alpha(t)\varphi(t) > \qquad \text{(II-11)}$$

We will in fact use (II-11) to define αT whenever this can be defined, that is not only for infinitely differentiable α.

Example II-11:

The familiar sampling property

$$\alpha(t)\delta(t) = \alpha(0)\delta(t) \qquad \text{(II-12a)}$$

of the impulse, δ, is an immediate consequence of (II-11).[26] Similarly[27]

$$\alpha(t)\delta'(t) = \alpha(0)\delta'(t) - \alpha'(0)\delta(t) \qquad \text{(II-12b)}$$

These show that for $\alpha\delta$ we only require α continuous at zero while for $\alpha\delta'$ we require at most that α be one time continuously differentiable at the origin.

Further we have[28]

$$(\alpha T)' = \alpha' T + \alpha T' \qquad \text{(II-12c)}$$

a special case of which is

$$(gu)' = g'(t)u(t) + g(0)\delta(t) \qquad \text{(II-12d)}$$

One should in general use care in using distributional multiplication since it isn't always associative. For instance, one finds,

$$(\delta \cdot t) \cdot PF(\tfrac{1}{t}) = 0 \text{ since } \delta \cdot t = t \cdot \delta = 0$$

$$\delta \cdot (t \cdot PF(\tfrac{1}{t})) = \delta \text{ since } t \cdot PF(\tfrac{1}{t}) = 1$$

The usual commutitivity and associativity rules do hold, however, if all except at most one of the terms are infinitely differenitable functions.[29]

We recall that for two functions, g and h, the convolution, $g*h$, is defined by

$$g*h(t) = \int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau$$

If we now revert to (II-2a) we see that in the functional case the distributional interpretation is[30]

$$< g*h(t),\ \varphi(t) > = \int_{-\infty}^{\infty} g(t)[\int_{-\infty}^{\infty} h(\tau)\varphi(t+\tau)d\tau]dt$$

This is the natural form to use for the distributional extension, which, as with multiplication, can't always be defined. In contrast to multiplication where the problems were of a local type, the problems are here of a global nature.

Definition II-7:

For S, $T\in \mathcal{D}'$, whenever the convolution $S*T\in \mathcal{D}'$ can be defined it is given by

$$< S*T(t),\ \varphi(t) > = < S(t),\ < T(\tau),\ \varphi(t+\tau) > > \qquad \text{(II-13)}$$

We note that the problem is here that $\psi(t) = < T(\tau),\ \varphi(t+\tau) >$ is a function whose support is in general non-compact. However, ψ is infinitely differentiable, since $\psi' = < T(\tau),\ d\varphi(t+\tau)/dt >$ by the linearity and continuity of the functional. Thus (II-13) does define a distribution, S*T, whenever ψ is of compact support or if the intersections of the supports of ψ and S are compact for all testing functions φ. If T has compact support then so does ψ, and in this case S*T always exists. If both S, $T\in \mathcal{D}'_+$ then again S*T always exists, and is in $\mathcal{D}'_+$, since $< T(\tau),\ \varphi(t+\tau) >$ will be zero for t larger than some contant while S(t) is zero for t smaller than some constant.

As can be seen by a short but somewhat tricky argument[31]

$$S*T = T*S \qquad \text{(II-14)}$$

Example II-12:

Since δ has compact support it can be convoluted with any $T\in \mathcal{D}'$. We have in fact by an immediate application of (II-13) using (II-4)

$$T*\delta = T \qquad \text{(II-15a)}$$

Likewise, by further using (II-6),

$$T(t)*\delta(t+b) = T(t+b) \qquad \text{(II-15b)}$$

A similar property holds for higher derivatives[32]

$$T*\delta^{(k)} = T^{(k)} \tag{II-15c}$$

Example II-13:

We have if C is any complex constant[33]

$$(\mathrm{PF}\tfrac{1}{t})*C = 0 \tag{II-16}$$

which shows that the convolution may be zero even though neither term is zero.

Although the convolution is always commutative, that is (II-14) generally holds, it need not be associative. That is for any three distributions R, S and T, in general

$$(R*S)*T \neq R*(S*T)$$

as is seen by[34]

$$(1*\delta')*u = 0*u = 0 \neq 1*(\delta'*u) = 1*\delta = 1$$

However, if R, S, $T \in \mathcal{D}'_+$ then

$$(R*S)*T = R*(S*T)$$

and further, under these conditions $S*T = 0$ if and only if $S = 0$ or $T = 0$. That is, $\mathcal{D}'_+$ forms an algebra under convolution which has no divisors of zero.[35] As an application of this result we show how to find a primitive, as mentioned at the end of paragraph 4, for any $T \in \mathcal{D}'_+$. Such is $T*u$ which we write

$$T*u = \int_{-\infty}^{t} T(\tau)d\tau$$

since, by (II-15c) and associativity

$$(T*u)' = (T*u)*\delta' = T*u*\delta' = T*\delta = T$$

7. Some Important Results

A consequence of some importance for physical systems is the fact that $\mathcal{D}$ is dense in $\mathcal{D}'$. That is any distribution can be arbitrarily closely approximated by testing functions, or more precisely[36]

Theorem II-2:

For every $T \in \mathcal{D}'$ there is a sequence with $T_n \in \mathcal{D}$ such that

$$T = \lim_{n\to\infty} T_n$$

Physically this result means the following. In the laboratory it appears that we can only encounter testing functions in the measurement of physical variables. This because measurements start and end at finite times and all variables could be passed through differentiators.[37] However, it is theoretically convenient to build our theories upon distributions, such that quantities such as impulse responses can be considered. Doing this, theorem II-2 shows that any theoretical results can be obtained to arbitrarily good accuracy in the laboratory.

Example II-14:

For both linear and nonlinear systems the unit step response is of interest. This can be determined by applying inputs which are sequences of testing functions of the form of Fig. II-5, where ϵ_n converge to zero.

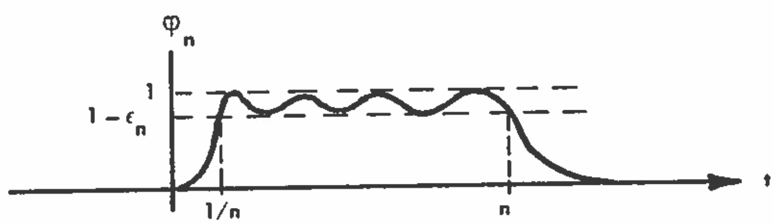


FIG. II-5. $\varphi \epsilon \mathfrak{D}$ CONVERGING TO u.

A result of some importance is the following.[38]

Theorem II-3:

On every open set whose closure is compact, every distribution T is the k th derivative of a continuous function.

In other words, locally $T = g^{(k)}$ where g is a continuous function. For instance $\delta(t) = [tu(t)]^{(2)}$. This allows us to replace some arguments concerning distributions by arguments concerning continuous functions. In general the continuous function, g, and the derivative, k, will vary with the open set under consideration. Of especial interest, however, are those T for which $T(t) = g^{(k)}(t)$ for $-\infty \leq t \leq \infty$; such T are said to be of finite order. δ is of finite order while

$$\sum_{k=0}^{\infty} \delta^{(k)}(t-k) \text{ is not.}$$

We do not present any proof of theorem II-3. However, we can check its validity by considering an ordinary function, everywhere differentiable. This function g can be considered as a distribution defined by the integral (II-2a) and its generalized derivative (II-7) equals the ordinary derivative interpreted as a distribution.

The study of linear physical systems customarily proceeds from the impulse response. The following theorem makes the conditions under which such a description is valid rigorous.[39]

Theorem II-4:

Every operation L, $L(\varphi)\in \mathcal{D}'$ when $\varphi\in\mathcal{D}$, for which, with α and β complex constants,

a) $L(\alpha\varphi_1+\beta\varphi_2) = \alpha L(\varphi_1)+\beta L(\varphi_2)$ (linearity)

b) $\lim_{n\to\infty} L(\varphi_n) = L(\lim_{n\to\infty} \varphi_n)$ (continuity)

c) $L'(\varphi) = L(\varphi')$ (permutation with differentiation)

is of the form

$$L(\varphi) = S*\varphi \tag{II-17a}$$

for a fixed $S\in\mathcal{D}'$.

Here in fact

$$S = L(\delta) = \lim_{n\to\infty} L(\delta_n) \tag{II-17b}$$

$$\lim_{n\to\infty} \delta_n = \delta, \ \delta_n\in\mathcal{D} \tag{II-17c}$$

That is, S is the impulse response; such a sequence converging to δ exists by theorem II-2. Further, by theorem II-2, L can be extended to as much of $\mathcal{D}'$ as possible by taking distributional limits in (II-17a). In theorem II-4, condition a) limits the result to linear systems, condition b) is a continuity condition, which seems to be frequently overlooked in engineering theory, and condition c) limits the result to time-invariant systems.[40] In other words, every linear, continuous, time-invariant system can be characterized by its impulse response.

Example II-15:

A linear, time-invariant system has an output given by

$$L(\delta_n) = \frac{10}{1+e^{-nt}}$$

for inputs $\delta_n \in \mathfrak{D}$, $\lim_{n\to\infty} \delta_n = \delta$. But

$$\lim_{n\to\infty} L(\delta_n) = 10u(t)$$

and thus L can not be described by a convolution operator, since

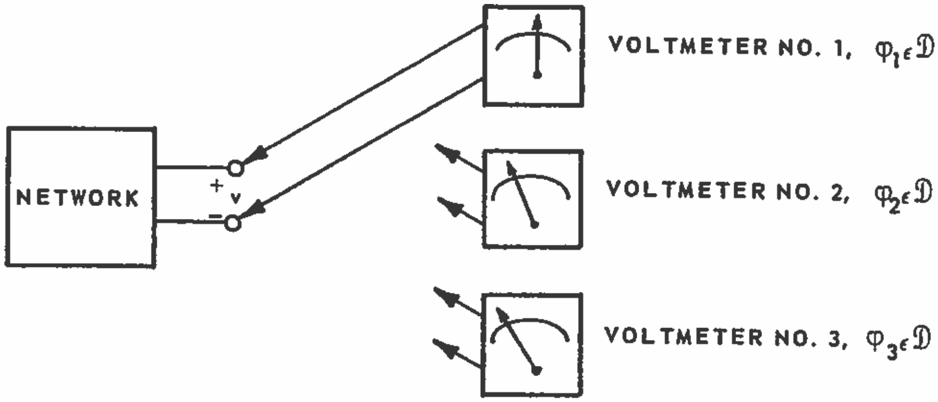
$$\delta_n * \lim_{n\to\infty} L(\delta_n) \neq \frac{10}{1+e^{-nt}}$$

The term on the left of this must be in $\mathfrak{D}'_+$, δ_n and u both being in $\mathfrak{D}'_+$; the right hand term clearly isn't in $\mathfrak{D}'_+$, since its support is not bounded on the left.

8. Naive Physical Interpretation

Before continuing further we present a concise possible physical interpretation of the meaning of the distributional representation of physical variables.

Consider a physical variable which we can mathematically represent by a function, v, which we assume at this point to be in $\mathfrak{D}$. One, but certainly not the only, way of determining v is by the totality of responses of all possible measuring instruments excited by v. In fact it is sufficient to excite only those instruments, each characterized by a separate testing function φ, as a glance at paragraph 3 will show, whose responses are of the form $\int_{-\infty}^{\infty} v(t)\varphi(t)dt$; perhaps such an instrument is best called a weighted integrator. This is illustrated in Fig. II-6, where the variable v can be considered as the voltage at the terminals of a network.



nth METER READING, $< v, \varphi_n > = \int_{-\infty}^{\infty} v(t)\, \varphi_n(t) dt$ WITH $v \epsilon \mathfrak{D} \subset \mathfrak{D}'$

FIG. II-6. EXPERIMENTAL MEANING OF DISTRIBUTIONS.

Theoretically such measurements would completely characterize v, although practically it would be somewhat difficult to "graph" v in the customary sense from such data. It is from the theoretical advantages gained, in terms of formal operations, that the theory of distributions stems. The picture is somewhat analogous to the frequency domain descriptions of great use in engineering, where manipulations are generally easy, but recovery of the time functions is sometimes difficult.

NOTES TO II

1. For a treatment of the Stieltjes integral see Widder, [WI 2, p. 126]. Using the Stieltjes integral a representation of δ yielding the sifting property is

$$\int_{-\infty}^{\infty} \varphi(t)du(t) = \varphi(0)$$

where u is the unit step function, that is u(t) is zero for $t < 0$ and unity for $t > 0$.

2. For a thorough preview of the theory of distributions the introduction of Schwartz's book is of interest, [SC 1, pp. 3-11].

3. By the closure of a set is meant the points in the set as well as limit points of the set; thus the closure of $-1 < t \leq 0$ is the set $-1 \leq t \leq 0$. For example the support of the unit step function is the closed set $0 \leq t \leq \infty$.

4. The word "compact" in the definition of $\mathfrak{D}$ is equivalent to the word "bounded" in the single variable case which we are treating. The words "testing functions" signify an analogy with the Fourier series representation of periodic functions, where sines and cosines serve to test for the coefficients of the series. Another example of a testing function is

$$\varphi(t) = e^{-1/(t-a)^2} e^{-1/(t-b)^2} u(t-a)u(b-t), \; b > a$$

which is depicted in Fig. N-II-1.

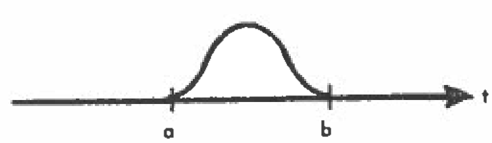


FIG. N-II-1. TESTING FUNCTION.

The requirement that testing functions be infinitely differentiable is closely connected to the consideration of derivatives of impulses. To allow treatment of $\delta^{(k)}$, which mathematically can be thought to represent a physical k-pole electric moment, [ST 1, p. 177], we must take the testing functions to be infinitely differentiable.

5. The introduction of a notion of convergence makes $\mathfrak{D}$ into a topological space, [PA 2, p. 34], in which notions such as continuity can be rigorously defined. The real usefulness comes into play as a consequence of the fact that these notions are naturally extended to the topologically dual space of distributions. The concept of a topological space is fundamental to much of modern mathematics. However, a thorogh understanding of topology is not necessary to grasp the meaning and applications of distributions.

6. By a functional is essentially meant a mapping of a set of functions, in this case $\mathfrak{D}$, into the complex plane, see for instance [KO 1, p. 62]. For a given testing function φ and a fixed linear functional T, this mapping is a fixed complex number, in this text denoted by $< T, \varphi >$. Suitable examples of this mapping are given by (II-2a), (II-4), (II-8b) and (II-9a)

7. We adhere to the standard set theory symbols, thus $\varphi \epsilon \mathfrak{D}$ means φ "is contained in the set" $\mathfrak{D}$

   As stated in note 6, $< T, \varphi >$ is a fixed number when T and φ are fixed, but when T is fixed, $< T, \varphi >$ is a complex number which varies with φ. The notation strives to emphasize that we are dealing with a scalar product of T with φ. A similar attempt is clear in the use of the notation $T \cdot \varphi$, from vector analysis. Both notations are used by Schwartz and they have the same meaning. Other notations in use are $T(\varphi)$, [SC 1], and $T < \varphi >$, [ER 1]. For our purposes the notation $< T, \varphi >$ seems to be the most symbolic since it correctly suggests that the space of distributions is dual to the space of testing functions. This notation is favored by Schwartz in his lectures on mathematical physics, [SC 7, p. 4] and follows that of Friedman, [FR 1, p. 6].

8. Vector spaces and their duals are well treated in Halmos, [HA 2, pp. 3 & 20].

9. A topological dual space is, by definition, the subset of linear functionals which are continuous in terms of the notion of convergence defined by the topology of the original space. That is, if $\mathfrak{D}$ is any topological vector space, then the elements of the topological dual space $\mathfrak{D}'$ satisfy the conditions 1) and 2) below definition II-1.

There are of course discontinuous linear functionals on $\mathfrak{D}$, but these seem to have no importance, [SC 7, p. 4].

10. By a locally integrable function is meant a function g(t) for which $\int_a^b |g(t)|dt$ exists and is finite for every <u>finite</u> a and b. $g(t) = t$, $g(t) = 1/\sqrt{t}$ and $g(t) = \exp(t)$ are locally integrable but $g(t) = 1/t$ is not. Whenever it is necessary to specify the type of integral used, we will assume it to be taken in the Lebesgue sense, [BU 1], since the mathematical literature is based on this. In most cases assuming the integral to be in the normal (that is, Riemann) sense will cause no difficulty. We see that (II-2) requires two functions g_1 and g_2 to be identified if

$$\int_{-\infty}^{\infty} |g_1(t)-g_2(t)|dt = 0$$

Thus

$$u_1(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \text{and} \quad u_2(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

are identified with u(t); in other words the value of the unit step function at the origin doesn't matter.

11. g(t) is square integrable if it is "measurable" (that is, essentially can be integrated) and

$$\int_{-\infty}^{\infty} |g(t)|^2 dt < \infty$$

Over any bounded interval g satisfies

$$\left[\int_a^b |g(t)|dt\right]^2 \leq (b-a)^2 \int_a^b |g(t)|^2 dt < \infty$$

by Holder's inequality, [BU 1, p. 65]. Thus g is locally integrable.

12. To see (II-5b) we have

$$< g(at+b),\ \varphi(t)> = \int_{-\infty}^{\infty} g(at+b)\varphi(t)dt;\ x = at+b$$

$$= \int_{-\infty}^{\infty} g(x)\varphi(\frac{x-b}{a})\frac{dx}{|a|}$$

$$= < \frac{1}{|a|}g(x),\ \varphi(\frac{x-b}{a}) >$$

but x, being a dummy variable, can now be replaced by t.

13. We have

$$< \delta(at+b),\ \varphi(t) > = < \frac{1}{|a|}\delta(t),\ \varphi(\frac{t-b}{a}) > = \frac{1}{|a|}\varphi(-b/a)$$

$$= < \frac{1}{|a|}\delta(t),\ \varphi(t-\frac{b}{a}) >$$

$$= < \frac{1}{|a|}\delta(t+\frac{b}{a}),\ \varphi(t) >$$

14. This is a consequence of the fact that in the theory of distributions the integral in

$$< g,\ \varphi > = \int_{-\infty}^{\infty} g\varphi dt$$

is taken in the Lebesgue sense where two functions are in fact identified if they are equal "almost everywhere," the last two words having a precise mathematical meaning, [BU 1, p. 37].

15. We have for functions

$$< h',\ \varphi > = \int_{-\infty}^{\infty} h'(t)\varphi(t)dt = - \int_{-\infty}^{\infty} h(t)\varphi'(t)dt$$

$$= - < h,\ \varphi' >$$

where in the integration by parts we used the fact that φ vanishes at $\pm\infty$. Notice also the need of infinite differentiability of $\varphi\epsilon\mathcal{D}$, if we wish h to have derivatives of all orders, as a distribution.

16. That is, (II-7) agrees with the functional derivative by the very manner in which it was derived by generalizing the result of note 14.

Maybe one of the most startling points of this theory is apparent here. We are evaluating derivatives (distributionally) of discontinuous functions! This is one of the points where the conventional theory fails; to speak of derivatives of u, call them δ, and more to speak of δ', δ", etc., in the customary sense is certainly to go too far.

17. To obtain the scale change in δ' we have

$$< \delta'(at+b), \varphi(t) > = < \frac{1}{|a|}\delta'(t), \varphi(\frac{t-b}{a}) >$$

$$= - < \frac{1}{|a|}\delta(t), d\varphi(\frac{t-b}{a})/dt >$$

$$= - \frac{1}{|a|}\cdot\frac{1}{a}\cdot\varphi'(-\frac{b}{a})$$

$$= < \frac{1}{a|a|}\delta'(t+\frac{b}{a}), \varphi(t) >$$

18. See [SC 1, p. 37] or [SC 7, p. 14]. Thus the function of Fig. N-II-2(a), where

$$g = e^{t}u(-t)+3e^{-t}u(t)$$

has

$$g' = e^{t}u(-t)-3e^{-t}u(t)+2\delta(t)$$

as shown in Fig. N-II-2(b).

