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A hysteretic RC line for solitary pulses

1. INTRODUCTION

Solitary waves, of which solitons are special cases, are wave packets which have constant wave form and velocity [1,p.2]. Consequently they hold some promise as information carriers in electronic systems. For example one can conceive of pulse signal processing systems in very large scale integrated circuits, VLSI, where the pulse signals are solitary waves. Since we have previously shown the convenience of hysteresis in the generation of pulses of importance to solitary wave systems [2], it is of interest to investigate transmission systems based upon hysteresis that can support solitary wave pulses. Here we show that through the use of distributed shunt "broken-sloping" hysteresis a resistor-capacitor, RC, line can support solitary pulse waves. In doing this we develop the theory for such lines with design criteria and then show by numerical example that the design criteria can be met, thus proving the desired existence.

2. THE LINE AND ITS OPERATION

The transmission line to be considered will be taken as having one space dimension, of coordinate x. At any point x along the line Fig.1a shows the assumed equivalent circuit to be used to represent a segment of the line of differential length dx; for the present we will assume the line to extend uniformly over $-\infty<x<\infty$. Figure 1b shows the i-v characteristic assumed for the nonlinear resistive element shunting the capacitor. The characteristic of Fig.1b we will call "broken-sloping" hysteresis, this designating the fact that it consists of two broken linear portions. However, Fig.1b has three specializations that we would not ascribe to all sloping hystereses, these being that (1) one portion, having positive slope, passes through the origin and is zero to the left of it, (2) the other portion has an endpoint at the voltage origin, (3) the slopes of both nonzero portions are identical. Specialization (1) is in order to obtain a zero resting state, specialization (2) is to ensure an eventual return to the zero resting state while (3) is to simplify the rather extensive calculations which follow in Section 3.

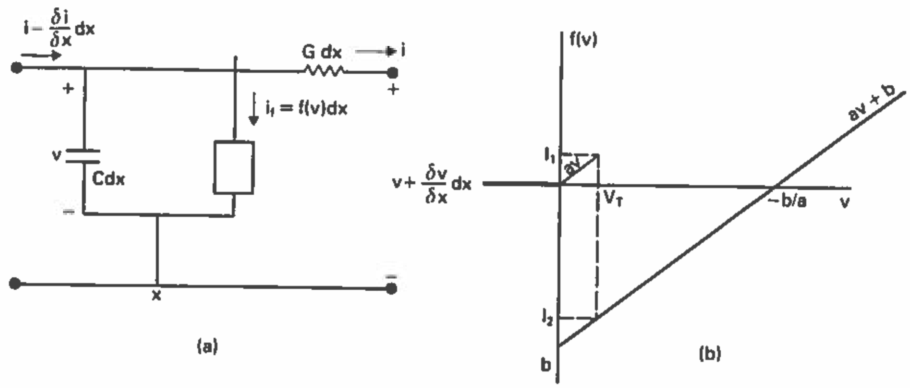

Figure 1. Hysteretic RC transmissions line
(a) Differential section equivalent circuit. (b) Broken-sloping hysteresis assumed $a > 0$, $b < 0$, $V_T > 0$.

The philosophy of operation is as follows. Prior to the arrival of a solitary pulse at position x, the differential section at x rests at zero voltage and current; hence the hysteretic nonlinearity rests in a stable manner on its upper branch at the origin. When a positive pulse arrives, the current i_f increases, drawing energy from previous sections. The increase in voltage at x charges the capacitor C whose dynamics ensures smooth changes in the voltage. However, if the incoming pulse contains sufficient energy, the voltage at x will reach the threshold value, V_T of Fig.1b, in which case the current i_f jumps to the value $I_2 < 0$ on the lower hysteresis branch. The hysteresis element then acts as a current source to excite succeeding sections while, if the solitary input pulse simultaneously begins to decay, the voltage v begins to decrease. When v reaches zero the current i_f again jumps from $b < 0$ to zero and the differential section returns to rest, having passed the solitary pulse on to succeeding sections.

As this description indicates, the system requires careful design to ensure appropriate matching of voltage levels and timing. For this reason we develop the mathematics of the systems operation in the next section.

3. MATHEMATICS OF THE PULSE DESIGN

Let $v(x,t)$ and $i(x,t)$ be the voltage and current at position x and time t on a line having the differential section of Fig.1a. We will assume that the broken-sloping hysteresis can be adequately described by the characterization

$$f(v) = \begin{cases} 0 & \text{if } v(x,t) \le 0 \\ av \text{ (upper branch)} & , 0 \le v(x,t) < V_T \text{ and } f(v(x,t\_)) \ge 0 \\ av + b \text{ (lower branch)} & , 0 < v(x,t) < V_T \text{ and } f(v(x,t\_)) < 0 \\ & \text{or } V_T \le v(x,t) < \infty \end{cases} \tag{3.1}$$

where $a > 0$, $b < 0$ and $V_T > 0$.

By writing Kirchhoff's Laws we have

$$C \frac{\partial v}{\partial t} + f(v) = -\frac{\partial i}{\partial x}, \quad i = -G \frac{\partial v}{\partial x}. \tag{3.2ab}$$

Since, by definition, solitary waves are travelling waves whose shapes remain constant, we assume

$$v(x,t) = v(ct - x) = V(y), \quad y = ct - x, \tag{3.3a}$$

$$i(x,t) = I(y), \tag{3.3b}$$

where c is the velocity of the wave. Using a prime to denote differentiation with respect to y, and substituting (3.2b) into (3.2a) we get

$$V'' - \frac{cC}{G} V' - \frac{1}{G} f(V) = 0. \tag{3.4}$$

To simplify matters we can next scale impendances, to bring $G = 1$, and scale time so that $cC = 1$. Assuming from now on, unless otherwise stated, that $V \ge 0$, the scaled version of (3.4) is

$$V'' - V' - aV = \varepsilon b, \quad \varepsilon = \begin{Bmatrix} 0 \text{ on upper branch} \\ 1 \text{ on lower branch} \end{Bmatrix}. \tag{3.5}$$

On both branches of the hysteresis the same characteristic polynomial results, this being

$$P(s) = s^2 - s - a = (s - s_1)(s - s_2) \tag{3.6}$$

where

$$s_1 = \frac{1}{2}\left[1 + \sqrt{1 + 4a}\right] = \alpha + \beta > 0, \quad s_2 = \frac{1}{2}\left[1 - \sqrt{1 + 4a}\right] = \alpha - \beta < 0. \tag{3.7ab}$$

For future reference we note

$$s_1 + s_2 = 1 = 2\alpha, \quad s_1 s_2 = -\alpha, \quad s_1 - s_2 = \sqrt{1+4a} = 2\beta, \tag{3.8abc}$$

$$0 < -s_2 < s_1 < s_1 - s_2. \tag{3.9}$$

The solution to (3.5) takes the form

$$V(y) = \begin{cases} V_u(y) = A_1 e^{s_1 y} + A_2 e^{s_2 y} & \text{on upper branch} \\ V_\ell(y) = \hat{A}_1 e^{s_1 y} + \hat{A}_2 e^{s_2 y} - \frac{b}{a} & \text{on lower branch} \end{cases} \tag{3.10}$$

where the A_1, A_2, $\hat{A}_1$, $\hat{A}_2$ are determined by initial conditions for (3.5). On the upper branch

$$\begin{bmatrix} V(0) \\ V'(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \frac{-1}{2\beta} \begin{bmatrix} s_2 & -1 \\ -s_1 & 1 \end{bmatrix} \begin{bmatrix} V(0) \\ V'(0) \end{bmatrix} \tag{3.11ab}$$