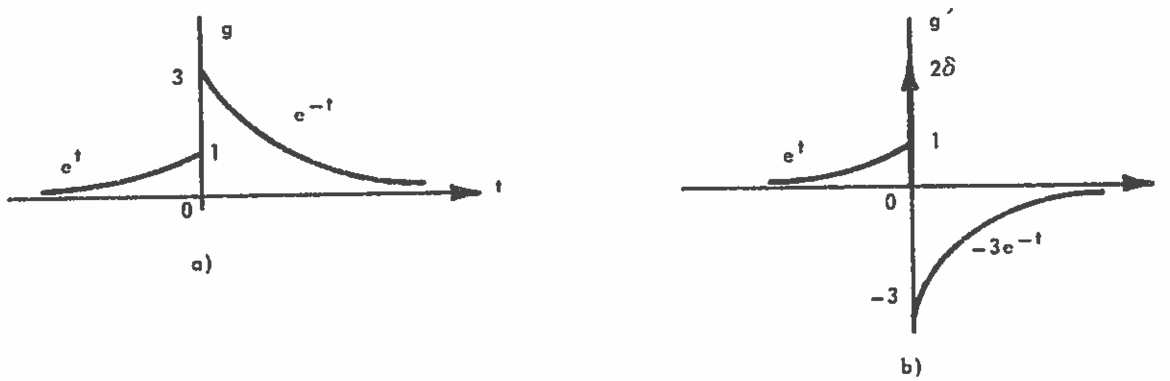


FIG. N-II-2. DISTRIBUTIONAL DERIVATIVES.

19. PF stands for "pseudo-function" or what is equivalent in the theory of distributions, "la partie finie de Hadamard," see [SC 1, p. 38]. In the case of 1/t Hadamard's finite part reduces

to the principal value of Cauchy, and thus, in this case PF is sometimes written VP, see [SC 7, p. 16]; we use PF since VP is not valid for $t^{-\ell}$, $\ell > 1$. $PF(\frac{1}{t})$ defines a distribution since the conditions of definition II-1 are seen to be satisfied. Linearity is obvious and only continuity need be checked. Since φ_n has compact support we can choose the interval (-A, A) to contain the support of φ_n after writing

$$\lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon} \frac{\varphi(t)}{t}dt+\int_{\epsilon}^{\infty} \frac{\varphi(t)}{t}dt\right] = PF\int_{-\infty}^{\infty} \frac{\varphi(t)}{t}dt$$

we then have

$$\lim_{n\to\infty} < PF(\frac{1}{t}),\ \varphi_n(t) > = \lim_{n\to\infty} PF\int_{-A}^{A} \frac{\varphi_n(t)}{t}dt$$

But

$$PF\int_{-A}^{A} \frac{\varphi_n(t)}{t}dt = \varphi_n(0)PF\int_{-A}^{A} \frac{1}{t}dt+PF\int_{-A}^{A} \frac{\varphi_n(t)-\varphi_n(0)}{t}dt$$

In this the first term on the right is zero since 1/t is odd while the integrand in the second term is bounded in magnitude by the maximum value of φ_n'. Taking absolute values gives

$$\left|PF\int_{-\infty}^{\infty} \frac{\varphi_n(t)}{t}dt\right| \leq 2A[\max|\varphi_n'|]$$

By the definition of convergence in $\mathfrak{D}$ this term tends to zero as $n\to\infty$. Thus

$$\lim_{n\to\infty} < PF(\frac{1}{t}),\ \varphi_n(t) > = 0$$

when φ_n converges to zero, which proves the continuity.

20. $PF(1/t^{\ell})$ is also given by a Hadamard's finite part, see [SC 1, p. 41].

21. Precisely

$$< (\ln|t|)';\ \varphi(t) > = - < \ln|t|,\ \varphi'(t) >$$
$$= -\int_{-\infty}^{\infty} \ln|t|\varphi'(t)dt$$

But

$$-\int_{0}^{\infty} \ln t\ \varphi'(t)dt = -\lim_{\epsilon\to 0}\int_{\epsilon}^{\infty} \ln t\ \varphi'(t)dt$$
$$= \lim_{\epsilon\to 0}\left[(-\ln t\ \varphi(t)|_{\epsilon}^{\infty}+\int_{\epsilon}^{\infty}\frac{\varphi(t)}{t}dt\right]$$

$$= \lim_{\epsilon \to 0} [\ln \epsilon\varphi(0) + \int_{\epsilon}^{\infty} \frac{\varphi(t)}{t} dt]$$

since $\ln \epsilon[\varphi(\epsilon)-\varphi(0)]$ tends to zero as $\epsilon \to 0$ because $\epsilon \ln \epsilon[\frac{\varphi(\epsilon)-\varphi(0)}{\epsilon}]$ tends to zero since $\epsilon \ln \epsilon$ does, $\frac{\varphi(\epsilon)-\varphi(0)}{\epsilon}$ remaining finite being an approximation to the derivative at zero. Thus

$$< (\ln|t|)', \varphi(t) > = \lim_{\epsilon \to 0} \{\ln \epsilon\varphi(0) - \ln \epsilon\varphi(0) + \int_{\epsilon}^{\infty} \frac{\varphi(t)}{t} dt + \int_{-\infty}^{-\epsilon} \frac{\varphi(t)}{t} dt\}$$

$$= PF \int_{-\infty}^{\infty} \frac{\varphi(t)}{t} dt$$

$$= < PF(\frac{1}{t}), \varphi(t) >$$

22. To see that every distribution has a primitive we merely reverse (II-7) for $k = 1$. However, the integral of φ may not have compact support, so we first restrict to those $\varphi = \mathcal{X}$ which are exact differentials, and hence have compact support, since $\mathcal{X} = \psi'$ and $\int_{-\infty}^{t} \mathcal{X}(\tau)d\tau = \psi$ are both in $\mathcal{D}$,

$$< \int^{t} T(\tau)d\tau, \mathcal{X}(t) > = < -T(t), \psi(t) >$$

Then for any $\varphi \epsilon \mathcal{D}$ we choose a fixed $\varphi_o \epsilon \mathcal{D}$ for which $\int_{-\infty}^{\infty} \varphi_o(t)dt = 1$ and find $\lambda = \int_{-\infty}^{\infty} \varphi(t)dt$ and an exact differential $\mathcal{X}$ such that

$$\varphi(t) = \lambda\varphi_o(t) + \mathcal{X}(t)$$

Then

$$< \int^{t} T(\tau)d\tau, \varphi(t) > = < \lambda \int^{t} T(\tau)d\tau, \varphi_o(t) > + < -T(t), \int_{-\infty}^{t} \mathcal{X}(\tau)d\tau >$$

Thus, by assigning $\int^{t} T(\tau)d\tau$ at φ_o, $\int^{t} T(\tau)d\tau$ is determined for for all $\varphi \epsilon \mathcal{D}$. Letting P_1 and P_2 be two different assignments at φ_o, then

$$< P_1(t)-P_2(t), \varphi(t) > = \lambda < P_1-P_2, \varphi_o > =$$
$$= \lambda C = \int_{-\infty}^{\infty} C \cdot \varphi(t)dt$$

where C is the constant value of $< P_1-P_2, \varphi_o >$. But by (II-2a) this means that $P_1 - P_2 = C$, or two primitives differ by a constant. The above follows [SC 1, p. 52].

23. The proof of this limit is as follows:

For Fig. II-3(a)

$$2\delta(t) = \lim_{n\to\infty} [n\{u(t-\frac{1}{n})-u(t+\frac{1}{n})\}]$$

since

$$< n\{u(t-\frac{1}{n})-u(t+\frac{1}{n})\}-2\delta(t),\ \varphi(t) > = n\int_{-1/n}^{1/n} \varphi(t)dt-2\varphi(0)$$

which is zero in the limit. Similarly, for Fig. II-2(b). Observe that the usual argument of taking δ as the limit (in the usual sense) of a sequence of pulses shaped as in Fig. II-3 as $n\to\infty$ is meaningless (i.e. the usual limit does not exist). An approach like in [AS 1, p. 22] is to be avoided.

24. See [KA 1, p. 393].

25. See [SC 1, p. 117]. Note that $g = h = 1/\sqrt{t}$ are functions but $g\cdot h = 1/t$ is not. König, [KO 3] has defined a multiplication for any two distributions, but one generally obtains entities which are not distributions, such as $\delta\cdot u \neq u\cdot\delta$.

26. We have

$$< \alpha(t)\delta(t),\ \varphi(t) > = < \delta(t),\ \alpha(t)\varphi(t) > = \alpha(0)\varphi(0)$$
$$= < \alpha(0)\delta(t),\ \varphi(t) >.$$

Thus, for example $t\delta(t) = 0$, $e^t\delta(t) = \delta(t)$.

27. We have, by (II-7),

$$< \alpha(t)\delta'(t),\ \varphi(t) > = < \delta'(t),\ \alpha(t)\varphi(t) > = (-1)[\alpha'(0)\varphi(0)+\alpha(0)\varphi'(0)]$$
$$= < \delta(t),\ -\alpha'(0)\varphi(t) > + < \delta'(t),\ \alpha(0)\varphi(t) >$$
$$= < \alpha(0)\delta'(t)-\alpha'(0)\delta(t),\ \varphi(t) >$$

Thus for example $e^{at}\delta'(t) = \delta'(t)-a\delta(t)$.

28. Here, [SC 1, p. 120],

$$< (\alpha T)',\ \varphi > = (-1) < \alpha T,\ \varphi' > = < T,\ -\alpha\varphi' >$$
$$< \alpha' T,\ \varphi > = < T,\ \alpha'\varphi >$$
$$< \alpha T',\ \varphi > = < T',\ \alpha\varphi > = < T,\ -\alpha'\varphi-\alpha\varphi' >$$
$$= < -\alpha' T+(\alpha T)',\ \varphi >$$

where the last follows by subtracting the first two results.

29. This result as well as the non-associative example are stated in [SC 1, p. 121].

30. For the distributional expression of the functional convolution we have

$$< g*h(t), \varphi(t) > = \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau]\varphi(t)dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y-x)\varphi(y)dxdy; \; x = \tau, \; y = t$$

$$= \int_{-\infty}^{\infty} g(t)[\int_{-\infty}^{\infty} h(\tau)\varphi(t+\tau)d\tau]dt; \; x = t, \; y-x = \tau$$

which is the desired expression. The integral interchanges are fully justified by Fubini's theorem, [BU 1, p. 63].

31. To see that $S*T = T*S$ we first look at the case when $\varphi(t+\tau) = \theta(t)\xi(\tau)$. Then

$$< S*T, \varphi > = < S(t), \theta(t) < T(\tau), \xi(\tau) > > = < S(t), \theta(t) > \cdot < T(\tau), \xi(\tau) >$$
$$= < T*S, \varphi >$$

by linearity. But any $\varphi(t+\tau)$ can be written as

$$\varphi(t+\tau) = \lim_{j\to\infty} \sum_k \theta_{jk}(t)\xi_{jk}(\tau)$$

for suitably chosen θ_{jk} and ξ_{jk}, [SC 1, p. 108], and thus

$$< T*S, \varphi > = \lim_{j\to\infty} \sum_k (< S, \theta_{jk}> < T, \xi_{jk} >) = < S*T, \varphi >$$

In Schwartz, [SC 2, p. 10], this is obtained by defining the convolution by the use of the tensor product and (II-13) is derived as a consequence.

32. To see that $T*\delta^{(k)} = T^{(k)}$ form

$$< T*\delta^{(k)}(t), \varphi(t) > = < T(t), < \delta^{(k)}(\tau), \varphi(t+\tau) > >$$
$$= < T(t), (-1)^k\varphi^{(k)}(t) > = < T^{(k)}(t), \varphi(t) >$$

33. To see that $(PF\frac{1}{t})*C = 0$ form

$$< PF(\tfrac{1}{t}), < C, \varphi(t+\tau)> > = < PF(\tfrac{1}{t}), \int_{-\infty}^{\infty} \varphi(t+\tau)d\tau >$$

$$= \lim_{\epsilon\to 0} [\int_{-\infty}^{-\epsilon} \frac{1}{t}(\int_{-\infty}^{\infty} \varphi(t+\tau)d\tau)dt + \int_{\epsilon}^{\infty} \frac{1}{t}(\int_{-\infty}^{\infty} \varphi(t+\tau)d\tau)dt]$$

$$= \lim_{\epsilon\to 0} \left[\int_{-\infty}^{\epsilon} \frac{1}{x}\left(\int_{-\infty}^{\infty} \varphi(-x+\tau)d\tau\right)dx + \int_{\epsilon}^{\infty} \frac{1}{t}\left(\int_{-\infty}^{\infty} \varphi(t+\tau)d\tau\right)dt \right]$$

$$= \lim_{\epsilon\to 0} \left[-\int_{\epsilon}^{\infty} \frac{1}{t}\left(\int_{-\infty}^{\infty} \varphi(y)dy\right)dt + \int_{\epsilon}^{\infty} \frac{1}{t}\left(\int_{-\infty}^{\infty} \varphi(y)dy\right)dt \right]$$

$$= 0$$

34. This example is due to Schwartz, [SC 2, p. 27].

35. These results are stated in Schwartz, [SC 2, p. 29]. For the definition of an algebra see Hille and Philips, [HI 1, p. 19].

36. See [SC 1, p. 75]. One can pick a sequence of infinitely differentiable functions, g_n, perhaps not of compact support, converging to $T \epsilon \mathcal{D}'$ by forming

$$g_n(t) = T * \delta_n(t)$$

where $\delta_n \epsilon \mathcal{D}$ but $\lim_{n\to\infty} \delta_n = \delta$, [SC 2, p. 22].

37. Of course ideal differentiators don't exist and thus one can't really tell if measured functions are infinitely differentiable or not. It seems that this may possibly be a postulate of physical systems which allows a simple theory yielding all measurable results.

38. See [SC 1, p. 82].

39. See [SC 1, pp. 18-20, 53].

40. The generalization of this, valid for time-varying systems, is given by Schwartz, [SC 12, p. 223].

III. FOURIER TRANSFORMS - GOOD FUNCTIONS

Summary: In this section the theory of Fourier transforms of a certain class of functions is given. We are not concerned here with the most general Fourier - transformable functions, but with those functions for which the theory takes its simplest form. The results obtained will be conveniently extended in the subsequent sections to more general classes of functions and distributions.

We define in paragraph 1 the "good" functions which are infinitely differentiable, continuous (as well as all their derivatives) functions that vanish at infinity faster than the reciprocal of any polynomial. The good functions constitute a space, $\mathcal{S}$, whose topological dual, $\mathcal{S}'$, consists of the Fourier - transformable distributions. For transforming series the notion of convergence is given.

The Fourier transformation is defined in paragraph 2 in the standard form which appears in the engineering literature,

$\mathcal{F}[\psi(t)] = \int_{-\infty}^{\infty} \psi(t)e^{-j\omega t}dt$. A correspoding expression is given for

the inverse transform, $\overline{\mathcal{F}}[\psi(\omega)]$. Of key interest is the fact that $\mathcal{F}[\psi]$ is a good function if ψ is a good function.

Paragraph 3 gives two properties of the transformation, which justify the existence of the section. The first of these is one form of a Parseval's theorem, which allows the shifting of the Fourier transform of distributions onto the Fourier transform of good functions. This result, to be used extensively in this text, is $< \mathcal{F}[\psi_1], \psi_2 > = < \psi_1, \mathcal{F}[\psi_2] >$. The second property consists of the transformation of time differentiation into frequency multiplication, $\mathcal{F}[\psi^{(k)}] = (j\omega)^k \mathcal{F}[\psi]$, and is needed to obtain the corresponding result for Laplace transforms.

1. Good Function

For distributions the most general Laplace transform is obtained by extending the distributional Fourier transform. This latter is in turn an extension of the familiar functional Fourier transform. However, in place of considering arbitrary functions it is sufficient to work with the limited class of "good" functions. This allows us to use only simple arguments from the very beginning; the usual tricky manipulations necessary to work with the classical theory need not be considered.[1]

The set of "infinitely differentiable functions of rapid decrease," which are merely called "good functions," will be denoted by $\mathcal{S}$ and is defined as follows.[2]

Definition III-1:

ψ is a good function, $\psi \epsilon \mathcal{S}$, if

(a) ψ is (continuously) infinitely differentiable and

(b) $\lim_{|t| \to \infty} |t^m \psi^{(k)}(t)| = 0$ for all integer k, $m \geq 0$.

Thus the good functions vanish at infinity as well as all their derivatives, faster than the reciprocal of any polynomial. However, their support need not be compact, as the following example shows.

Example III-1:

α) $\psi(t) = t^m \cdot \exp[-t^2] \epsilon \mathcal{S}$ for any integer $m > 0$.

β) Since condition (a) is violated,

$$\eta(t) = \exp[-|t|] \notin \mathcal{S}.$$

Since condition (b) is violated,

$$\eta(t) = \exp[t] \notin \mathcal{S}.$$

Clearly no rational, non-zero, function of t can qualify as a good function.

Again a matter of theoretical importance for obtaining the Fourier transformable distributions, which form the topological dual space of $\mathcal{S}$, is the notion of convergence in $\mathcal{S}$. Precisely we say that a sequence of $\psi_n \epsilon \mathcal{S}$ converges to zero if the sequence of $t^m \psi_n^{(k)}(t)$ converges uniformly to zero for all integer k, $m \geq 0$.[3]

2. Fourier Transform Definition

With the definition of the good functions the basic Fourier transform pair can be introduced.

Definition III-2:

(a) If $\psi(t) \epsilon \mathcal{S}$, then by definition,

$$\mathcal{F}_\omega[\psi(t)] = \Psi(\omega) = \int_{-\infty}^{\infty} \psi(t) e^{-j\omega t} dt \tag{III-1a}$$

(b) If $\Psi(\omega) \epsilon \mathcal{S}$, then by definition

$$\overline{\mathcal{F}}_t[\Psi(\omega)] = \psi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(\omega) e^{j\omega t} d\omega \tag{III-1b}$$

In this definition $\mathcal{F}$ is called the direct Fourier transform, or just the "Fourier transform," and $\overline{\mathcal{F}}$ the inverse Fourier transform.[4]

Here $j = \sqrt{-1}$ and ω is a real variable, physically to be interpreted as radian frequency, $\omega = 2\pi f$. For later convenience the variables ω and t upon which $\mathfrak{F}$ and $\overline{\mathfrak{F}}$ depend are indicated as subscripts. These will usually be omitted but are sometimes necessary for clarity, as in convolution expressions (see, for instance, notes III-7 and IV-15).

As a consequence of $\psi \epsilon \mathcal{S}$, $\mathfrak{F}[\psi]$ exists for every real ω and $\mathfrak{F}_\omega[\psi] \epsilon \mathcal{S}$, that is the Fourier transform of a good function is again a good function. This is in fact the real justification for considering definition III-1.[5] The operators $\mathfrak{F}$ and $\overline{\mathfrak{F}}$ are actually inverse to each other, since[6]

$$\overline{\mathfrak{F}}[\mathfrak{F}[\psi]] = \psi \tag{III-2a}$$

$$\mathfrak{F}[\overline{\mathfrak{F}}[\Psi]] = \Psi$$

whenever ψ and Ψ are good functions. We also note that from (II-2a) we can write (III-1a) in the form

$$\mathfrak{F}_\omega[\psi(t)] = \langle \psi(t), \exp[-j\omega t] \rangle$$

Example III-2:[7]

$$\psi(t) = \exp[-t^2] \text{ has } \Psi(\omega) = \frac{1}{\pi} \exp[-(\tfrac{\omega}{2\pi})^2]$$

3. Two Basic Properties

Although all of the standard properties could now be given, there are essentially only two required for further generalizations. The most important property is one form of Parseval's theorem.

Theorem III-1:

If $\psi_1, \psi_2 \epsilon \mathcal{S}$ then

$$\int_{-\infty}^{\infty} \Psi_1(x)\psi_2(x)dx = \int_{-\infty}^{\infty} \Psi_2(y)\psi_1(y)dy \tag{III-3a}$$

Here, by (III-1a), Ψ_1 and Ψ_2 are the Fourier transforms of ψ_1 and ψ_2. The result is seen to be true by inserting the integral for Ψ_2 into the left integral of (III-3a) and converting to a double integral.[8] By (II-2a) we can rewrite (III-3a) in the more useful form

$$\langle \mathfrak{F}[\psi_1], \psi_2 \rangle = \langle \psi_1, \mathfrak{F}[\psi_2] \rangle \tag{III-3b}$$

That is, if we treat ψ_1 and ψ_2 as "sampling" functions, then sampling $\mathfrak{F}[\psi_1]$ with ψ_2 is equivalent to sampling ψ_1 with $\mathfrak{F}[\psi_2]$, (or $\mathfrak{F}[\psi_2]$ with ψ_1). This notion will then allow us to shift $\mathfrak{F}$ off of distributions onto good functions.

The second, but somewhat less vital property concerns differentiation.

Theorem III-2:

If ψ, $\Psi \in \mathcal{S}$, then

$$\mathfrak{F}[\psi^{(k)}] = (j\omega)^k \mathfrak{F}[\psi];\ k = 0, 1, 2, \ldots \qquad \text{(III-4a)}$$

$$\overline{\mathfrak{F}}[\Psi^{(k)}] = (-jt)^k \overline{\mathfrak{F}}[\Psi];\ k = 0, 1, 2, \ldots \qquad \text{(III-4b)}$$

This is seen by integrating in $\mathfrak{F}[\psi']$ by parts and applying induction on k to the result.[9] As is well recognized, this is the fundamental property which makes the Fourier transform useful in the analysis of lumped physical systems; that is, the operation of differentiation is converted to frequency multiplication. Eqs. (III-4) will be needed to obtain this result for the Laplace transform. However, they also give a method for finding transforms of certain functions, as the following example shows.

Example III-3:

$$\mathfrak{F}[-2t\ \exp(-t^2)] = \frac{2j\omega}{\pi}\exp[-(\frac{\omega}{2\pi})^2]$$

since this results by applying (III-4a) to example III-2.

NOTES TO III

1. For instance the "limit in the mean" considerations necessary to work with general square integrable functions need not bother us here, [BO 1, p. 108]. Of course such concepts do not disappear completely, but show up later when actually evaluating distributional transforms. Essentially, then one advantage of the distributional theory is to remove such complications from the general theory; classical difficulties still remain in many numerical calculations.
2. The definition is that of Schwartz, [SC 2, p. 89], with the name "good" function coming from Lighthill, [LI 1, p. 15]. These ψ are analogous to the $\varphi \epsilon \mathfrak{D}$ and will serve to test the general Fourier transform. Such ψ may be complex, but as pointed out, need not have compact support. The essential reason for requiring condition b) is that the Fourier transform of $\psi \epsilon \mathcal{S}$ is also a function in $\mathcal{S}$.
3. See [SC 2, p. 90] or [SC 10, p. 12].
4. Other possible forms for the transform pair are commonly met. For instance, [WI 3, p. 3]

$$\mathfrak{F}[\psi] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-jux} dx$$

$$\overline{\mathfrak{F}}[\psi] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(u) e^{jux} du$$

or [SC 2, p. 87]

$$\mathfrak{F}[\psi] = \int_{-\infty}^{\infty} \psi(t) e^{-j2\pi ft} dt$$

$$\overline{\mathfrak{F}}[\psi] = \int_{-\infty}^{\infty} \psi(f) e^{j2\pi ft} dt$$

These have the advantage of symmetry but are quite inconvenient for obtaining inverse Laplace transforms in the distributional case.
5. We should properly index $\mathcal{S}$ with t or ω but omit this. If $\psi \epsilon \mathcal{S}$ then $\mathfrak{F}[\psi]$ clearly exists, for all real ω, since

$$|\mathcal{F}[\psi]| \leq \int_{-\infty}^{\infty} |\psi(t)|dt < \infty$$

The last inequality holds since $|\psi|$ has a finite integral over every finite interval, ψ being continuous, and the contributions near infinity are small, since $|\psi|$ is bounded by every integral power of $1/t$ for large t. In other words, if $\psi \epsilon \mathcal{S}$, then ψ is summable over the whole line, [BU 1, p. 65].

To see that $\mathcal{F}[\psi] \epsilon \mathcal{S}$ if $\psi \epsilon \mathcal{S}$, we have, [SC 2, p. 105]:

a) $$d\Psi/d\omega = \int_{-\infty}^{\infty} -jt\psi(t)e^{-j\omega t}dt$$

which also exists for all ω, since $t\psi \epsilon \mathcal{S}$. Differentiating any number of times then shows that $\Psi(\omega)$ is infinitely differentiable.

b) Integrating $\mathcal{F}[d\psi/dt]$ by parts gives, since $\psi(\pm\infty) = 0$,

$$j\omega\Psi(\omega) = \int_{-\infty}^{\infty} [d\psi/dt]e^{-j\omega t}dt$$

Since the right is finite for all ω, so is the left. Repeating the process multiplies the left by $j\omega$ again, but $\omega^2 \Psi(\omega)$ must again be finite for all ω and thus $|\omega \Psi(\omega)| \to 0$ as $|\omega| \to \infty$. Q.E.D.

6. This is proven in [BO 1, p. 10]. We comment that even though j is replaced by -j in the exponent, $2\pi \overline{\mathcal{F}}[g]$ need not be the complex conjugate of $\mathcal{F}[g]$ (with ω and t interchanged). This would, however, be the case for a real good function, g.

7. See [SC 10, p. 7] where it is shown that

$$\mathcal{F}_{2\pi f}[\exp(-\pi t^2)] = \exp(-\pi f^2)$$

The result of the example follows by a simple change of variable using

$$\mathcal{F}_{\omega}[\psi(at)] = \frac{1}{|a|}\mathcal{F}_{\frac{\omega}{a}}[\psi(t)]$$

which immediately follows from (III-1a).

8. In detail, [BO 1, p. 2],

$$\int_{-\infty}^{\infty} \Psi_1(x)\psi_2(x)dx = \int_{-\infty}^{\infty}\psi_2(x)[\int_{-\infty}^{\infty}\psi_1(y)e^{-jxy}dy]dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_1(y)\psi_2(x)e^{-jxy}dxdy$$

$$= \int_{-\infty}^{\infty} \psi_1(y)[\int_{-\infty}^{\infty} \psi_2(x)e^{-jxy}dx]dy$$

$$= \int_{-\infty}^{\infty} \Psi_2(y)\psi_1(y)dy$$

All the integral interchanges are justified by Fubini's theorem, [BU 1, p. 63], since all the integrals exist as a consequence of ψ being absolutely integrable if $\psi \epsilon \mathcal{S}$. This result combines a transform with an original in contrast to the convolution which combines two originals or two transforms. Parseval's name is assigned in [SC 2, p. 87].

9. The first step is seen in note 5b). The result for $\mathfrak{F}$ follows by obvious replacements in that for $\mathcal{F}$.

IV. FOURIER TRANSFORMS - TEMPERED DISTRIBUTIONS

Summary: This section consists of the definition and properties of the Fourier transform of distributions. We cannot assure that the Fourier transform of an arbitrary distribution exists (e.g. those with the wrong behavior at infinity, such as [exp(t)u(t), $\exp(t^2)$ do not have Fourier transforms); only those distributions which are in a space $\mathcal{S}'$ are Fourier transformable. This space $\mathcal{S}'$ is the topological dual of the space $\mathcal{S}$ of good functions treated in section 3. The Fourier-transformable distributions are here designated "tempered". The tempered distributions are distributions of slow increase at infinity. In paragraph 2 a theorem (IV-1) is given which shows that a tempered distribution is the derivative of a function which behaves as a polynomial at infinity. Of course, the definition of tempered distributions is that of linear (continuous) functionals on $\mathcal{S}$.

In paragraph 3 the Fourier transform is defined for a tempered distribution. The scalar product $< \mathcal{F}_{\omega}[T(t)], \psi(\omega) > = < T(\omega), \mathcal{F}_{\omega}[\psi(t)] >$ defines the direct Fourier transform; an analogous expression exists for the inverse transformation. These definitions are shown to coincide with those usually given for functions.

Paragraph 4 consists of the multiplication and convolution in $\mathcal{S}'$. For this we define the "fairly good functions," which are slowly increasing, infinitely differentiable functions. These functions constitute a space which is denoted by $\mathcal{O}_M$; the Fourier transforms of functions in $\mathcal{O}_M$ constitute a space of distributions denoted by $\mathcal{O}'_c$. From the definition of fairly good functions some results can be drawn:

1) A tempered distribution can always be multiplied by a fairly good function and the Fourier transform of this product is the convolution of the individual Fourier transforms, eq. (IV-10).

2) We can always convolve any distribution in $\mathcal{S}'$ with the Fourier transform of a fairly good function. The Fourier transform of this convolution is then the product of the Fourier transforms of the separate terms being convolved, eq. (IV-11).

The results on multiplication and convolution are valid in other contexts, which are sometimes of interest in physical problems. The supplementary result at the end of the section is concerned with one of these results, which in fact can be deleted on a first reading of the text; it is, however, important because it refers to energy relations and establishes a natural link with the Hilbert transform.

1. Introduction

In this section the definition of the Fourier transform of some distributions is presented. The restriction to a certain class of distributions is so important that for this purpose, a new space $\mathcal{S}'$, which consists of the Fourier-transformable distributions is introduced. These are called tempered distributions and a theorem will be given which establishes a criterion to determine whether a given distribution belongs to $\mathcal{S}'$ or not.

A question might arise at this point, why consider separately Fourier transforms of functions (section III) and of distributions (section IV)? Since these are generalized functions, as we have seen in section II, the treatment of Fourier transforms of distributions should contain that of functions as a particular case.

A simple reasoning might clear this aspect. In section III we have considered the simplest class of functions which are Fourier-transformable. Those few pages contain a simple, yet complete, theory. There we introduced a space $\mathcal{S}$ which will be extensively used in this section. This space contains some simple functions which are Fourier-transformable (they were called good functions).

This space $\mathcal{S}$ consists of functions which act as the testing functions for defining the Fourier transform of a certain class of distributions (these are the tempered distributions). Reason enough, we believe, to consider the two ideas separately. Obviously, the results of this section have as a very particular case those of the preceding one.

Two comments seem to be in order. The first is concerned with the complexity introduced by the consideration of so many spaces. Are they really necessary? A mathematician might consider them essential to a precise statement of the theory. For those interested in evaluating some transform they are not so critical but do serve to give the conditions under which evaluations can occur. The reader might find it useful to refer to the appendix on spaces at any moment when their meaning or definition seem to be obscure or imprecise - and we do hope this will cause an improvement of the understanding of the theory, not the contrary. Second, this section is concerned with the definition and important properties of the Fourier transform, $\mathcal{F}[T]$, of a distribution T. Since the idea of a distribution may be somewhat new to the reader, it may well be worth pointing out that this section is concerned with a rigorous generalization of the scalar product $< T(t), \exp[-j\omega t] >$, which, it will be recalled, is a complex number whose value depends upon the variable ω. However, some of the details of the theory are concerned with the fact that $< T(t), \exp[-j\omega t] >$ doesn't exist for every distribution T for which a Fourier transform exists, for instance when $T(t) = 1$.

2. Tempered Distributions

Just as the space $\mathcal{D}'$ of distributions is defined as the topological dual of the space $\mathcal{D}$ of testing functions, so is the space $\mathcal{S}'$ of Fourier transformable distributions defined as the topological dual of the space $\mathcal{S}$ of good functions. That is, the space $\mathcal{S}'$, the dual of $\mathcal{S}$, of "slowly increasing distributions" or simply tempered distributions,[1] is the set of linear functionals, defined over $\mathcal{S}$, which are continuous in terms of the notion of convergence[2] in $\mathcal{S}$. We can see that every testing function is a good function, since the good functions need not have compact support. The relation $\mathcal{D} \subset \mathcal{S}$ implies that $\mathcal{D}' \supset \mathcal{S}'$, because of the notion of a dual space. This means that every tempered distribution is itself a distribution, and $< T, \psi >$ is well defined for every $\psi \in \mathcal{D}$ as well as for every $\psi \in \mathcal{S}$.

The space $\mathcal{S}'$ consists of the Fourier transformable distributions, the tempered distributions. The reason for this name, as well as a convenient test to tell if a distribution deserves it, are the subject of the following theorem.[3]

Theorem IV-1:

A necessary and a sufficient condition for T to be a tempered distribution (i.e., $T \in \mathcal{S}'$), is that there exists some bounded function, h, such that for some real m and integer $k' \geq 0$

$$T(t) = \frac{d^{k'}}{dt^{k'}}[(1+t^2)^m h(t)] \qquad \text{(IV-1)}$$

In other words, $T\epsilon \mathcal{S}'$ is such that its behavior is somewhat smooth, being a derivative of a function which behaves as a polynomial at infinity.

For taking transforms of series it is necessary to have a notion of convergence in $\mathcal{S}'$. We will say, by analogy with definition II-5, that a sequence $\{T_n\}$, $T_n \epsilon \mathcal{S}'$, converges to $T \epsilon \mathcal{S}'$ if, for all $\psi \epsilon \mathcal{S}$,

$$\lim_{n\to\infty} < T-T_n, \psi > = 0$$

Example IV-1:

We list several important types of tempered distributions.

a) $\delta^{(k)} \epsilon \mathcal{S}'$ as $h = u$, $m = 0$, $k' = k$

b) If $g(t)$ is a summable function, over $-\infty \leq t \leq \infty$, then $g \epsilon \mathcal{S}'$ as

$$h(t) = \int_a^t g(\tau)d\tau$$

with $m = 0$, $k' = 1$.[4]

c) If $g(t)$ is a square integrable function, over $-\infty \leq t \leq \infty$, then $g \epsilon \mathcal{S}'$.[5]

d) $g(t) = at \epsilon \mathcal{S}'$, a = constant, as $h = a/2$, $m = k' = 1$. Note that here g is not a good function.

By observing (IV-1) it is clear that if T is tempered then $T^{(k)}$ is tempered as well as any primitive.

Example IV-2:

$\exp[t] \notin \mathcal{S}'$ since none of its integrals behaves as $(1+t^2)^m h(t)$ with h bounded for all t. This is equivalent to the fact that the partial sums of $\sum_{i=0}^{\infty} t^i/i!$ do not define a convergent sequence in $\mathcal{S}'$; these partial sums are not bounded by a fixed polynomial.[6]

3. Fourier Transform Definition

The Fourier transform theory can now proceed by applying the results of section III. This follows from the fact that if T is tempered then for every good function ψ, $< T(t), \psi(t) >$ is well defined, and, hence, since $\mathcal{F}[\psi]$ is also a good function, so is $< T(\omega), \mathcal{F}_\omega[\psi(t)] >$ well defined. The Parseval's relation, as given by (III-3b), then shows that there is only one logical definition for the Fourier transform of a distribution.

Definition IV-1:

If $T \epsilon \mathcal{S}'$, then we define $\mathcal{F}[T]$ and $\overline{\mathcal{F}}[T]$ by, for all $\psi \epsilon \mathcal{S}$,

$$< \mathcal{F}_{\omega}[T(t)], \psi(\omega) > = < T(\omega), \mathcal{F}_{\omega}[\psi(t)] > \quad \text{(IV-2a)}$$

$$< \overline{\mathcal{F}}_{t}[T(\omega)], \psi(t) > = < T(t), \overline{\mathcal{F}}_{t}[\psi(\omega)] > \quad \text{(IV-2b)}$$

As in section III, $\mathcal{F}$ is called the (direct) Fourier transform and $\overline{\mathcal{F}}$ the inverse Fourier transform. Also, as in section III, we will normally take the variable for $\mathcal{F}$ to be the radian frequency ω and the variable for $\overline{\mathcal{F}}$ to be time t, and, thus will frequently drop the subscripts.[7]

It is still true that in the distributional case $\overline{\mathcal{F}}$ is the inverse of $\mathcal{F}$, and vice versa since by (IV-2) and (III-2)

$$< \overline{\mathcal{F}}[\mathcal{F}[T]], \psi > = < \mathcal{F}[T], \overline{\mathcal{F}}[\psi] > = < T, \mathcal{F}[\overline{\mathcal{F}}[\psi]] > = < T, \psi >$$

That is, for any tempered distribution

$$\overline{\mathcal{F}}[\mathcal{F}[T]] = T \quad \text{(IV-3a)}$$

$$\mathcal{F}[\overline{\mathcal{F}}[T]] = T \quad \text{(IV-3b)}$$

An obvious, but quite important, conclusion from this is that the Fourier transform of a tempered distribution T can only be zero if T is zero.

Although $< T(t), \exp[\mp j\omega t] >$ is not always defined for a tempered distribution, whenever this is defined we can calculate the Fourier transforms by[8]

$$\mathcal{F}[T] = < T(t), \exp[-j\omega t] > \quad \text{(IV-4a)}$$

$$\overline{\mathcal{F}}[T] = \frac{1}{2\pi} < T(t), \exp[j\omega t] > \quad \text{(IV-4b)}$$

That is, when the right of (IV-4) exists, we can think of the Fourier transform as being given by the same formula used for functions, since by our symbolical convention of (II-2b), (IV-4a) becomes

$$\mathcal{F}[T] = \int_{-\infty}^{\infty} T(t) e^{-j\omega t} dt$$

Unfortunately the expression in (IV-4) can not be evaluated for all tempered T and one must rely upon the definitions of (IV-2) to evaluate the Fourier transforms. The expressions of (IV-2) do exist for all T which are good functions, as well as for δ and its derivatives. $T(t) = 1$ is an example of a tempered distribution for which $< 1, \exp[-j\omega t] >$ doesn't exist, see (IV-5d).

Example IV-2:

By the last comments the functional Fourier transform and the distributional one coincide, when the former exists. Thus the pulse of unit height and of width 2τ,

$$T(t) = u(t+\tau)-u(t-\tau)$$

has

$$\mathfrak{F}[T] = [2\sin \omega\tau]/\omega$$

Example IV-3:

Using (IV-4) with the property of the impulse's derivatives, (II-8b), immediately gives, for integer $k \geq 0$,

$$\mathfrak{F}[\delta^{(k)}] = (j\omega)^k \qquad \text{(IV-5a)}$$

$$\overline{\mathfrak{F}}[\delta^{(k)}] = \frac{1}{2\pi}(-jt)^k \qquad \text{(IV-5b)}$$

Then by (IV-3), we directly get

$$\mathfrak{F}[t^k] = 2\pi j^k \delta^{(k)}(\omega) \qquad \text{(IV-5c)}$$

$$\overline{\mathfrak{F}}[(j\omega)^k] = \delta^{(k)}(t) \qquad \text{(IV-5d)}$$

Example IV-4:

We have, by a simple application[9] of example II-6.

$$\mathfrak{F}[PF(\frac{1}{t})] = -j\pi \text{sgn } \omega \qquad \text{(IV-6a)}$$

$$\overline{\mathfrak{F}}[PF(\frac{1}{\omega})] = \frac{j}{2}\text{sgn } t \qquad \text{(IV-6b)}$$

Then by (IV-3), we directly get

$$\overline{\mathfrak{F}}[\text{sgn } \omega] = PF(\frac{j}{\pi t}) \qquad \text{(IV-6c)}$$

$$\mathfrak{F}[\text{sgn } t] = PF(\frac{2}{j\omega}) \qquad \text{(IV-6d)}$$

An important consequence of these equations is found by writing

$$2u(t) = 1+\text{sgn } t$$
$$2u(-t) = 1-\text{sgn } t$$

and using (IV-5c) with k = 0 and (IV-6d)

$$\mathcal{F}[u(t)] = \pi\delta(\omega)+PF(\frac{1}{j\omega}) \quad \text{(IV-7a)}$$
$$\mathcal{F}[u(-t)] = \pi\delta(\omega)-PF(\frac{1}{j\omega}) \quad \text{(IV-7b)}$$

4. Multiplication and Convolution

One of the basic reasons for the use of the Fourier transform is that under such a transformation multiplication and convolution become interchanged. Unfortunately any two tempered distributions can't be convoluted or multiplied, and, thus, we must put some restrictions on these operations.[11] In order to state precisely when these operations can be performed it is necessary to introduce the classes of distributions mentioned below.