Consider at t=0, x=0, a rising pulse being formed from the resting state. Then

$$V(0) = 0, \quad V'(0) > 0 \tag{3.12}$$

and (3.11ab) gives

$$A_1 = -A_2 = \frac{1}{s_1 - s_2} V'(0). \tag{3.13}$$

Since $A_1 > 0$ and $s_1 > 0$, $s_2 < 0$, $V(y) = V_u(y)$ will eventually increase with y (as seen from (3.10); that is, at a given x, v(x,t) will eventually increase with t. Hence there is some $y = y_T$ such that the threshold voltage V_T will be reached with positive slope, that is $V(y_T) = V_u(y_T) = V_T, v_u'(y_T) > 0$.

$$V(y_T) = V_u(y_T) = A_1(e^{s_1 y_T} - e^{s_2 y_T}) = V_T. \tag{3.14}$$

Since A_1 is known from (3.13), in principle y_T can be determined when a and V_T are specified, but for design purposes we will find it more convenient to specify y_T. In any event switching to the lower hysteresis branch necessarily takes place at $y = y_T$ at which y there is a jump in I_f from I_1 to I_2,

$$I_f \text{ jump} = I_2 - I_1 = b. \tag{3.15}$$

Since this is a finite jump, no impulses of current occur and the presence of the capacitor consequently forces V(y) to be continuous at $y = y_T$. Further, it shows that there is at most a jump in I' in which case I(y) is continuous at y_T, the implication of which is that V'(y) is also continuous at y_T, as seen from (3.2b). In summary,

$$A\_(e^{s_1 y_T} - e^{s_2 y_T}) = V_u(y_T) = V_T = V_\ell(y_T) = \hat{A}_1 e^{s_1 y_T} + \hat{A}_2 e^{s_2 y_T} - \frac{b}{a}, \tag{3.16}$$

$$A_1(s_1 e^{s_1 y_T} - s_2 e^{s_2 y_T}) = V_u'(y_T) = V_\ell'(y_T) = \hat{A}_1 s_1 e^{s_1 y_T} + \hat{A}_2 s_2 e^{s_2 y_T}. \tag{3.17}$$

These two equations determine $\hat{A}_1$ and $\hat{A}_2$, given the other parameters for the system. From (3.16),

$$\hat{A}_2 = (V_T + \frac{b}{a})e^{-s_2 y_T} - \hat{A}_1 e^{(s_1-s_2)y_T} \tag{3.18}$$

while substitution of (3.18) into (3.17), with replacement of V_T by (3.14) gives

$$\hat{A}_1 = A_1 - \frac{s_2}{s_1-s_2}\,\frac{b}{a}\,e^{-s_1 y_T} \tag{3.19}$$

which, on substitution in (3.18), gives

$$\hat{A}_2 = -A_1 + \frac{s_1}{s_1-s_2}\,\frac{b}{a}\,e^{-s_2 y_T}. \tag{3.20}$$

Since $A_1 > 0$, $\hat{A}_2 < 0$ but $\hat{A}_1$ may be positive or negative. Considering $V_\ell(y)$ in (3.10) we see that the voltage will certainly reach zero if $\hat{A}_1 < 0$, which is

$$\frac{s_2}{s_1-s_2}\,\frac{b}{a}\,e^{-s_1 y_T} > A_1. \tag{3.21}$$

By the choice of a small enough, (3.21) can be obtained and under it we are insured that there is a y_0 such that there is a return to zero voltage on the lower hysteresis branch, that is,

$$V(y_0) = V_\ell(y_0) = 0 = \hat{A}_1 e^{s_1 y_0} + \hat{A}_2\, e^{s_2 y_0} - \frac{b}{a}\,. \tag{3.22}$$