By the formula for multiplication, (II-11), we write

$$< \alpha(t)T(t), \psi(t) > = < T(t), \alpha(t)\psi(t) > \quad \text{(IV-8)}$$

If T is an arbitrary tempered distribution and $\psi \epsilon \mathcal{S}$, this requires that α be such that $\alpha\psi$ by a good function. This will always be the case if α is a slowly increasing infinitely differentiable function or a fairly good function. The space of fairly good functions is denoted by $\mathcal{O}_M$ and characterized by the fact that $\alpha(t) \epsilon \mathcal{O}_M$ if the absolute value of the k th derivative, $|\alpha^{(k)}(t)|$, for every integer $k \geq 0$, is bounded by a polynomial in t (whose degree may depend upon k);[12] as examples we can state that t, sin t, exp[jt] are fairly good functions. Then, if $\alpha \epsilon \mathcal{O}_M$ and $\psi \epsilon \mathcal{S}$, we have[13] $\alpha\psi \epsilon \mathcal{S}$ and (IV-8) can be formed for every $T \epsilon \mathcal{S}'$.

Since $\alpha(t) = t^k$ is a fairly good function, we can multiply any tempered distribution by t^k. This easily allows us to show the important result concerning the Fourier transform of the derivative.[14]

$$\mathcal{F}[T^{(k)}] = (j\omega)^k \mathcal{F}[T] = \mathcal{F}[\delta^{(k)} * T]; \; k = 0, 1, 2\cdots \quad \text{(IV-9a)}$$
$$\overline{\mathcal{F}}[T^{(k)}] = (-jt)^k \; \overline{\mathcal{F}}[T] = \overline{\mathcal{F}}[\delta^{(k)} * T]; \; k = 0, 1, 2\cdots \quad \text{(IV-9b)}$$

The results on multiplication can be used to obtain the desired properties for the convolution. By a straightforward application of the definitions of multiplication, convolution and the Fourier transform, we see[15] that for $\alpha \epsilon \mathcal{O}_M$ and $T \epsilon \mathcal{S}'$,

$$\mathcal{F}[\alpha T] = \frac{1}{2\pi} \mathcal{F}[\alpha] * \mathcal{F}[T] \qquad \text{(IV-10a)}$$

$$\overline{\mathcal{F}}[\alpha T] = \overline{\mathcal{F}}[\alpha] * \overline{\mathcal{F}}[T] \qquad \text{(IV-10b)}$$

Consequently, since $\mathcal{F}[T] = S$ can be chosen arbitrarily in $\mathcal{S}'$, that is, given S let $T = \overline{\mathcal{F}}[S]$, we see that we can always convolute distributions of the form of $\mathcal{F}[\alpha]$, with α a fairly good function, with arbitrary distributions in $\mathcal{S}'$. The space of these distributions, such that their (inverse) Fourier transform is a fairly good function, is called the space of distributions of <u>rapid decrease</u> and denoted by $\mathcal{O}_c'$. One characterization[16] of $\mathcal{O}_c'$ is by the fact that $T \epsilon \mathcal{O}_c'$ if T is the finite sum of derivatives of continuous functions, the latter being such that their product by $(1+t^2)^m$ is bounded for every $m \geq 0$.

By taking the inverses of (IV-10) we immediately see that if S is of rapid decrease, i.e. $S \epsilon \mathcal{O}_c'$, and $T \epsilon \mathcal{S}'$, then[17]

$$\mathcal{F}[S*T] = \mathcal{F}[S] \cdot \mathcal{F}[T] \qquad \text{(IV-11a)}$$

$$\overline{\mathcal{F}}[S*T] = 2\pi \overline{\mathcal{F}}[S] \cdot \overline{\mathcal{F}}[T] \qquad \text{(IV-11b)}$$

These concepts are best illustrated by some simple examples.

Example IV-5:

$\delta^{(k)}$ is a distribution of rapid decrease, since

$$\delta^{(k)} = g^{(k+2)}; \ k \geq 0$$

where

$$g(t) = (e^{-t} - e^{-2t})u(t)$$

This has $(1+t^2)^m g(t)$ bounded for every m and all t. Then for any $T \epsilon \mathcal{S}'$ we have

$$\mathcal{F}[\delta^{(k)} * T] = \mathcal{F}[\delta^{(k)}] \cdot \mathcal{F}[T] = (j\omega)^k \mathcal{F}[T]$$

by (IV-11a) and (IV-5a). This gives an alternate justification of (IV-9a), as well asanother way of seeing that $(j\omega)^k \epsilon \mathcal{O}_M$.

Example IV-6:

Since $e^{-at}u(t) \epsilon\, \mathcal{O}_c'$ when Re a > 0, this has a Fourier transform,[18] which by (IV-4a) is

$$\mathcal{F}[e^{-at}u(t)] = \frac{1}{a+j\omega}\ ,\ \text{Re } a > 0$$

This is indeed in $\mathcal{O}_M$. Thus, since

$$[e^{-at}u]*[e^{-bt}u] = \frac{e^{-bt}-e^{-at}}{a-b}\cdot u$$

we obtain from (IV-11)

$$\mathcal{F}[\frac{e^{-bt}-e^{-at}}{a-b}\ .u] = \frac{1}{a+j\omega}\cdot\frac{1}{b+j\omega}\ ;\ \text{Re } a > 0,\ \text{Re } b > 0$$

Example IV-7:

$\exp[jt^2]$ belongs to both $\mathcal{O}_c'$ and $\mathcal{O}_M$ and[19]

$$\mathcal{F}[\exp jt^2] = (1+j)\sqrt{\frac{\pi}{2}}\ \exp[-j\frac{\omega^2}{4}]$$

Supplementary Result:

Although in the general case one must restrict one of the members of a convolution to be a distribution of rapid decrease, there are special, and important, cases where this restriction need not be kept. For a case of much interest for physical systems, because of energy relations, we introduce the $\mathcal{D}_{L_q}'$ spaces. For these we recall that L_q is the set of (measurable) functions, g, for which

$$\int_{-\infty}^{\infty} |g(t)|^q dt < \infty$$

The space $\mathcal{D}_{L_q}'$ consists of those distributions which are finite sums of derivatives of functions in L_q. The result of interest is then the following.[20]

Theorem IV-2:

If $S \epsilon\ \mathcal{D}_{L_q}'$, $T \epsilon\ \mathcal{D}_{L_r}'$, $1 \le q \le 2$, $1 \le r \le 2$, then

$$S*T \epsilon\ \mathcal{D}_{L_s}',\ (1/s) = (1/q) + (1/r) - 1$$

$$\mathcal{F}[S*T] = \mathcal{F}[S]\cdot\mathcal{F}[T]$$

with all three transforms being functions. In particular $\mathcal{F}[S]$ is the product of a polynomial by a function in $L_{q'}$ where $q' = q/(q-1)$.

Example IV-8:

Consider

$$S(t) = T(t) = PF(\frac{1}{t})$$

We have[21] $S \epsilon L_2 \epsilon \mathcal{D}'_{L_2}$, but S is not a distribution of rapid decrease, since at infinity the behavior is as 1/t. By the last theorem we have, using (IV-6a)

$$\mathcal{F}[PF(\frac{1}{t})*PF(\frac{1}{t})] = (-j\pi \operatorname{sgn} \omega)(-j\pi \operatorname{sgn} \omega)$$
$$= -\pi^2$$

But $-\pi^2$ is the Fourier transform of $-\pi^2\delta(t)$ and hence we obtain, by the uniqueness of the Fourier transform.

$$PF(\frac{1}{t})*PF(\frac{1}{t}) = -\pi^2\delta(t) \qquad \text{(IV-12)}$$

This yields the Hilbert transform pairs, since, letting $T \epsilon \mathcal{D}'_{L_q}$, $1 \leq q \leq 2$, with

$$S = -\frac{j}{\pi}[PF(\frac{1}{t})]*T \qquad \text{(IV-13a)}$$

shows, by use of (IV-12), that

$$T = -\frac{j}{\pi}[PF(\frac{1}{t})]*S \qquad \text{(IV-13b)}$$

Then S and T are a Hilbert transform pair. Note that one can add constants to both S and T and the result is unchanged, since $1*PF(\frac{1}{t}) = 0$ by direct calculation using (II-13).

NOTES TO IV

1. In Schwartz, [SC 2, p. 93], the tempered distributions are called "tempérées" as well as "distributions à croissance lent," that is, distributions of slow increase. We recall that the notion of topological dual is mentioned in note-II-9.
2. That is, if $T \epsilon \mathcal{S}'$ and $\psi \epsilon \mathcal{S}$ then $< T, \psi >$ satisfies conditions 1) and 2) listed after definition II-1 of section II. For this reason we have to introduce the notion of convergence, which follows example III-1, in $\mathcal{S}$.
3. See [SC 2, p. 95], where this theorem is proven as part 1^o of Theorem VI. Although Schwartz doesn't so state, $m \geq 0$ can be assumed, as otherwise $(1+t^2)^m$ can be absorbed in g. Schwartz writes (IV-1) as $T(x) = d^p[(1+x^2)^{k/2}f(x)]/dx^p$, the factor 1/2 associated with k apparently stemming from the fact that $(1+x^2)^{1/2}$ is associated with a distance.
4. Recall that if g is a summable function then $\int_a^t g(\tau)d\tau$ is an absolutely continuous function and hence is bounded, [BU 1, p. 55]. It only makes sense that summable $g \epsilon \mathcal{S}'$, since all such functions have a Fourier transform in the functional sense, [BO 1, p. 1]. That is

$$| \mathfrak{F} [g]| = | \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt| \leq \int_{-\infty}^{\infty} |g(t)|dt$$

with the right side finite when g is summable, this being essentially the definition of summability, [BO 1, p. 32].
5. We first recall that, on the whole line $-\infty \leq t \leq \infty$, summable functions, such as $g(t) = 1/(1+t^2)^{1/2}$, need not be square integrable and thus example IV-1c) is not a special case of example IV-1b). The fact that square integrable functions are tempered distributions follows as a special case of the result to be given in note-IV-20; the square integrable functions are defined in note-II-11.
6. See [SC 2, p. 95]. Note that exp[t] is locally summable but not summable on the whole line, and thus example IV-1b) doesn't apply. Since neither exp[t] nor u(t)·exp[t] are tempered they do not have

Fourier transforms and thus raise a need for the Laplace transform. This is no longer the case for u, since $u \in \mathcal{S}'$, and thus the standard argument that the Laplace transform is introduced to give u, or $u \cdot \cos t$, a transform breaks down in the distributional case [GA 1, p. 100].

7. Occasionally, as in convolution expressions, it becomes necessary to pay strict attention to all the parameters, as done in (IV-2). We comment that in the scalar product form of (IV-2) all the parameters are dummy and can be replaced by any other appropriate symbols. In (IV-2) we use T for the "operand" of both $\mathcal{F}$ and $\overline{\mathcal{F}}$, since all that is of interest is that we operate on a tempered distribution. This is somewhat in contrast to section III where we wished to emphasize the relationship between ψ and Ψ .

8. To see that $\mathcal{F}[T] = \langle T(t), \exp[-j\omega t] \rangle$ when this exists we proceed as follows.

$$\langle \mathcal{F}_\omega[T], \psi(\omega) \rangle = \langle T(\omega), \mathcal{F}_\omega[\psi(t)] \rangle \text{ by (IV-2a)}$$

$$= \langle T(y), \int_{-\infty}^{\infty} \psi(x) e^{-jyx} dx \rangle \text{ by (III-1a) and a variable change}$$

$$= \int_{-\infty}^{\infty} \psi(x) [\langle T(y), e^{-jyx} \rangle] dx \text{ by linearity}$$

$$= \int_{-\infty}^{\infty} \psi(\omega) [\langle T(t), e^{-j\omega t} \rangle] d\omega \text{ by a variable change}$$

Identifying this with (II-2a) we get (IV-4a) whenever $\mathcal{F}[T]$ is a function. The result for $\overline{\mathcal{F}}$ follows by simply replacing j by -j in the exponential and multiplying by $1/2\pi$.

9. Here

$$\operatorname{sgn} t = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0 \end{cases}$$

To see that (IV-6a) is true we can use (II-9a) with (IV-4a).

$$\mathcal{F}[PF(\tfrac{1}{t})] = \langle PF(1/t), \exp[-j\omega t] \rangle$$

$$= \lim_{\epsilon \to 0} \{ \int_{\epsilon}^{\infty} (1/t) \exp[-j\omega t] dt + \int_{-\infty}^{-\epsilon} (1/t) \exp[-j\omega t] dt \}$$

$$= \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} (1/t) \{ \exp[-j\omega t] - \exp[j\omega t] \} dt$$

0].

$$= -2j \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} (1/t)\sin[\omega t]dt = -j\pi \operatorname{sgn} \omega$$

since $\int_0^{\infty} (1/t)\sin \omega t \, dt = (\pi \operatorname{sgn} \omega)/2$ by [DW 1, p. 198].

The result for $\overline{\mathcal{F}}$ follows by dividing by 2π and conjugating. The dissymmetry of $\mathcal{F}$ versus $\overline{\mathcal{F}}$ clearly shows up in these equations.

10. In obtaining (IV-7) we have used the fact that $\mathcal{F}[T+S] = \mathcal{F}[T] + \mathcal{F}[S]$ which follows from the linearity of $< T+S, \mathcal{F}[\psi] >$.

11. For instance $u(t)*u(-t)$ is not defined but $u \in \mathcal{S}'$. Likewise $\delta \cdot \delta$ is not defined but $\delta \in \mathcal{S}'$.

12. See [SC 2, p. 99] for this characterization. The name "fairly good" is ascribed by Lighthill, [LI 1, p. 15].
Clearly $\mathcal{S} \subset \mathcal{O}_M$, since, definition III-1b), $\lim_{|t| \to \infty} |t^m \psi^{(k)}(t)| = 0$, in conjunction with continuity, shows that $\psi \in \mathcal{S}$ satisfies the requirement for $\mathcal{O}_M$ that $|\alpha^{(k)}(t)|$ be bounded by a polynomial in t.

13. If $\alpha \in \mathcal{O}_M$ and $\psi \in \mathcal{S}$ then $\alpha\psi$ is an infinitely differentiable continuous function. Since

$$(\alpha\psi)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} \alpha^{(i)} \psi^{(k-i)}; \quad \binom{k}{i} = \frac{k!}{(k-i)!\,i!}$$

we have

$$|t^m (\alpha\psi)^k| \le \sum_{i=0}^{k} \binom{k}{i} |t^m \alpha^{(i)} \psi^{(k-i)}|$$

$$\le \sum_{i=0}^{k} |t^{m+p} \psi^{(k-i)}| \qquad \text{for large } |t|$$

where p is defined by $|\alpha^{(i)}| \le |t^p|$ for large $|t|$. Since $\lim_{|t| \to \infty} |t^{m+p} \psi^{(k-i)}| = 0$ because $\psi \in \mathcal{S}$, this shows that $\alpha\psi \in \mathcal{S}$.

14. To see that $\mathcal{F}[T^{(k)}] = (j\omega)^k \mathcal{F}[T]$ we proceed as follows:

$$< \mathcal{F}[T^{(k)}], \psi > = < T^{(k)}, \mathcal{F}[\psi] > \text{ by (IV-2a)}$$
$$= (-1)^k < T, d^k \mathcal{F}_{\omega}[\psi]/d\omega > \text{ by (II-7)}$$
$$= (-1)^k < T, \mathcal{F}_{\omega}[(-jt)^k \psi(t)] > \text{ by note III-5a)}$$
$$= < \mathcal{F}_{\omega}[T], (j\omega)^k \psi(\omega) > \text{ by (IV-2a)}$$

$= < (j\omega)^k \mathcal{F}_\omega[T], \psi(\omega) >$ by (IV-8)

The result for $\overline{\mathcal{F}}$ merely replaces j by -j in the multiplier. Clearly $\mathcal{F}[T^k] = \mathcal{F}[\delta^{(k)} * T]$, since $T^k = \delta^{(k)} * T$ by (II-15c).

15. To see that for $\alpha \epsilon \mathcal{O}_M$ and $T \epsilon \mathcal{S}'$, $\mathcal{F}[\alpha T] = \mathcal{F}[\alpha] * \mathcal{F}[T]$, we have

$< \mathcal{F}_\omega[\alpha(t)T(t)], \psi(\omega) > = < \alpha(\omega)T(\omega), \mathcal{F}_\omega[\psi(t)] >$ by (IV-2a)

$= < T(\omega), \alpha(\omega) \mathcal{F}_\omega[\psi(t)] >$ by (IV-8)

$= < T(\omega), \overline{\mathcal{F}}_\omega[\mathcal{F}_\tau[\alpha(t)]] \cdot \mathcal{F}_\omega[\psi(t)] >$ by (IV-3a)

$= < T(\omega), \mathcal{F}_\omega[\psi(t)] \cdot \frac{1}{2\pi} < \mathcal{F}_\tau[\alpha(t)], \exp[j\omega\tau] > >$ by (IV-4b)

$= < T(\omega), \frac{1}{2\pi} < \mathcal{F}_\tau[\alpha(t)], \mathcal{F}_\omega[\psi(t)] \cdot \exp[j\omega\tau] > >$ by linearity of <,>

$= < T(\omega), \frac{1}{2\pi} < \mathcal{F}_\tau[\alpha(t)], \mathcal{F}_\omega[\psi(t+\tau)] > >$ since

$$\int_{-\infty}^{\infty} \psi(t+\tau) e^{-j\omega t} dt = e^{j\omega\tau} \int_{-\infty}^{\infty} \psi(t) e^{-j\omega t} dt$$

$= < T(\omega), \frac{1}{2\pi} \mathcal{F}_\omega[< \mathcal{F}_\tau[\alpha(t)], \psi(t+\tau) >] >$ by linearity of <,>

$= < \mathcal{F}_\omega[T(t)], \frac{1}{2\pi} < \mathcal{F}_\tau[\alpha(t)], \psi(\omega+\tau) > >$ by (IV-2a)

$= < \frac{1}{2\pi} \mathcal{F}_\omega[T(t)] * \mathcal{F}_\omega[\alpha(t)], \psi(\omega) >$ by (II-13)

The proof for $\overline{\mathcal{F}}$ is similar except that no multiplier $1/2\pi$ appears at the fourth step.

16. In the French "rapid decrease" is "décroissance rapide". The condition stated is necessary and sufficent for $T \epsilon \mathcal{O}'_c$, [SC 2, p. 100]. $\mathcal{O}'_c$ is not the dual of $\mathcal{O}_M$, [SC 2, p. 101], and hence we don't write $\mathcal{O}'_M$ for $\mathcal{O}'_c$, but $\mathcal{O}'_c$, and $\mathcal{O}_M$ are in a one to one correspondence. That is, every distribution $T \epsilon \mathcal{O}'_c$ is the Fourier transform of some $\alpha \epsilon \mathcal{O}_M$, $T = \mathcal{F}[\alpha]$, and vice versa, [SC 2, p. 124]. Schwartz's proof of this seems quite delicate, but we essentially get around this problem by defining $\mathcal{O}'_c$ to have this property.