Even if (3.21) is not true we possibly may achieve $V_\ell(y_0) = 0$, but in any event (3.21) is a guarantee. Again (3.22) may be solved for y_0, given the other parameters, but for design purposes it is convenient to note that choosing y_0 can lead to the avoidance of solving transcendental equations. Consequently, at this point we use (3.22) in conjunction with (3.19)-(3.20) to determine

$$A_1 = \frac{1}{s_1-s_2}\,\frac{b}{a}\,\frac{[s_1(1-e^{s_2(y_0-y_T)})-s_2(1-e^{s_1(y_0-y_T)})]}{e^{s_1 y_0} - e^{s_2 y_0}} \tag{3.23}$$

The condition for A_1 to be positive can be rewritten from (3.23), by noting that the bracketed term must be negative, to be

$$\frac{1-e^{s_1(y_0-y_T)}}{1-e^{s_2(y_0-y_T)}} < \frac{s_1}{s_2}\,. \tag{3.24}$$

At $y = y_0$ we switch to the upper hysteresis branch. On it $V_u(y_0) = 0$

by continuity though the derivative may be nonzero. Since our desire is to have the system return to the resting state of zero, we next require $V'_u(y_0) \leq 0$, which by continuity of $V'(y_0)$ means

$$0 \geq V'_\ell(y_0) = \hat{A}_1 s_1 e^{s_1 y_0} + \hat{A}_2 s_2 e^{s_2 y_0}. \tag{3.25}$$

Again expressing this in terms of (3.10)-(3.20) and using (3.23) gives

$$0 \geq V'_\ell(y_0) = \frac{b}{a}\left[\frac{s_1 e^{s_1 y_0}(1-e^{s_2(y_0-y_T)}) - s_2 e^{s_2 y_0}(1-e^{s_1(y_0-y_T)})}{e^{s_1 y_0} - e^{s_2 y_0}}\right] \tag{3.26}$$

which is

$$\frac{s_1}{s_2} < e^{(s_2-s_1)y_0}\left[\frac{1-e^{s_1(y_0-y_T)}}{1-e^{s_2(y_0-y_T)}}\right]. \tag{3.27}$$

Since $V'_\ell(y_T) = 0$, the maximum of $V_\ell(y)$ is at $y = y_T$ and the pulse monotonically decreases.

When (3.24) and (3.27) are simultaneously satisfied the response has $I_f(y_{0+}) = 0$ with $V(y_{0+}) \leq 0$. Hence, by (3.1) I_f will remain at zero and (3.2ab) becomes $V'(y_{0+}) = V(y_{0+}) \equiv 0$, in which case the system returns to rest at the origin. The realizability constraint for the existence of a solitary pulse wave can therefore be expressed as

$$\frac{1-e^{s_1(y_0-y_T)}}{1-e^{s_2(y_0-y_T)}} < \frac{s_1}{s_2} < e^{(s_2-s_1)y_0}\left[\frac{1-e^{s_1(y_0-y_T)}}{1-e^{s_2(y_0-y_T)}}\right] < 0. \tag{3.28}$$

In conclusion, when (3.28) is satisfied we can design our system by choosing $V'(0) = (s_1-s_2)A_1$ such that a solitary pulse wave is guaranteed. The velocity of propagation of the pulse is given, from our normalization of $cC = 1$, by

$$c = 1/C. \tag{3.29}$$

Thus, we can also design for a given pulse velocity by appropriate choice of C.

4. A DESIGN EXAMPLE

We normalize time and admittance level to choose $cC = G = 1$. Let it be desired to have the pulse peak time, y_T, and the pulse turn-off time, y_0, chosen as

$$y_T = 1, \ y_0 = 2 \tag{4.1}$$

these being true (normalized) times at $x = 0$ for a pulse that is zero before $y = 0$ and after $y = y_0$. We are free to choose a. For numerical convenience, set

$$a = 4/9. \tag{4.2}$$

Then $P(s) = s^2-s-4/9$ and $s_1 = 4/3$, $s_2 = -1/3$, in which case all the exponentials of interest are determined

$$\begin{aligned} e^{s_1 y_T} &= e^{4/3} = 3.7936678, \qquad e^{s_2 y_T} = e^{-1/3} = 0.7165313, \\ e^{s_1 y_0} &= e^{8/3} = 14.391916, \qquad e^{s_2 y_0} = e^{-2/3} = 0.5134171. \end{aligned} \tag{4.3}$$