17. To see that if $S \epsilon \mathcal{O}_c'$ and $T \epsilon \mathcal{S}'$ then

$\mathcal{F}[S*T] = \mathcal{F}[S] \cdot \mathcal{F}[T]$, we rewrite (IV-10b) as

$\overline{\mathcal{F}}[\alpha U] = \overline{\mathcal{F}}[\alpha] * \overline{\mathcal{F}}[U]$ and then let $\alpha = \mathcal{F}[S]$, $U = \mathcal{F}[T]$

giving

$$\overline{\mathcal{F}}[\mathcal{F}[S] \cdot \mathcal{F}[T]] = \overline{\mathcal{F}}[\mathcal{F}[S]] * \overline{\mathcal{F}}[\mathcal{F}[T]] = S*T$$

Applying $\mathcal{F}$ to this gives the desired result. The result for $\overline{\mathcal{F}}$ follows in the same manner.

18. Note that if $a = j\beta$, β real, then $e^{-j\beta t}u(t) \notin \mathcal{O}_c'$ since $(1+t^2)e^{-j\beta t}$ is not bounded for large t. However, $e^{-j\beta t}u(t)$ does have a Fourier transform, since $e^{-j\beta t}u(t) \epsilon \mathcal{S}'$, as is seen by (IV-1) with $m = k' = 0$. Noting (IV-4a) with (II-11) we need merely replace $\omega+\beta$ in (IV-7a) to get

$$\mathcal{F}[e^{-j\beta t}u(t)] = \pi\delta(\omega+\beta) + PF\frac{1}{j(\omega+\beta)}$$

19. This result is stated in [SC 2, p. 126] and proven in [LI 1, p. 49].

20. See [SC 2, pp. 112 & 126] for a proof of theorem IV-2. The properties of $\mathcal{D}_{L_q}'$ are as follows. As is customary we define L_q as the set of measurable functions, g, for which, [BU 1, p. 65],

$$\int_{-\infty}^{\infty} |g(t)|^q dt < \infty$$

Then define $\mathcal{D}_{L_q}$, $1 \leq q \leq \infty$, as the space of infinitely differentiable functions (of arbitrary support) for which all derivatives are also in L_q, [SC 2, p. 55]. One also must define a notion of convergence in $\mathcal{D}_{L_q}$ for reasons which should now be clear; $\varphi_j \epsilon \mathcal{D}_{L_q}$ converge to zero (in $\mathcal{D}_{L_q}$) if for every non-negative integer k,

$$\lim_{j\to\infty} \int_{-\infty}^{\infty} |\varphi_j^{(k)}(t)|^q dt = 0$$

Letting q' be defined by $q' = q/(q-1)$ we define $\mathcal{D}_{L_{q'}}'$, $1 < q' \leq \infty$, as the topological dual of $\mathcal{D}_{L_q}$. Now clearly $\mathcal{S} \subset \mathcal{D}_{L_q}$ ($\subset$ denotes "is contained in") and thus $\mathcal{D}_{L_{q'}}' \subset \mathcal{S}'$. The distributions in

$\mathcal{D}'_{L_q}$ are characterized by the following result, [SC 2, p. 57].

Theorem:

1^o. A necessary and sufficient condition for $T \epsilon\ \mathcal{D}'_{L_q}$ is that T be the finite sum of derivatives of functions in L_q.

2^o. A necessary and sufficient condition for $T \epsilon\ \mathcal{D}'_{L_q}$ is that for every $\alpha(t) \epsilon\ \mathcal{D}$, $\alpha * T \epsilon L_q$.

Thus $\delta \epsilon\ \mathcal{D}'_{L_q}$ for any q, since $\delta = g' + g$ with $g = e^{-t}u(t)$. Further $L_2 \subset \mathcal{D}'_{L_2}$ which gives the result of example IV-1c).

One of the most important of these spaces for physical problems is $\mathcal{D}'_{L_2}$, because of the relation of square integrable, L_2, functions to energy.

21. This is a special case of the statement of [SC 2, p. 127] that $PF(t^q) \epsilon\ \mathcal{D}'_{L_2}$ for Re $q < -1/2$.

V. LAPLACE TRANSFORMS

Summary: The definition and main properties of the Laplace transform, $\mathcal{L}[T]$, of a distribution, T, are the subject of this the principal section.

The first paragraph considers the types of distributions which have a Laplace transform; these being the distributions, T, for which $T(t)\exp(-\sigma t)$ is tempered for certain σ, the pertinent σ's being denoted by $\sigma \epsilon \Gamma(T)$. In paragraph 2 the Laplace transform is defined by the use of the Fourier transform of section IV, $\mathcal{L}[T] = \mathcal{F}_{\omega}[T(t)\exp(-\sigma t)]$. $\mathcal{L}[T]$ is then seen in paragraph 3 to be an analytic function of the complex variable $p = \sigma + j\omega$ for p in the interior, $\mathring{\Gamma}(T)$, of $\Gamma(T)$. In this interior the magnitude of $\mathcal{L}$ is bounded by the magnitude of a polynomial in p, eq. (V-10), a fact which means that one need consider the complex inversion formula only for continuous T(t), eq. (V-20d). As shown by $T(t) = u(t)$, the Laplace transform on the boundary of the convergence strip may be a non-functional distribution and will reduce to the Fourier transform when $\sigma = 0$ if $0 \epsilon \Gamma(T)$. Of course $\mathcal{L}[T'] = p\mathcal{L}[T]$, eq. (V-7).

The subject of paragraph 4 is the Laplace transform of the convolution and of the product, as with the Fourier transform, these two operations are transformed into each other. Under suitable restrictions one has eq. (V-12a), $\mathcal{L}[S*T] = \mathcal{L}[S]\cdot\mathcal{L}[T]$ while, eq. (V-16a), $\mathcal{L}[\alpha T] = \mathcal{L}[\alpha]*\mathcal{L}[T]$, this latter requiring a proper interpretation, since a complex convolution is involved. From these two results most of the important properties follow, such as $\mathcal{L}[T(t+b)] = \mathcal{L}[T]\cdot\exp(bp)$ and $\mathcal{L}[tT(t)] = -d\mathcal{L}[T]/dp$. Paragraph 5 is concerned with inverting $\mathcal{L}[T]$ to get T. A general form of Jordan's lemma is used to allow the application of the residue calculus to $\mathcal{L}[T]$ which are single-valued with isolated singularities.

1. Transformable Distributions

The most profitable means of defining the Laplace transform for distributions is familiar to engineers. This consists of making a transition from the Fourier transform by inserting a convergence factor $\exp[-\sigma t]$ and then working with a complex variable in place of a real variable.[1] This process allows one to obtain more general, and apparently more useful, results than by other methods, such as taking distributional limits of sequences of integrals.[2]

Let the complex frequency variable be $p = \sigma + j\omega$. The following argument shows that for a distribution T, the set of p for which $e^{-pt}T(t)$ is a tempered distribution is a converge strip: by observing that $e^{-j\omega t} \epsilon \mathcal{O}_M$, we see that $e^{-j\omega t}S \epsilon\ \mathcal{S}'$ if $S \epsilon\ \mathcal{S}'$, see note IV-13.

Thus, if $e^{-pt}T \epsilon \mathcal{S}'$ for a given σ and ω, this must hold for all ω at this given σ.

The convergence strip[3] can be defined by the connected set of real points $\Gamma(T)$ for which this strip intersects the σ axis; for example as shown in Fig. V-1.

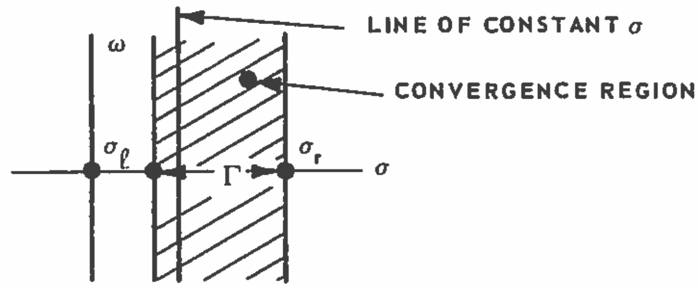


FIG. V-1. CONVERGENCE STRIP.

Here $\Gamma(T)$ may be closed or open on either the right or left and it may be empty; the left and right boundaries will be respectively denoted[4] by σ_ℓ and σ_r. For example for $T = e^{at}u(t)$ the convergence strip is defined by $\sigma_\ell = \text{Re } a \leq \sigma \leq \infty = \sigma_r$, for $T = \delta$ the entire plane forms the convergence strip, while for $T = e^{t^2}u(t)$, Γ is empty.

Since the customary situation treated in engineering texts limits σ to the interior of $\Gamma(T)$, the interior, denoted by $\mathring{\Gamma}(T)$, is of some importance; for $T = e^{at}u(t)$, $\mathring{\Gamma}$ is $\sigma > \sigma_\ell = \text{Re } a$. By the very meaning of the convergence strip, for every $\sigma \epsilon \Gamma(T)$, $e^{-\sigma t}T(t)$ has a Fourier transform, being tempered. For a given distribution, T, there is a fixed convergence strip, $\Gamma(T)$, conversely given a real (convex) set, Γ, there is a fixed set of distributions, called $\mathcal{S}'(\Gamma)$, whose convergence strips contain that defined by Γ; that is, if $T_o \epsilon \mathcal{S}'(\Gamma)$ has its convergence strip defined by $\Gamma_o(T_o)$, then $\Gamma \cap \Gamma_o(T_o) = \Gamma$. We will generally make a distinction between a Γ defining $\mathcal{S}'(\Gamma)$ and the $\Gamma(T)$ defining the convergence strip for a fixed T in the manner illustrated, that is, by explicitely exhibiting T in brackets for $\Gamma(T)$.

It is necessary for evaluating Laplace transforms of sequences or series to state what is meant by convergence in $\mathcal{S}'(\Gamma)$. However, since for each $\sigma \epsilon \Gamma$, $e^{-\sigma t} T \epsilon \mathcal{S}'$ if $T \epsilon \mathcal{S}'(\Gamma)$, we revert to the definition given for $\mathcal{S}'$. That is, a sequence $\{T_n\}$, $T_n \epsilon \mathcal{S}'(\Gamma)$, converges to $T \epsilon \mathcal{S}'(\Gamma)$, if for each $\sigma \epsilon \Gamma$ and all $\psi \epsilon \mathcal{S}$,

$$\lim_{n \to \infty} < e^{-\sigma t}[T - T_n], \psi > = 0 \qquad \text{(V-1)}$$

As we shall see at the end of paragraph 3, example V-5-b), $\sum_{k=0}^{\infty} \delta^{(k)}(t-k)$ is not a convergent series in $\mathcal{S}'(\Gamma)$ for any non-empty Γ.

2. <u>Laplace Transform Definition</u>

The principal definition is now conveniently given.

Definition V-1:

Given Γ and $T \epsilon \mathcal{S}'(\Gamma)$ the Laplace transform, $\mathcal{L}_p[T]$ is defined, for each $p = \sigma + j\omega$ with $\sigma \epsilon \Gamma$, by

$$\mathcal{L}_p[T(t)] = \mathcal{F}_{\omega}[e^{-\sigma t} T(t)] \qquad \text{(V-2)}$$

that is, for every $\psi \epsilon \mathcal{S}$,

$$< \mathcal{L}_p[T(t)], \psi(\omega) > = < \mathcal{F}_{\omega}[e^{-\sigma t} T(t)], \psi(\omega) >$$

In this definition the scalar product is formed with respect to ω for each $\sigma \epsilon \Gamma$; σ is then a parameter which is allowed to vary such that p can be considered as a complex variable. As with ω on $\mathcal{F}_{\omega}$, the subscript p on $\mathcal{L}_p$ will normally be omitted.

Clearly the Laplace transform reduces to the Fourier transform when $\sigma = 0$. We also note that given T we can choose Γ to be a subset of $\Gamma(T)$, in which case the Laplace transform is only given for a subset of its possible convergence strip. When $< T, \exp[-pt] >$ exists, we have[5]

$$\mathcal{L}[T] = < T(t), e^{-pt} >, \qquad \sigma \epsilon \Gamma \qquad \text{(V-3a)}$$

which shows more clearly the dependence upon p than (V-2). Consequently, if T is a function this reduces to the normal definition of the Laplace transform as

$$\mathcal{L}[T] = \int_{-\infty}^{\infty} T(t) e^{-pt} dt \qquad \text{(V-3b)}$$

Thus the tables valid for transforms of functions remain true when applied to this new definition. We note, however, that the definition coincides with the bilateral, or two-sided, transform, which of course reduces to the one-sided transform if $T(t)$ has support in $t \geq 0$. But now one has no trouble finding the Laplace transform for generalized functions, and one doesn't have to worry about different sequences defining δ, such as in Fig. II-3, giving different results for $\mathcal{L}[\delta]$. Note that the same comments hold concerning this form of $\mathcal{L}[T]$ as were given for (IV-4a); that is (V-3) can not always be used and we must use (V-2) in the general case.

Example V-1:

Since, by (II-12a), $e^{-\sigma t}\delta(t) = \delta(t)$, we see that $\Gamma(\delta)$ yields the entire p plane as the convergence strip for δ. Then by either (V-2) with (IV-5a) or (V-3a) with (II-4) we immediately get[6]

$$\mathcal{L}[\delta(t)] = 1 \qquad \text{(V-4a)}$$

By the use of (V-3a) with (II-6) we also have

$$\mathcal{L}[\delta(at+b)] = \frac{1}{|a|}e^{pb/a}, \quad a \neq 0 \qquad \text{(V-4b)}$$

Example V-2:

The unit step function, u, has $\Gamma(u)$ defined by $\sigma \geq 0$. Since the Laplace transform for u is normally given only for $\sigma > 0$, the customary theory is somewhat incomplete. By (IV-7a) and (V-3b) we get

$$\mathcal{L}[u(t)] = \begin{cases} \mathcal{F}[u] = \pi\delta(\omega) + PF(\frac{1}{j\omega}) & \sigma = 0 \\ \int_{0}^{\infty} e^{-pt}dt = 1/p & \sigma > 0 \end{cases} \qquad \text{(V-5)}$$

If we restrict to $\sigma > 0$, this reduces to the customary result. In fact we can easily see that the $\sigma > 0$ result reduces in the (distributional) limit to the $\sigma = 0$ result.[7]

Example V-3:

$T(t) = \exp[p_o t]$, $p_o = \sigma_o + j\omega_o$, has for its $\Gamma(T)$ the single point $\sigma = \sigma_o$. By a simple argument[8]

$$\mathcal{L}[e^{p_o t}] = 2\pi\delta(\omega-\omega_o), \qquad \sigma = \sigma_o$$

3. Differentiation, Analyticity Properties

The conversion of differentiation to multiplication has long been recognized as one basic property of the Laplace transform. This carries over to the distributional case. By using an argument identical to that used for the Fourier transform,[9] or what is essentially the same, replacing $j\omega$ by p, we see that

$$\mathcal{L}[T^{(k)}] = p^k\mathcal{L}[T];\ \sigma\epsilon\Gamma(T);\ k = 0, 1, 2, \ldots \qquad \text{(V-7)}$$

In the special case where $\mathcal{L}[T] = < T, \exp[-pt] >$ this is easily checked by applying the very definition of the distributional derivative, (II-7). Equation (V-7) again shows that the distributional Laplace transform is to be considered as bilateral. But, if T is a function which is zero for $t < 0$, we can write, using (II-12d),

$$T(t) = g(t)u(t)$$

$$T'(t) = g'(t)u(t) + g(0)\delta(t)$$

from which we get the familiar unilateral result

$$\mathcal{L}[g'u] = \mathcal{L}[gu]-g(0) \qquad \text{(V-8)}$$

Note that $\Gamma(T)$ and $\Gamma(T^{(k)})$ may differ, as seen from $T = u$, $T^{(1)} = \delta$, but (V-7) only holds for $\sigma\epsilon\Gamma(T)$.

Turning to properties which depend on analyticity requirements we first define

$$\frac{\partial}{\partial p} = \frac{1}{2}\left[\frac{\partial}{\partial\sigma} - j\frac{\partial}{\partial\omega}\right] \qquad \text{(V-9a)}$$

$$\frac{\partial}{\partial p*} = \left[\frac{1}{2}\frac{\partial}{\partial\sigma} + j\frac{\partial}{\partial\omega}\right] \qquad \text{(V-9b)}$$

For a given T, in the interior of the convergence strip, defined by $\Gamma(T)$, we find that the Cauchy-Riemann equations

$$\frac{\partial\mathcal{L}[T]}{\partial p*} = 0$$

are satisfied.[10] As a consequence, $\mathcal{L}[T]$ is an analytic, or more precisely "holomorphic", function in the interior of its strip of convergence;[11] in this interior an expression for the derivative is $\partial\mathcal{L}[T]/\partial p = d\mathcal{L}[T]/dp$.

Now considering Γ to be given, for each $\sigma\epsilon\Gamma$ $\mathcal{L}[\ ]$ is a mapping of $\mathcal{S}'(\Gamma)$ into $\mathcal{S}'$, since the Laplace transform is merely a Fourier transform. However, in the interior of the convergence strip the image of this map, being holomorphic in p, is a <u>function</u> of ω for each σ; this need not be the case on the boundary, as shown by example V-2. Consequently in this interior of Γ, $\mathcal{S}'(\mathfrak{P})$ is mapped into a functional subset of $\mathcal{S}'$; for each σ this[12] turns out to be $\mathcal{O}_M$.

Since every function of $\mathcal{O}_M$ is characterized by the fact that every derivative is bounded in absolute value by a polynomial, we see that if T has a Laplace transform, then for every closed subset, Γ', of $\mathfrak{P}$ we have[13]

$$|\mathfrak{L}[T]| < M|p|^m , \qquad \sigma \in \Gamma' \subset \mathfrak{P} \tag{V-10}$$

where M and m are non-negative constants, m an integer. This is immediately seen to be verified in the finite order case by writing

$$T = g^{(k)}$$

with the functional Laplace transform converging absolutely, and using

$$\mathfrak{L}[T] = p^k \mathfrak{L}[g]$$

In this case, since $g(t)e^{-pt}$ is absolutely integrable

$$M \geq \int_{-\infty}^{\infty} |g(t)| e^{-\sigma t} dt$$

Equation (V-10) has some importance for inversion, since we can invert $\mathfrak{L}[T]$ by first considering $\mathfrak{L}[\ ]/p^k$ which corresponds to the Laplace transform of a function.

Example V-4:

We have for real a

$$\mathfrak{L}[e^{at}u(t)] = \begin{cases} \mathfrak{F}[u] = \pi\delta(\omega) + PF(\frac{1}{j\omega}) , & \sigma = a \\ \int_0^{\infty} e^{-(p-a)t} dt = \frac{1}{p-a} , & \sigma > a \end{cases} \tag{V-11}$$

In $\sigma > a$, $\mathfrak{L}$ is clearly holomorphic. Then, (V-7),

$$\mathfrak{L}[(e^{at}u(t))^{(2)}] = \begin{cases} a^2\pi\delta(\omega) + (2a+j\omega) + PF(\frac{a^2}{j\omega}), & \sigma = a \\ \frac{p^2}{p-a} , \ \sigma > a \end{cases}$$

In $\sigma \geq a+\epsilon$, for any $\epsilon > 0$, we have

$$|\mathfrak{L}[\{e^{at}u(t)\}^{(2)}]| < M|p|\,;\ M > \frac{a+\epsilon}{\epsilon}$$

In the study of passive systems the square integrable functions, L_2 functions, are of extreme importance. Here we give a result which contains the L_2 results as a special case. Defining, as at the end of section IV, $\mathcal{D}'_{L_q}$ to be the space of finite sums of (distributional) derivatives of L_q functions, where we recall, $g \epsilon L_q$ if $\int_{-\infty}^{\infty} |g(t)|^q dt < \infty$, we can obtain the following result.[14] We first recall that the space of $\mathcal{D}'_{L_2}$ distributions contains the L_2 functions as a subspace. This theorem allows a rigorous proof of the fact that a passive system is "stable".[15]

Theorem V-1:

If $T \epsilon \mathcal{D}'_{L_q}$, $1 \leq q \leq 2$, has support in $t \geq 0$ then $\Gamma(T)$ contains the closed right half line $\sigma \geq 0$. Consequently, $\mathfrak{L}[T]$ is holomorphic at least in the open right half plane, Re $p > 0$.