The realizability check of (3.28) is satisfied, as

$$-9.8552961 < \frac{s_1}{s_2} = -4 < -0.3515777. \tag{4.4}$$

We wish to choose $V'(0)$, V_T and b such that a solitary wave results. From the equations of Section 3, namely 23, 14, 19, 20 and 13, we have

$$\begin{aligned} A_1 &= -A_2 = -(0.0538175)b \quad 0 \\ V_T &= (3.0771365)A_1 = -(0.1656037)b \\ \hat{A}_1 &= A_1 + (0.1186187)b = (0.0648012)b < 0 \\ \hat{A}_2 &= -A_1 + (2.5121023)b = (2.5659198)b \\ V'(0) &= \frac{5}{3} A_1. \end{aligned} \tag{4.5}$$

As seen, $b < 0$ is a free parameter which can be used to assist in designing with physically realizable components. Thus, let us choose the threshold voltage to be one volt; $V_T = 1$ gives $b = -6.0385124$. Choosing this value of b gives

$$\begin{aligned} &V_T = 1, \qquad A_1 = 0.3249776 = -A_2, \qquad \hat{A}_1 = -0.3913028, \\ &\hat{A}_2 = -15.494338, \; -b/a = 13.586653, \qquad I_1 = aV_T = 0.444444, \end{aligned} \tag{4.6}$$

$$V(0) = 0, \; V'(0) = 0.5416293.$$

Figure 2 gives the resulting pulse and broken-sloping hysteresis, the pulse being described by

$$v(y)\Big|_{y=ct-x} = \begin{cases} 0 & y \le 0 \\ (0.3249776)\left[e^{\frac{4}{3}y} - e^{\frac{-1}{3}y}\right], & 0 \le y \le y_T = 1 \\ (-0.3913028)e^{\frac{4}{3}y} - (15.494338)e^{\frac{-1}{3}y} + 13.586653, & 1 \le y \le y_0 = 2 \\ 0 & 2 \le y \end{cases} \tag{4.7}$$

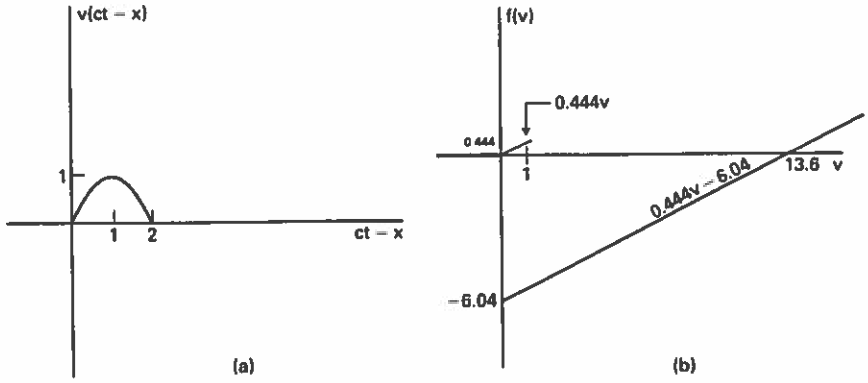

Figure 2. Design example results; $b = -6.04$, $y_T = 1$, $y_O = 2$.
(a) Solitary pulse. (b) Broken-sloping hysteresis.

5. BROKEN-SLOPING HYSTERESIS CIRCUIT

Important to our design is the existence of electronic circuits for the realization of the required broken-sloping hysteresis. This can be achieved by putting in series a linear resistor and a broken piecewise linear negative resistor.

The analysis can be carried out graphically as illustrated in Figs.3a and b, which show the voltage-controlled characteristics of the two components, these being curves which are most conveniently realized by electronic devices. Figures 3c and d show these characteristics inverted so that their voltages may be added, as is done in part e. Again taking the inverse of the characteristic gives the final broken-sloping hysteresis of Fig.3f. We see from Fig.3 that we desire

$$\psi = \gamma, \quad V_\delta = V_T, \quad I_\psi = b, \quad \frac{\delta\gamma}{\delta+\gamma} = a, \qquad (5.1)$$

which become the design equations for specifying the positive and negative resistors. Most convenient is the choice of equal positive and negative slopes in the negative resistor, $\delta = \psi$, in which case $a = \delta/2$.

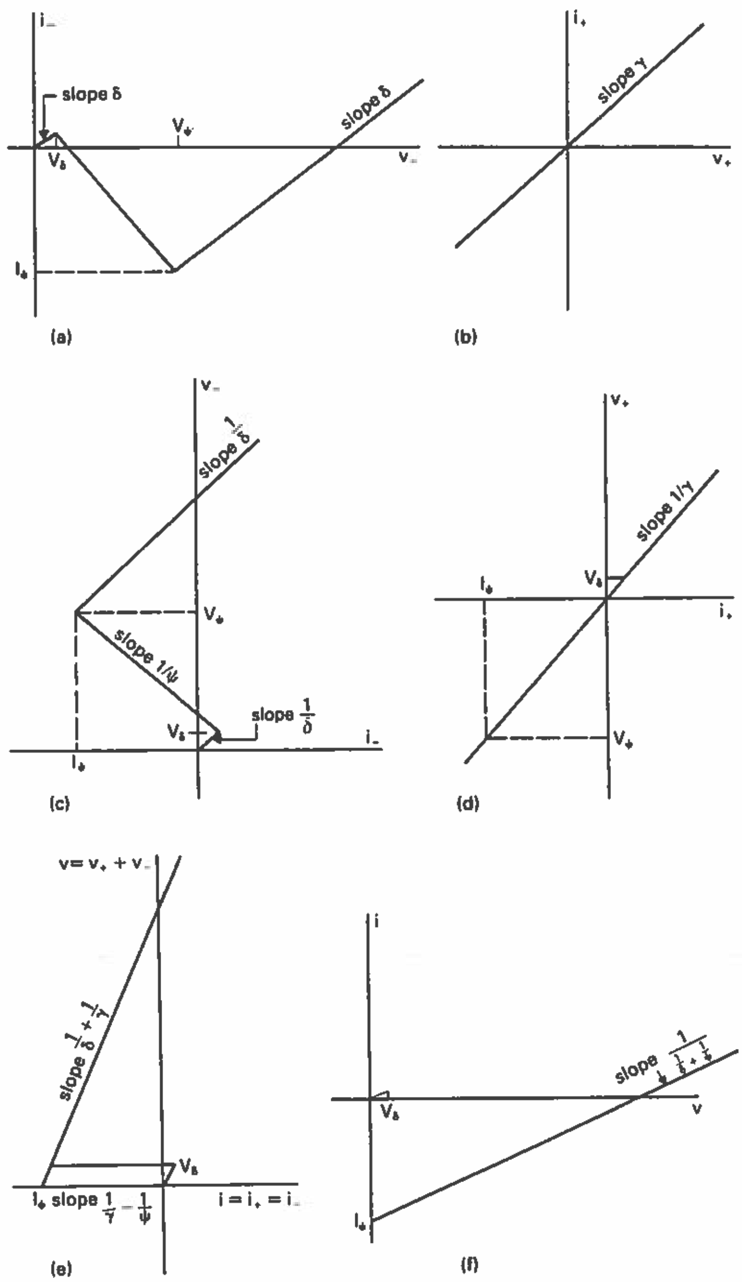

Figure 3. Design of broken-sloping hysteresis

(a) Negative resistor (b) Positive resistor

(c) Inverse of (a) (d) Inverse of (b)