Example V-5:

a) The Laplace transform

$$\mathfrak{L}[T] = \sqrt{p-1}$$

can't arise from $T \epsilon \mathcal{D}'_{L_2}$ with support in $t \geq 0$, since $\sqrt{p-1}$ has a singularity (a branch point) at $p = 1$. A physical consequence of this is that $\sqrt{p-1}$ can't be an impedance (driving-point or transfer) associated with a passive network.

b) The distribution

$$T(t) = \sum_{k=0}^{\infty} \delta^{(k)}(t-k)$$

has no convergence strip, that is $\Gamma(T)$ is empty, since we would require, by (V-4) and (V-7), that its Laplace transform be

$$\sum_{k=0}^{\infty} p^k e^{-kp}$$

By the ratio test this converges in the region $\omega^2 < e^{2\sigma} - \sigma^2$

which is as shown in Fig. V-2 and is not a convergence "strip". Consequently the series for T is not convergent in $\mathcal{S}'(\Gamma)$ for any non-empty Γ.

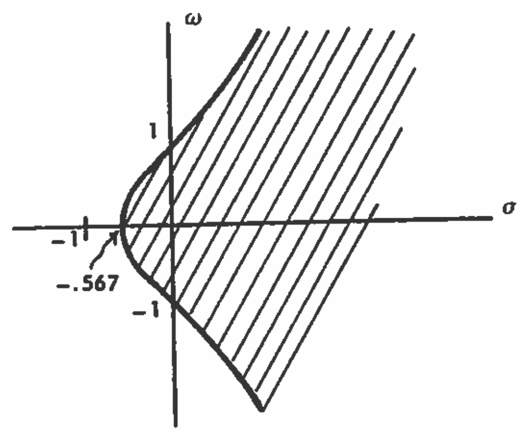


FIG. V-2. CONVERGENCE REGION FOR $\sum_{k=0}^{\infty} p^k e^{-kp}$.

4. Convolution and Multiplication

To a large extent, the usefulness of the Laplace transform is due to its ability to transform the convolution of two functions into the product of its transforms. We obtain here similar results concerning distributions, derived as before from those for the Fourier transform.

Fundamental here is the fact that $\mathcal{L}[T]$ is a fairly good function, as a function of ω, for each $\sigma \epsilon \Gamma(T)$. Using this in conjunction with the fact that the fairly good functions are Fourier transforms of distributions in $\mathcal{O}_c'$, we see that for each $\sigma \epsilon \Gamma(T)$, $e^{-\sigma t} T \epsilon \mathcal{O}_c'$. But the main characteristic of $\mathcal{O}_c'$ is the fact that its members can be freely convoluted with members of $\mathcal{S}'$. Consequently, given two distributions, S and T, for which the convergence strip of one intersects the interior of the convergence strip of the other, we can convolute $e^{-\sigma t} S(t)$ with $e^{-\sigma t} T(t)$ for σ in such an "intersection strip".

Since,[16]

$$e^{-\sigma t}(S*T) = (e^{-\sigma t}S)*(e^{-\sigma t}T)$$

we see from the convolution result for the Fourier transform, (IV-11a), that[17]

$$\mathcal{L}[S*T] = \mathcal{L}[S]\cdot\mathcal{L}[T] \qquad \text{(V-12a)}$$

at least for $\sigma\epsilon\Gamma(S)\cap\Gamma(T)$. For example (V-12a) holds for each point in the shaded region of Fig. V-3a). But in cases such as shown in Fig. V-3b) equation (V-12a) need not hold on the common boundary. However, it will hold on such a common boundary if $e^{-\sigma_r t}S$ and $e^{-\sigma_r t}T$ are both rapidly decreasing distributions.

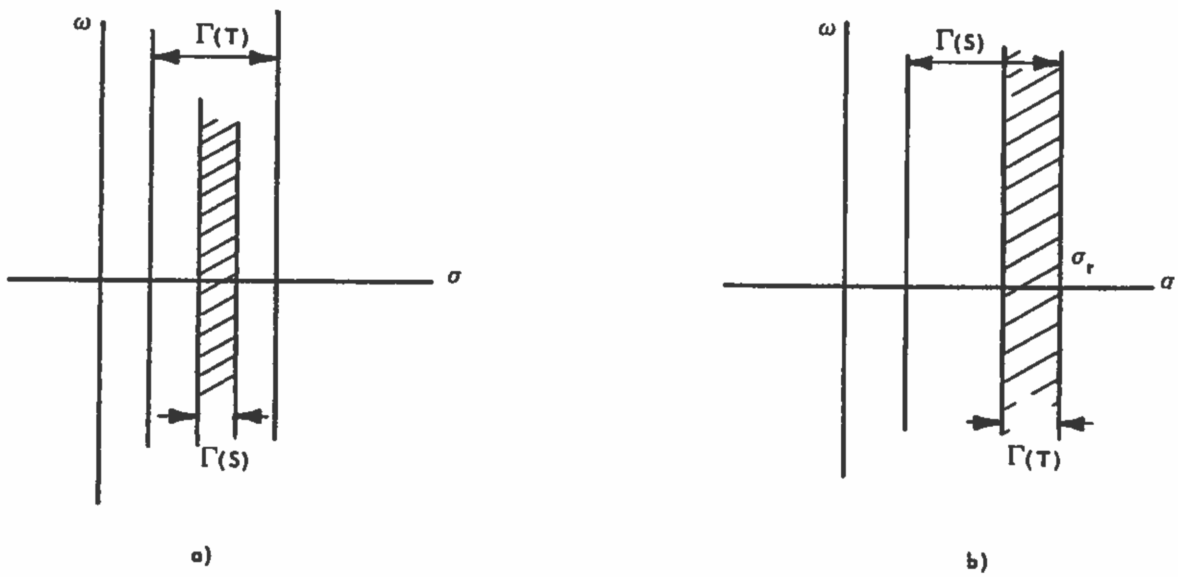


FIG. V-3. POSSIBLE CONVERGENCE REGIONS FOR $\mathcal{L}[S*T]$.

This further shows that if S, T, U have $\Gamma(S)$ intersecting a common strip of intersection of $\Gamma(T)$ and $\Gamma(U)$ then

$$(S*T)*U = S*(T*U) \qquad \text{(V-12b)}$$

since

$$\mathcal{L}[(S*T)*U] = \mathcal{L}[S]\cdot\mathcal{L}[T]\cdot\mathcal{L}[U] \qquad \text{(V-12c)}$$

In other words, in contrast to the general case for distributions, the order of convoluting the type of distributions under consideration is immaterial. Note that the conditions for (V-12b) are not satisfied for $S = 1$, $T = \delta'$, $U = u$.

By the uniqueness of the Fourier transform we know that $\mathcal{L}[T] = 0$ if and only if $T = 0$. Thus (V-12a) shows that, under the conditions of validity of (V-12a), $S*T$ is only zero if one of S or T is zero; that is the algebra of distributions with convolution treated as multiplication, has no zero divisors.

Example V-6:

Since $\mathcal{L}[\delta] = 1$, by (V-4a), then, (V-12a),

$$\mathcal{L}[\delta*T] = \mathcal{L}[T]$$

verifies that δ can be convoluted with every distribution.

Example V-7:

The convolution can be used to obtain results on integration. Thus if S has $\Gamma(S)$ with a point in $\sigma > 0$, we have by (V-5)

$$\mathcal{L}[S(t)*u(t)] = \mathcal{L}\left[\int_{-\infty}^{t} S(\tau)d\tau\right] = \frac{1}{p}\mathcal{L}[S] \qquad \text{(V-13a)}$$

Consequently, in this case, the primitive $\int_{-\infty}^{t} S(\tau)d\tau$ exists and can be found by inverting $p^{-1}\mathcal{L}[S]$. Similarly as for $\mathcal{L}[u(t)]$ of (V-5) one can show that

$$\mathcal{L}[u(-t)] = \begin{cases} \pi\delta(\omega) - PF\left(\frac{1}{j\omega}\right) & \sigma = 0 \\ -1/p & \sigma < 0 \end{cases} \qquad \text{(V-13b)}$$

Consequently, if S has $\Gamma(S)$ with a point in $\sigma < 0$, another primitive results from

$$\mathcal{L}[S(t)*u(-t)] = \mathcal{L}\left[\int_{t}^{\infty} S(\tau)d\tau\right] = -\frac{1}{p}\mathcal{L}[S] \qquad \text{(V-13c)}$$

Example V-8:

As we already know, example IV-1d), $S(t) = t^k \in \mathcal{S}'$. But, in fact $e^{-\sigma t} t^k \epsilon$ ' only for $\sigma = 0$. By (IV-5c) we see that

$$\mathcal{L}[t^k] = 2\pi j^k \delta^{(k)}(\omega), \quad \sigma = 0 \qquad \text{(V-14)}$$

If we let $T(t) = e^{-t}u(t)$, then by example V-4,

$$\mathcal{L}[e^{-t}u(t)] = 1/(p+1), \quad \sigma > -1$$

where $\sigma > -1$ defines $\mathcal{R}(e^{-t}u)$. Consequently, by (II-12b) using (V-12a)

$$\mathcal{L}[t*e^{-t}u(t)] = \frac{2\pi j\delta'(\omega)}{p+1}, \quad \sigma = 0$$
$$= 2\pi j\delta'(\omega) - 2\pi\delta(\omega)$$

This is just the Laplace transform of t-1 and thus, as can also be verified by direct calculation

$$t*e^{-t}u(t) = t-1$$

Example V-9:

If $S = \delta''-\delta$ and $T = e^{-|t|}$, then in the interior of their common strips of convergence, defined by $\sigma_\ell = -1$ and $\sigma_r = 1$, we have

$$\mathcal{L}[(\delta''-\delta)*e^{-|t|}] = (p^2-1)\cdot\frac{-2}{p^2-1} = -2$$

Thus, by the uniqueness of this result,

$$(\delta''-\delta)*e^{-|t|} = -2\delta$$

We comment that $\mathcal{L}[\delta]$ holds for the entire plane and not just $-1 < \sigma < 1$.

Example V-10:

We have, by (II-15b),

$$\delta(t+b)*T(t) = T(t+b)$$

and thus, by (V-4b) and (V-11a),

$$\mathcal{L}[T(t+b)] = e^{bp}\cdot\mathcal{L}[T] \qquad \text{(V-15)}$$

The Laplace transform of the product of two distributions can be defined. Although this yields some useful results it doesn't seem to have been treated in the literature. We can obtain the following result.

Let $e^{-\sigma t}\alpha(t)\epsilon\,\mathcal{O}_M$ for $\sigma\epsilon\Gamma(\alpha) = \Gamma_1$ and $e^{-\sigma t}T(t)\epsilon\,\mathcal{S}'$ for $\sigma\epsilon\Gamma(t) = \Gamma_2$, then $e^{-\sigma t}\alpha T\epsilon\,\mathcal{S}'$ for $\sigma\epsilon\Gamma(\alpha T) = \Gamma_3$ where[18]

$$\sigma_{3\ell} = \sigma_{1\ell}+\sigma_{2\ell}$$

$$\sigma_{3r} = \sigma_{1r}+\sigma_{2r}$$

define Γ_3. From the Fourier transform result, (IV-10a), we get[19]

$$\mathcal{L}_p[\alpha T] = \frac{1}{2\pi}\mathcal{L}[\alpha]\Big|_{\sigma_1} * \mathcal{L}[T]\Big|_{\sigma-\sigma_1} \quad ; \ \sigma_1\epsilon\Gamma_1,\ \sigma\epsilon\Gamma_3 \qquad \text{(V-16a)}$$

where, for example, $\mathcal{L}[\alpha]\Big|_{\sigma_1}$ means that $\mathcal{L}[\alpha]$ is evaluated as a Fourier transform, in ω, of $e^{-\sigma_1 t}\alpha(t)$ for any fixed $\sigma_1\epsilon\Gamma_1$. The convolution on the right of (V-16a) is to be thought of as that in ω of two Fourier transforms with the parameters σ_1 and $\sigma-\sigma_1$ held fixed. If T is a function we can rewrite (V-16a) as

$$\mathcal{L}_p[\alpha T] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty}\alpha(t)e^{-(\sigma_1+j\omega_1)t}dt\Big]\Big[\int_{-\infty}^{\infty}T(t)e^{-\{(\sigma-\sigma_1)+j(\omega-\omega_1)\}t}dt\Big]d\omega_1$$

In this last we can now let $\sigma_1\epsilon\Gamma_1$ vary, define $p_1 = \sigma_1+j\omega_1$, and convert to a complex integration along any path in the convergence strip of $\alpha(t)$, extending from $\omega_1 = -\infty$ to $\omega_1 = +\infty$, as shown in Fig. V-4, to get[20]

$$\mathcal{L}_p[\alpha T] = \frac{1}{2\pi j}\int_{\sigma_1\epsilon\Gamma_1}\mathcal{L}_{p_1}[\alpha]\cdot\mathcal{L}_{p-p_1}[T]dp_1 \qquad \text{(V-16c)}$$

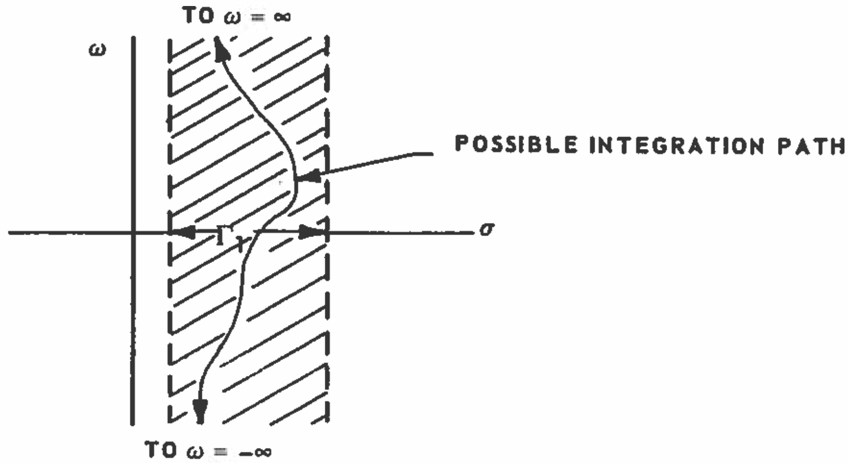


FIG. V-4. POSSIBLE PATHS FOR $\mathcal{L}[\alpha T]$.

Consequently, $\mathcal{L}_p[\alpha T]$ can be considered as a complex convolution, even when T is not a function, this interpretation being valid for (V-16a), if the convolution is interpreted in the distributional sense when the convergence strip for α reduces to a single line. Note that σ_1 completely vanishes from $\mathcal{L}_p[\alpha T]$ after the convolution is carried out.

Example V-11:

For $\sigma_1 = 0$ we have, by (V-14) and (V-16a) as $t^k \epsilon\, \mathcal{O}_M$,

$$\mathcal{L}[t^k T(t)] = j^k \delta^{(k)}(\omega) * \mathcal{L}[T] \; , \quad \sigma \epsilon \Gamma(T)$$

Now for the convolution σ is held constant and then $\delta^{(k)}$ merely differentiates with respect to ω. In a region where $\mathcal{L}[T]$ is holomorphic, with σ held constant, the derivative with respect to $j\omega$ is the same as that with respect to p and we get

$$\mathcal{L}[t^k T(t)] = (-1)^k \frac{d^k \mathcal{L}[T]}{dp^k} \; , \quad \sigma \epsilon \mathring{\Gamma}(T) \qquad \text{(V-17)}$$

This also holds along the boundaries of the convergence strip for T, if we interpret the derivative as being with respect to $j\omega$. As a special case we have

$$\mathcal{L}[t\delta'(t)] = (-1)\frac{dp}{dp} = -1$$

which checks $t\delta'(t) = -\delta(t)$ as given by (II-12b).

Example V-12:

Let $n(t)$ be a function for which $e^{-\sigma t}n(t) \in \mathcal{O}_M$ with $\Gamma(n)$ intersecting the origin and then let

$$n_1(t) = n(t)u(t)$$

Then, by (V-16a),

$$N_1(p) = \mathcal{L}[n_1] = \mathcal{L}[nu] = \frac{1}{2\pi}\mathcal{L}[n]\Big|_{\sigma_1=0} * \frac{1}{p}$$

this being valid for every $\sigma > 0$. Explicitely (V-16b) gives

$$N_1(p) = \frac{1}{2\pi}\int_{\infty}^{\infty} \frac{N(j\omega_1)}{p-j\omega_1}d\omega_1 , \qquad \sigma > 0$$

This gives $N_1(p)$ in terms of the boundary values of $N = \mathcal{L}[n]$. $N_1(p)$ can be thought of as an analytic extension of $N(j\omega)$ into the right half plane. The inverse of $N_1(p)$ is an "antecedal" function, that is, it vanishes for $t < 0$, while this need not be the case for $N(p)$. For instance if

$$n(t) = 1$$

then

$$N(j\omega) = 2\pi\delta(\omega)$$

$$N_1(p) = 1/p$$

and then, as expected from $n_1 = nu$,

$$n_1(t) = u(t)$$

Note that $N(j\omega)$ is not in general the limit of $N_1(p)$ as $\sigma \to 0$, this would be as in (V-5) for $N_1 = 1/p$, but the sum of two such limits; one from the right for $t > 0$ and one from the left for $t < 0$.

Example V-13:

The frequency shift result can be obtained from this convolution. We have $e^{-\sigma t}e^{at} \in \mathcal{O}_M$ for $\sigma = -\text{Re } a$. Then

$$\mathcal{L}_p[e^{at}g(t)] = \frac{1}{2\pi}\mathcal{L}[e^{at}]\Big|_{\sigma_1=\text{Re } a} * \mathcal{L}[g]\Big|_{\sigma-\text{Re } a}$$

$$= \delta(\omega-\text{Im } a)*G([\sigma-\text{Re } a]+j\omega) = G(p-a)$$

where $G(p) = \mathcal{L}_p[g]$. Thus

$$\mathfrak{L}_p[e^{at}g(t)] = \mathfrak{L}_{p-a}[g(t)] \qquad \text{(V-18)}$$

As with the Fourier transform the convolution results can be extended to $\mathfrak{D}'_{L_q}$ spaces. Thus, from theorem IV-2 we immediately see that if $e^{-\sigma t}S \epsilon \mathfrak{D}'_{L_q}$ and $e^{-\sigma t}T \epsilon \mathfrak{D}'_{L_r}$ for a common σ, $1 \leq q \leq 2$, $1 \leq r \leq 2$, then again the (real) convolution of (V-12a) holds for this σ.

If, after multiplication by $\exp[-\sigma t]$, $\alpha(t)$ and $T(t)$ are square integrable functions multiplied by polynomials in t, then the complex convolution of (V-16a) still holds.[21] Such a result has considerable importance for nonlinear system analysis where the following example is of interest.[22]

Example V-14:

Let

$$\alpha(t) = e^{at}u(t)\ , \qquad \mathfrak{L}[\alpha] = 1/(p-a)\ , \qquad \sigma > \text{Re } a$$

$$T(t) = e^{bt}u(t) \qquad \mathfrak{L}[T] = 1/(p-b)\ , \qquad \sigma > \text{Re } b$$

then, by the last comments

$$\mathfrak{L}[\alpha T] = \frac{1}{2\pi} \cdot \frac{1}{p-a} * \frac{1}{p-b} \qquad \sigma > \text{Re } a + \text{Re } b$$

$$= \mathfrak{L}[e^{(a+b)t}u(t)] = \frac{1}{p-(a+b)}$$

This can be checked by writing

$$\alpha T = \beta T;\ \beta = e^{at}$$

and using

$$\mathfrak{L}[\beta T] = \frac{1}{2\pi} \cdot [2\pi\delta(\omega - \text{Re } a)] * \frac{1}{p-b}$$

5. Inversion

Inversion of Laplace transform expressions stems from the fact that $\mathfrak{F}$ and $\overline{\mathfrak{F}}$ are inverses.