(e) Series connection characteristics (f) Inverse of (e)

Figure 4 shows a circuit which will realize the characteristics of Fig.3, the negative resistor circuit being that of Endo and Mori [3,p.15] in which an op-amp is used. In Fig.4 the current source shunted by diode D_2 is used to shift i_- of Fig.3a up by I_ψ when $v_- > 0$, D_3 (and D_4) is used to cut off the current i_- when $v_- < 0$ and the current source shunting D_1 shifts i_- back to the origin at $v_- = 0$; D_1 and D_2 provide paths for the I_ψ's when D_3 and D_4 are open. The current sources are readily realized by transistors [4,p.109], hence the circuit of Fig.4 is suitable for integrated circuit constructions. D_4 is a current inverted diode which can be realized by loading a current inverting NIC (negative impedance converter) by a normal diode.

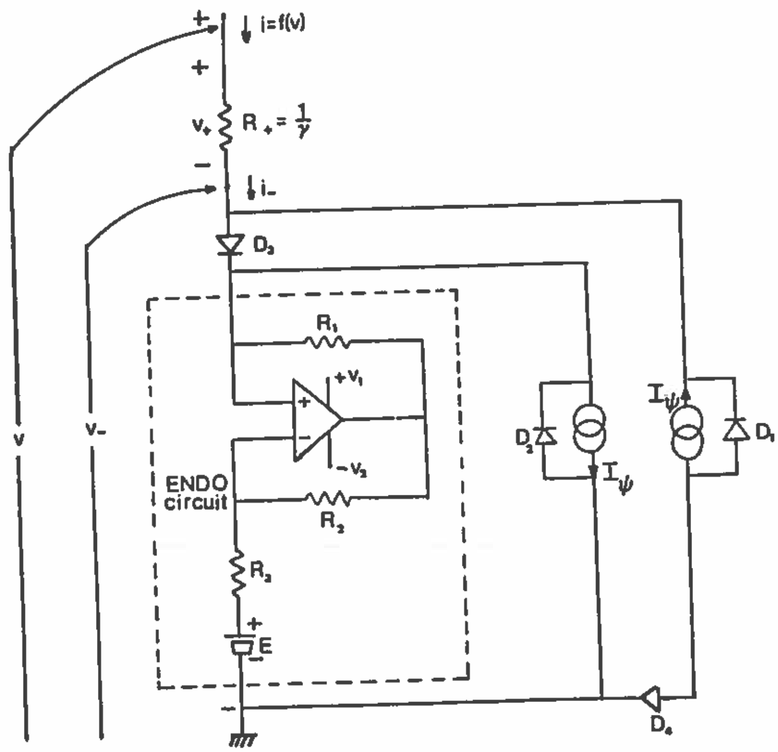

Figure 4. Circuit for broken-sloping hysteresis

6. DISCUSSION

Here we have shown that through the use of broken-sloping hysteresis as a nonlinear shunt element RC transmission lines can be made to support solitary pulse waves. In so doing we have been able to set up design equations which

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allow for certain specifications of the pulse shape, such as peak value, time to peak value and length of pulse, as well as pulse-wave velocity. The results, though, are by way of an existence proof since it still remains actually to construct such lines. Toward this construction, Section 4 gives an op-amp circuit which is a possibility for designing desired broken-sloping hysteresis characteristics. However, constructions will best take place in lumped-distributed structures, rather than fully distributed ones as treated here, where the hysteresis components load the distributed RC lines at discrete intervals. Consequently, a theory for such lumped distributed structures now seems in order, especially to see if solitons can be obtained in similar circuits. Also needed are stability studies and a more fully justifiable theoretical treatment of the multi-valued characteristics resulting from the hysteresis, say through semi-state theory. For sure the topic holds many fascinating areas for future study.

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