Given a Γ and a "Laplace transform", written[23] $F(p)$, defined in the convergence strip determined by Γ, we define the inverse Laplace transform, $\mathfrak{L}^{-1}$, as follows.

Definition V-2:

The inverse Laplace transform of $F(p)$ is defined by

$$\mathcal{L}_t^{-1}[F(p)] = e^{\sigma t}\ \overline{\mathcal{F}}_t[F(\sigma+j\omega)] \tag{V-19}$$

for any $\sigma\epsilon\Gamma$.

Here, as with $\mathcal{L}$ itself, σ is a parameter which actually disappears; $\exp[\sigma t]$ multiplies $\overline{\mathcal{F}}_t$ to get rid of the $\exp[-\sigma t]$ inside of $\mathcal{F}$ in the Laplace transform definition. Suitable restrictions put on $F(p)$ for it to be a Laplace transform are clear from the properties of $\mathcal{L}$, that is:[24]

1) If Γ is a point then $F(p)\epsilon\ \mathcal{S}'$; that is, for this one σ, F considered with ω as the variable is a tempered distribution or
2) If Γ is open then for $\sigma\epsilon\Gamma' \subset \Gamma = \mathring{\Gamma}$, where Γ' is any closed subset of Γ, $F(p)$ must be holomorphic in p and satisfy

   $$|F(p)| < M|p|^m$$

   for non-negative constants M and m (which may vary with Γ' but with m integer), by (V-10).

In the first of these cases $\overline{\mathcal{F}}[F]$ certainly must be in $\mathcal{S}'$, since $\overline{\mathcal{F}}$ maps $\mathcal{S}'$ into $\mathcal{S}'$. Here the result, $\mathcal{L}^{-1}$, need not be independent of the only $\sigma\epsilon\Gamma$, as the following example shows.

Example V-15:

If

$$F(p) = \delta(\omega-\omega_o)\ , \qquad \sigma = \sigma_o$$

then

$$\mathcal{L}_t^{-1}[F] = e^{\sigma_o t}\cdot\ \overline{\mathcal{F}}_t[\delta(\omega-\omega_o)] = e^{\sigma_o t}\frac{e^{j\omega_o t}}{2\pi}$$
$$= \frac{1}{2\pi}e^{p_o t} \qquad ;\ p_o = \sigma_o+j\omega_o$$

In case 2), however, $\mathcal{L}^{-1}$ is independent of σ and any $\sigma\epsilon\Gamma$ can be chosen for performing the inversion.[25] From (IV-4b) we see that when the scalar product is defined

$$\mathcal{L}_t^{-1}[F(p)] = \frac{1}{2\pi} < F(p),\ e^{pt} > , \qquad \sigma\epsilon\Gamma \tag{V-20a}$$

or using the integral representation of this

$$\mathcal{L}_t^{-1}[F(p)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)e^{pt}d\omega\ , \qquad \sigma\epsilon\Gamma \tag{V-20b}$$

That is, for each fixed σ, $\mathcal{L}^{-1}$ can be calculated by integrating over ω along a line of constant σ, for instance as shown in Fig. V-1. By letting σ vary we can convert this to a complex line integral, as was done to get (V-16c), the path being as in Fig. V-4,

$$\mathcal{L}_t^{-1}[F(p)] = \frac{1}{2\pi j}\int_{\sigma\epsilon\Gamma} F(p)e^{pt}dp \qquad \text{(V-20c)}$$

In obtaining this it is necessary to assume that F(p) vanishes at infinity in order to apply Cauchy's integral theorem.[26] Since for $\sigma\epsilon\mathring{\Gamma}$, F(p) is bounded by a polynomial we can divide by p^k for sufficiently large integer k to guarantee this. Consequently, when $\Gamma = \mathring{\Gamma}$, we can always obtain, by (V-7),

$$\mathcal{L}_t^{-1}[F(p)] = \frac{d^k}{dt^k}\{ \frac{1}{2\pi j}\int_{\sigma\epsilon\Gamma} \frac{F(p)}{p^k}e^{pt}dp\} \qquad \text{(V-20d)}$$

with k chosen such that the expression between brackets is a continuous function.[27] This shows that every Laplace transformable distribution is of finite order, being the k th derivative of a continuous function. This is substantiated by the fact, as seen in example V-5-b), that $\sum_{k=0}^{\infty} \delta^{(k)}(t-k)$, which is not of finite order, has no Laplace transform.

Since $\bar{\mathcal{F}}$ is unique and $\bar{\mathcal{F}}[\mathcal{F}[T]]$ is just T itself, we see that $\mathcal{L}^{-1}[F]$ is unique under the two conditions stated above. Further we have, as should now be clear,

$$\mathcal{L}^{-1}[\mathcal{L}[T(t)]] = T(t) \qquad \sigma\epsilon\Gamma(T) \qquad \text{(V-21a)}$$

$$\mathcal{L}[\mathcal{L}^{-1}[F(p)]] = F(p) \qquad \sigma\epsilon\Gamma \qquad \text{(V-21b)}$$

By the uniqueness, any means we have of obtaining $\mathcal{L}^{-1}$, or $\mathcal{L}$, gives the correct answer. Of course the most common way of doing this is by the use of tables.[28] However, the line integral (V-20d) can always be used, and sometimes it can be conveniently evaluated. As is commonly recognized, in the functional case, this evaluation may be carried out by the use of the calculus of residues. Here the theory rests upon Jordan's lemma, which we find in English in only a restricted form.

Before proceeding we quote the most general form of Jordan's lemma.[29]

Jordan's Lemma:

If G(p) is such that there exists a sequence of curves $C_{\ell n}$ and C_{rn}, as shown in Fig. V-5, with their rediitending to infinity and such that

1) $|G(p)|\to 0$ for p on $C_{\ell n}$ and C_{rn} as $n\to\infty$

and

2) G(p) is integrable on $C_{\ell n}$ and C_{rn}

then

$$\int_{C_{\ell n}} G(p)e^{pt}dp \longrightarrow 0 \text{ for } t > 0 \text{ as } n\to\infty \qquad \text{(V-22a)}$$

$$\int_{C_{rn}} G(p)e^{pt}dp \longrightarrow 0 \text{ for } t < 0 \text{ as } n\to\infty \qquad \text{(V-22b)}$$

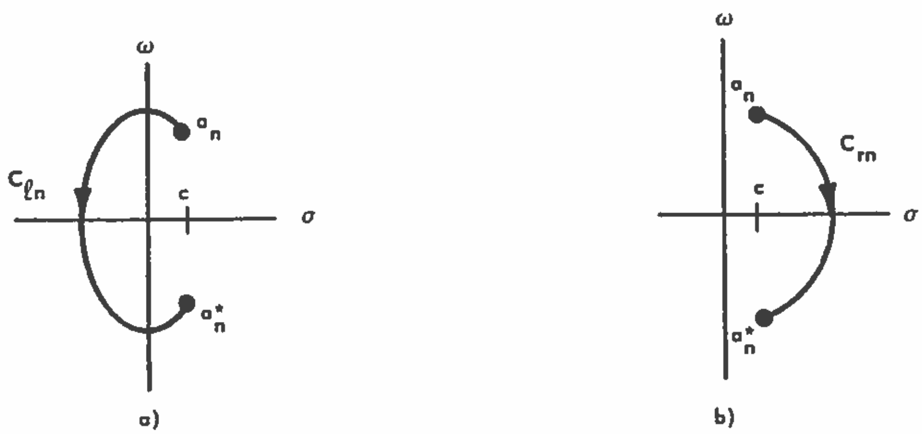


FIG. V-5. CURVES FOR JORDAN'S LEMMA.

In Jordan's lemma the curves can originate in either half plane; that is $c \geq 0$ or $c \leq 0$ is allowed. This can be applied with Cauchy's residue theorem[30] to evaluate $\mathcal{L}^{-1}$. Given a single-valued F(p) we divide by p^k such that

$$G(p) = F(p)/p^k$$

vanishes at infinity and then consider the two closed curves shown in Fig. V-6.

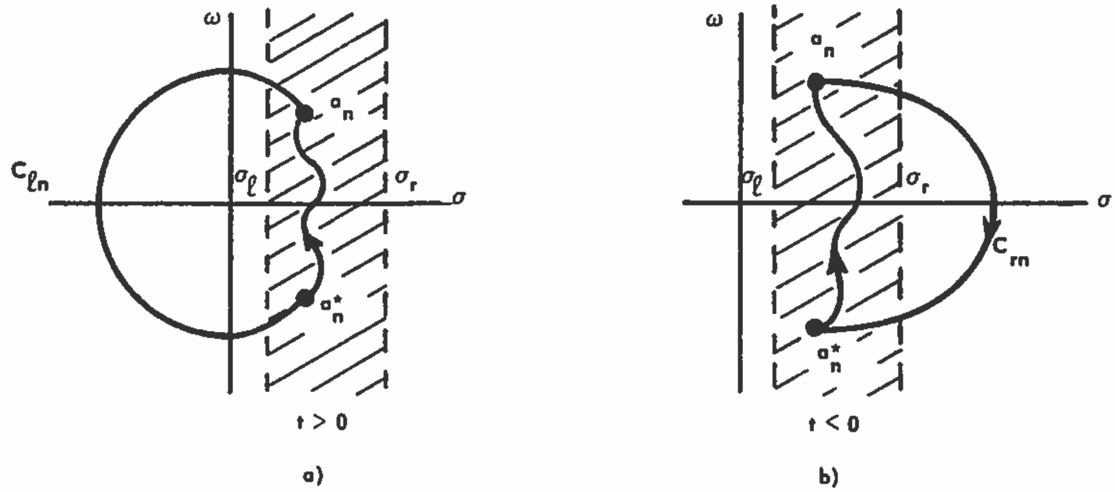


FIG. V-6. INTEGRATION PATHS FOR EVALUATION OF $\mathfrak{L}^{-1}$.

For $t > 0$ we consider the left hand curve and see by Jordan's lemma and Cauchy's integral theorem that the integral from a_n^* to a_n is given by the sum of the residues of $G(p)e^{pt}$ at singularities inside the curve. Similarly for the right hand curve for $t < 0$, except we use the negative of the residues, since we traverse the closed curve in the clockwise direction. Thus we obtain, with $r_{i\ell}(t)$ and $r_{ir}(t)$ the left and right residues of $G(p)e^{pt}$ at finite singularity by letting n (and a_n) tend to infinity,

$$\frac{1}{2\pi j}\int_{\sigma\epsilon\Gamma} \frac{F(p)}{p^k}e^{pt}dp = \sum_{i=0}^{\infty} r_{i\ell}(t)u(t) - \sum_{i=0}^{\infty} r_{ir}u(-t) \qquad \text{(V-23)}$$

$\mathfrak{L}^{-1}$ is then the k th derivative of this.

It should be observed that this form of Jordan's lemma allows an infinite number of singularities, which in fact can be essential. It merely requires that there be some space between singularities in which the curves $C_{\ell n}$ and C_{rn} can be drawn.

Example V-16:

A system has for the Laplace transform of its impulse response

$$F(p) = pe^{1/p} \quad , \qquad \sigma > 0$$

The impulse response can be found by (V-23). We form $G(p) = F(p)/p = e^{1/p}$ and then $G(p)e^{pt}$ has at its essential singularity at zero

$$r_\ell(t) = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!n!}$$

as can be seen by multiplying the series for $e^{1/p}$ and e^{pt}. Thus

$$\mathcal{L}^{-1}[F] = \frac{d}{dt}\{\sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!n!}u(t)\} = \delta(t) + \sum_{n=2}^{\infty} \frac{t^{n-2}}{(n-2)!n!}u(t)$$

NOTES V

1. This process, for functions, is familiar to the generations of engineers raised on the book of Gardner and Barnes, [GA 1, p. 103].
2. See for instance Rehberg [RE 1, p. 38], or Korevaar, [KO 4, p. 386]. In both of these an analytic continuation must be made to show that $\mathcal{L}[\delta] = 1$ can be made to hold for the entire p plane.
3. The set Γ is "convex", that is, given σ_1, $\sigma_2 \epsilon \Gamma$ then $\sigma_3 = x\sigma_1 + (1-x)\sigma_2 \epsilon \Gamma$ for all $0 \leq x \leq 1$, [SC 3, p. 97]. In other words given σ_ℓ and σ_r, as in Fig. V-1, every point between these two lines is in Γ.
4. Thus possible Γ are defined by $\sigma_\ell < \sigma \leq \sigma_r$, $\sigma_\ell \leq \sigma \leq \sigma_r$, $\sigma_\ell < \sigma < \sigma_r$, etc. Actually, σ_r is the supremum of points in Γ while σ_ℓ is the infinum. Strictly Γ defines a "region" only when $\sigma_\ell < \sigma < \sigma_r$, see note V-11.
5. To see that $\mathcal{L}[T] = < T, \exp[-pt] >$ we revert to the similar result for the Fourier transform, (IV-4a), and use the definition of $\mathcal{L}[T]$ with (II-11).

$$\mathcal{L}[T] = \mathcal{F}[e^{-\sigma t}T(t)] = < e^{-\sigma t}T(t), e^{-j\omega t} > = < T(t), e^{-(\sigma+j\omega)t} >$$
$$= < T(t), e^{-pt} >$$

6. $\mathcal{L}[\delta] = 1$ is immediately obtained for all p. One need not first obtain this for $\sigma > 0$ and then analytically continue to the left, as in other theories, [KO 4, p. 386]. Note that the one sided Laplace transform would give zero for $\mathcal{L}[\delta]$ defined by Fig. II-3b) and 1/2 for Fig. II-3a).
7. We have

$$\frac{1}{p} = \frac{\sigma}{\sigma^2+\omega^2} - j\frac{\omega}{\sigma^2+\omega^2}$$

Taking the limit as $\sigma = \frac{1}{n} \to 0$ gives $\mathcal{F}[u]$, since

$$\delta(\omega) = \frac{1}{\pi} \lim_{n\to\infty} \frac{n}{1+(n\omega)^2}$$

This results by integrating the sequence and showing that the integrals converge to u, see [NE 1, p. 9], where this result is proven and used to obtain a Hilbert transform theory.

8. To see that $\mathcal{L}[\exp p_o t] = 2\pi\delta(\omega-\omega_o)$ we note that

$$< \mathcal{F}_x[e^{-\sigma_o t} e^{p_o t}], \psi(x) > = < e^{j\omega_o x}, \mathcal{F}_x[\psi(t)] >$$

$$= < 1, e^{j\omega_o x} \mathcal{F}_x[\psi(t)] >$$

But, by (III-1a),

$$e^{j\omega_o x} \mathcal{F}_x[\psi(t)] = \int_{-\infty}^{\infty} \psi(t) e^{-jx(t-\omega_o)} dt$$

$$= \int_{-\infty}^{\infty} \psi(t+\omega_o) e^{-jxt} dt$$

Thus

$$< \mathcal{F}_x[e^{-\sigma_o t} e^{p_o t}], \psi(x) > = < 1, \mathcal{F}_x[\psi(t+\omega_o)] >$$

$$= < \mathcal{F}_\omega[1], \psi(\omega+\omega_o) >$$

$$< \mathcal{F}_{\omega-\omega_o}[1], \psi(\omega) > \text{ by (II-5b).}$$

Consequently, by (IV-5c)

$$\mathcal{L}[e^{p_o t}] = \mathcal{F}_{\omega-\omega_o}[1] = 2\pi\delta(\omega-\omega_o)$$

9. To see that $\mathcal{L}[T^{(k)}] = p^k \mathcal{L}[T]$ we have

$$< \mathcal{F}[e^{-\sigma t} T^{(k)}], \psi > = < e^{-\sigma\omega} T^{(k)}(\omega), \mathcal{F}_\omega[\psi] >$$

$$= < T^{(k)}(\omega), e^{-\sigma\omega} \mathcal{F}_\omega[\psi] > = (-1)^k < T(\omega), d^k\{e^{-\sigma\omega} \mathcal{F}_\omega[\psi]\}/d\omega^k >$$

$$= < T(\omega), \int_{-\infty}^{\infty} (\sigma+jt)^k \psi(t) e^{-(\sigma+jt)\omega} dt >$$

$$= < e^{-\sigma\omega} T(\omega), \int_{-\infty}^{\infty} (\sigma+jt)^k \psi(t) e^{-j\omega t} dt >$$

$$= < \mathcal{F}_\omega[e^{-\sigma t} T], (\sigma+j\omega)^k \psi(\omega) > = < p^k \mathcal{L}[T], \psi(\omega) >$$

10. We have

$$< \frac{\partial \mathcal{F}_\omega[e^{-\sigma t} T(t)]}{\partial\sigma}, \psi(\omega) > = \frac{\partial}{\partial\sigma} < \mathcal{F}_\omega[e^{-\sigma t} T(t)], \psi(\omega) >$$

$$= \frac{\partial}{\partial\sigma} < e^{-\sigma\omega} T(\omega), \mathcal{F}_\omega[\psi(t)] > = < \frac{\partial}{\partial\sigma}[e^{-\sigma\omega} T(\omega)], \mathcal{F}_\omega[\psi] >$$

$$= < e^{-\sigma\omega}\{-\omega T(\omega)\}, \mathcal{F}_\omega[\psi(t)] > = < \mathcal{F}_\omega[-e^{-\sigma t}\{tT(t)\}], \psi(\omega) >$$

or

$$\frac{\partial \mathcal{L}[T]}{\partial \sigma} = -\mathcal{L}[tT(t)]$$

and
$$< \frac{\partial \mathcal{F}_{\omega}[e^{-\sigma t}T(t)]}{\partial \omega}, \psi(\omega) > = (-1) < \mathcal{F}_{\omega}[e^{-\sigma t}T(t)], \psi'(\omega) >$$

$$= (-1) < e^{-\sigma \omega}T(\omega), \quad \mathcal{F}_{\omega}[\psi'(t)] > = (-1) < e^{-\sigma \omega}T(\omega), (j\omega) \mathcal{F}_{\omega}[\psi(t)] >$$

$$= < \mathcal{F}_{\omega}[-je^{-\sigma t}\{tT(t)\}], \psi(\omega) >$$

or
$$\frac{\partial \mathcal{L}[T]}{\partial \omega} = -j\, \mathcal{L}[tT(t)]$$

Consequently
$$2\frac{\partial \mathcal{L}[T]}{\partial p^*} = \frac{\partial \mathcal{L}}{\partial \sigma} + j\frac{\partial \mathcal{L}}{\partial \omega} = -\mathcal{L}[tT]-j^2\mathcal{L}[tT] = 0$$

Of course the derivatives can only be calculated inside the convergence strip, since on the boundary one can't define the derivative in the manner used in the interior. Observe that p^* can be considered as $p^* = \sigma - j\omega$ in the above expressions.

11. In English "analytic" is used somewhat loosely, thus $F(p) = 1/p$ is usually called an analytic function in the entire plane, even though it has a singularity at $p = 0$. The concept we need excludes any singularities in the region under consideration. Such is that of "holomorphic". Precisely, $F(p)$ is called holomorphic in a region if it has a derivative at each point of the region. Since, for such consideration, a region is defined as an open connected set, a function can only by holomorphic in the interior of a set. For precise concepts, not available in English, concerning complex variable theory, see Valiron, [VA 2, p. 321].

12. See Schwartz, [SC 3, pp. 199 & 202]. This rests upon showing that $e^{-\sigma t}T(t) \epsilon\, \mathcal{O}_c'$ when $\sigma \epsilon \Gamma$, since then, by the definition of $\mathcal{O}_c'$, we know the Fourier transform will be in $\mathcal{O}_M$. The reason for $e^{-\sigma t}T(t) \epsilon\, \mathcal{O}_c'$ is not, as yet, completely clear to us.

13. For each $\sigma \epsilon \Gamma' \subset \Gamma$ we have, since $\mathcal{L}[T] \epsilon\, \mathcal{O}_M$ as a function of ω (for each σ),
$$|d^k\mathcal{L}[T]/d\omega^k| < M_\sigma |\omega|^{n_\sigma} \leq M_\sigma |\sigma + j\omega|^{n_\sigma}$$

Taking M and m to be the maximum values (m is integer) of M_σ and n_σ over all $\sigma\epsilon\Gamma'$ gives the result. M and n are finite since we limit to a closed subset of $\mathfrak{P}$.

14. The $\mathcal{D}'_{L_q}$ spaces were discussed in note-IV-20.

15. See the companion report, [NE 2], for a proof of the fact that any passive network, if it is linear, time-invariant and solvable (i.e. somewhat continuous), has a scattering matrix which is holomorphic in the open right half plane.

The proof of theorem V-1 is as follows. We first consider $T\epsilon L_q \subset \mathcal{D}'_{L_q}$, $1 \leq q \leq 2$. Then if $\sigma = 0$ we know $\mathfrak{L}[T]$ exists, since the Fourier transform does, by theorem IV-2. If $\sigma > 0$ then we have

$$[\int_0^\infty |T(t)|e^{-\sigma t}dt]^q \leq [\int_0^\infty |T(t)|^q dt]\cdot[\int_0^\infty e^{-(\frac{q}{q-1})\sigma t}dt]^{q-1}$$

by Hölder's inequality, [BU 1, p. 65], and the support constraint on T. But the right side of this is finite, since $T\epsilon L_q$ and $\sigma > 0$, and thus $\mathfrak{L}[T]$ exists as an absolutely convergent integral. Since $\mathfrak{L}[T]$ exists in $\sigma > 0$ it is holomorphic in this region, as shown below (V-9).

Now if $T\epsilon\ \mathcal{D}'_{L_q}$ then

$$T = \sum_{i=1}^{m} T_i^{(k)}, \qquad T_i \epsilon L_q$$

by the definition of $\mathcal{D}'_{L_q}$. Then, by (V-7),

$$\mathfrak{L}[T] = \sum_{i=1}^{m} p^k \mathfrak{L}[T_i]$$

which is then holomorphic in $\sigma > 0$, being the sum of such functions. This also exists on $\sigma = 0$. Q.E.D.

16. We have

$$< e^{-\sigma t}(S(t)*T(t)), \psi(t) > = < S(t)*T(t), e^{-\sigma t}\psi(t) >$$

$$= < S(t), < T(\tau), e^{-\sigma(t+\tau)}\psi(t+\tau) > >$$

$$= < S(t), e^{-\sigma t} < e^{-\sigma\tau}T(\tau), \psi(t+\tau) > >$$

$$= < e^{-\sigma t}S(t), < e^{-\sigma\tau}T(\tau), \psi(t+\tau) > >$$

$$= < (e^{-\sigma t}S(t))*(e^{-\sigma t}T(t)), \psi(t) >$$

or

$$e^{-\sigma t}(S(t)*T(t)) = (e^{-\sigma t}S(t))*(e^{-\sigma t}T(t))$$

17. We have by (IV-11a)

$$\mathcal{F}[e^{-\sigma t}(S*T)] = \mathcal{F}[(e^{-\sigma t}S)*(e^{-\sigma t}T)]$$

$$= \mathcal{F}[e^{-\sigma t}S]\cdot\mathcal{F}[e^{-\sigma t}T]$$

We also assume that T is the distribution whose strip is intersected in the interior, if it isn't we can relabel.

18. We have

$$< e^{-\sigma t}\alpha T, \psi > = < e^{-(\sigma-\sigma_1)t}T(t), e^{-\sigma_1 t}\alpha(t)\psi(t) >$$

$$= < e^{-(\sigma-\sigma_1)t}T(t), \varphi_o(t) >$$

Since $e^{-\sigma_1 t}\alpha(t)\psi(t)\epsilon\,\mathcal{S}$ with $\psi\epsilon\,\mathcal{S}$. Thus the smallest σ that will yield $e^{-\sigma t}\alpha T\epsilon\,\mathcal{S}$ for all possible $\sigma_1\epsilon\Gamma_1$ and $\sigma_2\epsilon\Gamma_2$ is given by $\sigma-\sigma_1 = \sigma_{2\ell}$ or

$$\sigma_{3\ell} = \sigma_{1\ell}+\sigma_{2\ell}$$

Similarly for $\sigma_{3r} = \sigma_{1r}+\sigma_{2r}$

19. To see that $\mathcal{L}_p[\alpha T] = \frac{1}{2\pi}\mathcal{L}[\alpha]\Big|_{\sigma_1} * \mathcal{L}[T]\Big|_{\sigma-\sigma_1}$ we write

$$\mathcal{F}_\omega[e^{-\sigma t}\alpha T] = \mathcal{F}_\omega[\{e^{-\sigma_1 t}\alpha(t)\}\cdot\{e^{-(\sigma-\sigma_1)t}T(t)\}]$$

$$= \frac{1}{2\pi}\mathcal{F}_\omega[e^{-\sigma_1 t}\alpha(t)]*\mathcal{F}_\omega[e^{-(\sigma-\sigma_1)t}T(t)]$$

by (IV-10a).

20. To obtain this complex integration we can apply Cauchy's integral theorem, [KN 1, p. 48], that is if F(p) is holomorphic in a region R then for any closed path C in R

$$\oint_C F(p)dp = 0$$

Applying this to the region defined by Γ_1 requires that the integrand vanish at infinity, the path being as shown in Fig. N-V-1

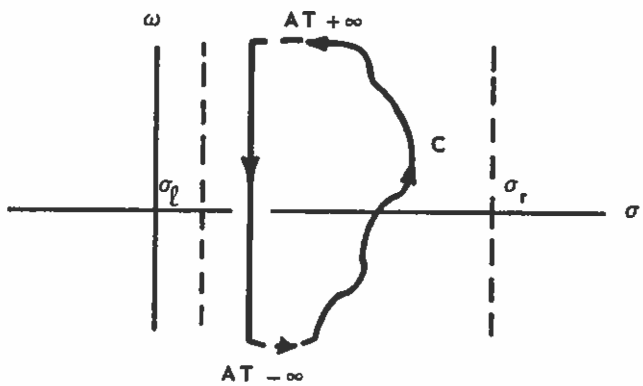


FIG. N-V-1. CLOSURE OF PATHS AT INFINITY.

21. To see that $\mathfrak{L}_p[\alpha T] = \frac{1}{2\pi}\mathfrak{L}[\alpha]*\mathfrak{L}[T]$ when $e^{-\sigma t}\alpha$, $e^{-\sigma t}T$ are functions in L_2 multiplied by polynomials in t, we first note that if α, $T \epsilon L_2$ then by the classical theory, [BO 1, p. 112], which still remains valid for distribution, α and T are the Fourier transforms of two L_2 functions. If α and T are L_2 functions multiplied by polynomials (in t), then, since time multiplication is equivalent to frequency differentiation, α and T are (inverse) Fourier transforms of distributions in $\mathfrak{D}'_{L_2}$, see the theorem IV-2. Thus

$$\alpha(t) = \overline{\mathfrak{F}}_t[A(\omega)] , \quad A\epsilon \ \mathfrak{D}'_{L_2}$$

$$T(t) = \overline{\mathfrak{F}}_t[B(\omega)] , \quad B\epsilon \ \mathfrak{D}'_{L_2}$$

Note that then $A*B\epsilon \ \mathfrak{D}'_{L_\infty}$ by theorem IV-2. Also by theorem IV-2, which can also be written from (IV-11) as

$$\overline{\mathfrak{F}} [A*B] = 2\pi \overline{\mathfrak{F}} [A]\cdot \overline{\mathfrak{F}} [B]$$

we see that

$$\alpha(t)\cdot T(t) = \frac{1}{2\pi} \overline{\mathfrak{F}} [A*B] = \frac{1}{2\pi} \overline{\mathfrak{F}} [\mathfrak{F}[\alpha]* \mathfrak{F}[T]]$$

Taking the Fourier transform immediately gives

$$\mathfrak{F}[\alpha \cdot T] = \frac{1}{2\pi} \mathfrak{F}[\alpha] * \mathfrak{F}[T]$$

Inserting the exponential convergence factors in the proper places again gives (V-16a).

22. See Weber, [WE 1, p. 414], where the complex convolution is used to solve nonlinear differential equations.
23. We put quotes on "Laplace transform" since F(p) must satisfy certain restrictions to be a Laplace transform.
24. If Γ is closed on one or both sides and contains more than one point, it appears, but does not seem definitely known, that as σ approaches σ_ℓ or σ_r, F(p) must tend to $F(\sigma_\ell + j\omega)$ or $F(\sigma_r + j\omega)$ respectively, in the distributional sense. This was certainly the case in (V-5).
25. The proof that $\mathfrak{L}^{-1}$ is independent of σ is given in [SC 3, p. 202] and is as follows. Since F(p) is holomorphic, we have, see (V-9a),

$$\frac{\partial F(p)}{\partial p^*} = 0$$

and then, from (V-19)

$$\frac{\partial \mathfrak{L}_t^{-1}}{\partial \sigma}[F] = te^{\sigma t}\, \overline{\mathfrak{F}}_t[F(p)] + e^{\sigma t}\, \overline{\mathfrak{F}}_t[\frac{\partial F}{\partial \sigma}] \text{ by (V-19)}$$

$$= te^{\sigma t}\, \overline{\mathfrak{F}}_t[F(p)] - je^{\sigma t}\, \overline{\mathfrak{F}}_t[\frac{\partial F}{\partial \omega}] \text{ by } \partial F/\partial p^* = 0$$

$$= te^{\sigma t}\, \overline{\mathfrak{F}}_t[F] - j(-j)te^{\sigma t}\, \overline{\mathfrak{F}}_t[F] \text{ by (IV-9b)}$$

$$= 0$$

That is $\mathfrak{L}_t^{-1}$ is constant as far as σ is concerned.

26. See note V-20.
27. That we can guarantee $\int \frac{F(p)}{p^k} e^{pt} dp$ to be a function is clear from the following.

F(p) considered in ω is in $\mathcal{O}_M$, and thus its inverse, being a Fourier transform, is in $\mathcal{O}_c'$. But distributions in $\mathcal{O}_c'$ are characterized as being finite sums of derivatives of continuous functions, see above (IV-11). Dividing F(p) by p^k removes these derivatives when k is chosen equal to the largest value of "derivation".

28. For two sided tables see Van der Pol and Bremmer, [VA 1]; for an extensive set of one sided tables of distributions and "pseudo-functions" see the small but excellent book of Lavoine, [LA 1].
29. See Doetsch, [DO 1, pp. 224, 272] for the proof of Jordan's lemma.
30. See [KN 1, p. 130] for a proof of Cauchy's residue theorem which is: If H(p) is single valued and analytic inside and on the simple closed curve C except for a finite number of singularities $p_1, p_2, \ldots, p_n$ inside C, then

$$\frac{1}{2\pi j}\oint H(p)dp = \sum_{k=1}^{n} r_k$$

where r_k is the residue of F(p) at p_k.

In our case each path enclosed only a finite number of singularities, even though $G(p)e^{pt}$ may have an infinite number. The single valued constraint puts a limitation on the use of the "residue" method of evaluating (V-20d).

VI. CONCLUSIONS

In the preceding pages the classical theory of Laplace transforms has been generalized, following the ideas of Schwartz, to allow the rigorous treatment of distributions.

As a preliminary to this it was found necessary to review the theory of distributions, Section I, giving the basic definitions and principle results. Some of the important results, such as theorem II-4, seem to be unavailable in English. With this in hand the Laplace transform could be defined by the use of the Fourier transform. Thus Section III gives the most elementary definition and properties of functional Fourier transforms, while Section IV treats the distributional case merely by working in a dual space.

The Laplace transform in the distributional case is again seen to be able to handle distributions which can not be treated by the Fourier transform, even though the (distributional) Fourier transform will now handle such functions as the unit step function. The main properties of the Laplace transform $\mathcal{L}$ are: 1) in every closed interior subset of its convergence strip, it is bounded in magnitude by a polynomial, 2) it is holomorphic in the interior of the convergence strip and reduces to the Fourier transform when the latter exists, 3) convolution and multiplication transform into each other as well as differentiation and variable multiplication (in both t and p), 4) inversion can always be converted to functional inversion, if the convergence strip is open.

In working with functions it is sometimes convenient to write the bilateral Laplace transform as the sum of two unilateral transforms, [WI 1, p. 237],

$$\mathcal{L}[f] = \int_{-\infty}^{\infty} f(t)e^{-pt}dt = \int_{0}^{\infty} f(t)e^{-pt}dt + \int_{0}^{\infty} f(-t)e^{pt}dt$$

The tables and properties which are well-known for unilateral transforms can then be carried over to the bilateral case. In the distributional theory, however, such a decomposition is out of place and can lead to erroneous answers. For instance letting $f(t) = \delta(t)$ then, since $\delta(t) = \delta(-t)$, the right side would yield two for many people, while the left would yield unity. Consequently, this splitting has been avoided in the text even though it is useful in many situations.

Except for the result of (V-16a), most of the material can be found in the literature, in scattered form. However, many of the concepts are illustrated by examples, which though quite simple, can only be properly treated by the distributional theory.

In the preceeding we have given a special case of the theory presented by Schwartz. Actually Schwartz treats the situation where the variables, t, p. etc., lie in n-dimensional spaces. Such a theory is of great importance in electromagnetic theory and nonlinear systems synthesis. However, most of the theoretical results are simple extensions of those given here; examples, though, are harder to carry out. For instance the relation of (II-2a) involves an n-dimensional integral with t replaced by an n-vector $\underline{t}$. The differentiation of (II-7) becomes a partial derivative and the Laplace transform of (V-3b) becomes

$$\mathcal{L}[T] = \int_{-\infty}^{\infty} T(\underline{t})e^{-\underline{p}\cdot\underline{t}}d\underline{t}$$

where $\underline{p}$ and $\underline{t}$ are n-vectors with $\underline{p}\cdot\underline{t}$ denoting the scalar product. Since this more general theory is easily understood once the one-dimensional case is mastered, we have limited the theory to the latter.

APPENDIX - SPACES

Because of the need for stating precisely the conditions under which various operations can be performed, many spaces have necessarily been introduced in the theory. Since the presence of some of these may lead to confusion, we append this section as an attempt to clarify some of the relationships.

The most basic space is $\mathfrak{D}$, the space of testing functions. As can be appreciated from section II, numerical calculations are seldom made with testing functions, which are infinitely differentiable functions taking the value zero outside of a bounded range of the variable. However, typical testing functions can be exhibited, such is $\varphi(t) = \{\exp[\frac{1}{t-1} + \frac{1}{2-t}]\}u(t-1)u(2-t)$. The real use for $\mathfrak{D}$ is in defining its (topological) dual $\mathfrak{D}'$, the space of all distributions. Thus between any testing function φ and any distribution T there is a "scalar product" defined, $< T,\varphi >$, which has the following properties, these being the same for the φ's as for the T's:

$$< T, \alpha\varphi_1+\beta\varphi_2 > = \alpha < T, \varphi_1 > +\beta < T, \varphi_2 >$$

$$< \alpha T_1+\beta T_2, \varphi > = \alpha < T_1, \varphi > +\beta < T_2, \varphi >$$

$$\lim_{n\to\infty} < T, \varphi_n > = 0 \text{ if } \varphi_n \text{ converge to zero in } \mathfrak{D}$$

$$\lim_{n\to\infty} < T_n, \varphi > = 0 \text{ if } T_n \text{ converges to zero in } \mathfrak{D}'$$

Since $\mathfrak{D}'$ contains all locally integrable functions, it contains $\mathfrak{D}$ as a subspace. However $\mathfrak{D}'$ contains distributions which are not locally integrable functions, such as impulses and doublets.

In order to work with the Fourier transform, one is forced to restrict considerations to special distributions, those forming the space $\mathcal{S}'$ of tempered distributions. To see why this must be the case we note that to define $\mathcal{F}[T]$ we would wish to have

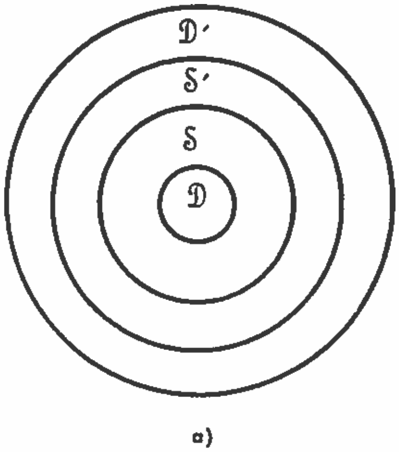
$$< \mathcal{F}[T], \varphi > = < T, \mathcal{F}[\varphi] >$$

which is seen by applying $\mathcal{F}$ to functions and using (II-2a). But, although $\mathcal{F}[\varphi]$ can be defined for all testing functions φ, $< T, \mathcal{F}[\varphi] >$

can not generally be defined for any distribution T, since it can not even be defined for $T = \exp[t^2]$. The most convenient way of obtaining the desired properties of the tempered distributions is to introduce the space $\mathcal{S}$ of good functions. The good functions are somewhat like the testing functions, being infinitely differentiable, but they need not have bounded support (they and their derivatives must vanish at infinity faster than the reciprocal of any polynomial). A typical good function is $\psi = \exp[-t^4]$.

The space $\mathcal{S}'$ is then obtained as the (topological) dual of $\mathcal{S}$, that is by extending the scalar product such that $< T, \psi >$ is defined for every good function ψ. Typical tempered distributions are the impulse and all its derivatives as well as the square integrable functions. However $T = e^t u(t)$ is not tempered. As a consequence of this latter example, a more general class of distributions, those tempered in a region, $\mathcal{S}'(\Gamma)$ is introduced. These are defined by requiring $e^{-\sigma t}T(t)$ to be tempered for some σ, $\sigma \epsilon \Gamma$, such that they possess a Laplace transform.

Since the good functions need not be zero outside a bounded set, $\mathcal{S}$ contains the space $\mathfrak{D}$ of testing functions as a subset. Since $\mathcal{S}$ is also contained in $\mathcal{S}'$, we can show the relation between the basic spaces by Fig. A-1.



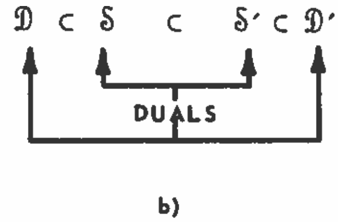


FIG. A-1. CONTAINMENT OF SPACES.

Several other spaces have been introduced. Since multiplication of distributions can't always be carried out, the space of fairly good functions, $\mathcal{O}_M$, and their Fourier transforms, the rapidly decreasing distributions, $\mathcal{O}_c'$, were introduced. Thus any tempered distribution, T, can be multiplied by any fairly good function, α, and convoluted with any rapidly decreasing distribution, S, while still preserving the relations

$$\mathcal{F}[\alpha T] = \mathcal{F}[\alpha] * \mathcal{F}[T]$$

$$\mathcal{F}[S*T] = \mathcal{F}[S] \cdot \mathcal{F}[T]$$

Although $\mathcal{O}_M$ and $\mathcal{O}_c'$ are not duals, one has $\mathcal{D} \subset \mathcal{S} \subset \mathcal{O}_M$, $\mathcal{O}_c' \subset \mathcal{S}' \subset \mathcal{D}'$.

Since one sometimes wishes to convolute distributions not in $\mathcal{O}_c'$ and still preserve the above relations for the Fourier transform, one introduces the $\mathcal{D}'_{L_q}$ spaces. Thus $\mathcal{D}'_{L_q}$ is the space of finite sums of derivatives of L_q functions and theorem IV-2 holds. Also one has $L_q \subset \mathcal{D}'_{L_q}$, $\mathcal{O}'_c \subset \mathcal{D}'_{L_q} \subset \mathcal{D}'_{L_r} \subset \mathcal{D}'$ if $q \leq r$.

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