Eqn. (24) as

$$\bar{\Delta}(s,g) = \det[gD(s) - N(s)]$$
$$= a_n'(g)s^n + a_{n-1}'(g)s^{n-1} + \cdots + a_0'(g) \quad (27)$$

The relationships between $g(s)$ and $s(g)$ have been established by Smith (1982), who has shown that $g(s)$ may be used instead of $s(g)$ in studies of root-locus behavior. By using the fractional power series expansions which are equivalent to those of Eqn. (26), the following definitions are obtained.

7.2 *Root-Locus Poles and Zeros* (Smith 1982)

Let $G(s) \in \mathbb{R}(s)^{m\times m}$, $s_0 \in \mathbb{C}_\infty$, and let $g_i = c_i z^{\alpha_i} + \ldots, i \in \{m\}$, be the local expansions of $g(s)$, where $z = s - s_0$ for $s_0 \in \mathbb{C}$ and $z = 1/s$ for $s_0 = \infty$. We say that $G(s)$ has a zero at $s = s_0$ of multiplicity $|\alpha_i|$ if $\alpha_i > 0$, and a pole of the root locus at $s = s_0$ of multiplicity $|\alpha_i|$ if $\alpha_i < 0$.

The α_i always give the rates of approach or departure, but not necessarily the orders of the Butterworth patterns, which are defined by the orders $|\mu_i|$ of the zeros or poles of the algebraic function $g(s)$. If $\#P\{A\}$ and $\#Z\{A\}$ denote the total numbers of poles and zeros at $s = s_0$ of the object A, then

$$\#P\{G(s)\} = \#Z\{a_m(s)\}$$
$$= \#P\{\mathrm{RL}\} + \#Z\{e(s)\} \quad (28)$$
$$\#Z\{G(s)\} = \#Z\{a_0(s)\}$$
$$= \#Z\{\mathrm{RL}\} + \#Z\{e(s)\} \quad (29)$$

8. *Invariance Properties of Zeros*

The importance of zeros in linear multivariable control systems is largely a result of their invariance properties under various transformations. Two of these important properties are briefly summarized as follows.

State-space transformations. The zero structure of $\mathcal{S}(A, B, C, D)$ (defined by the invariants of $P(s)$) is invariant under state, output feedback, output injection and input, state and output coordinate transformations.

Transfer function transformations. The finite (infinite) zeros and poles of $G(s)$ are invariant under left/right $\mathbb{R}[s](\mathbb{R}_{pr}(s))$ unimodular transformations

See also: Linear Systems: Kronecker Canonical Forms and Invariants; Smith–MacMillan Canonical Forms for Rational Matrices

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N. Karcanias

Linear Orderings

A linear ordering is a transitive, antisymmetric and total relation.

M. Pittarelli

Linear Passive Network Synthesis

Network synthesis concerns the mathematically precise design of electrical circuits from an input–output specification, a circuit being an interconnection of individual components (such as resistors, capacitors and transistors). Linear networks are usually synthesized by generating a network function matrix description from the specifications, and then decomposing the network function matrix into subnetwork functions, each corresponding to a circuit component. Passive networks

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are synthesized by using circuit components which contain no energy sources, such as positive resistors, capacitors and ideal transformers, at least when the network function satisfies necessary realizability conditions. The development of these realizability conditions is therefore an integral part of network synthesis, as are the techniques for designing a circuit from a network function matrix. Because its techniques are characteristic of engineering design, network synthesis has served as the background and impetus for much of modern mathematical systems theory. As such network synthesis has a very rich history and has branched in many directions. This article decribes the basic elements: characterization, realizability conditions, realization and equivalency.

1. Circuit Characterization

Consider n pairs of access points, called ports, associated with each of which there is a real-valued voltage v_j and a current i_j, $j = 1, \ldots, n$, with polarities such that the product $v_j(t)i_j(t)$ is the power into the jth port. Here n is assumed to be a finite positive integer, but one can generalize to continuous distributions of access points, and so obtain Hilbert ports. An electrical n-port network is characterized in time by the voltage $v(t)$ and current $i(t)$ n-vectors which the n-port allows; the total power input to the network at any instant t is then $p(t) = v^{\mathrm{T}}(t)i(t)$, where the superscript T denotes matrix transposition. The network is passive if its energy $\mathscr{E}(t)$ is nonnegative for all time; that is, if

$$\mathscr{E}(t) = \int_{-\infty}^{t} p(\tau)\,d\tau + \mathscr{E}(-\infty) \geq 0, \qquad -\infty < t < \infty \quad (1)$$

for all pairs of v and i possible at the ports. In terms of various network-function descriptions, realizability constraints are developed from the condition $\mathscr{E}(t) \geq 0$, usually $\mathscr{E}(-\infty) = 0$ being assumed. If, under appropriate square-integrable excitations, $\mathscr{E}(+\infty) = 0$ as well, a passive network is called lossless since all energy is eventually returned to the sources. Linearity is somewhat more difficult to define, but for our purposes we will take it to mean that there exist linear operators L_i and L_v such that the network is completely described by

$$L_i[i] = L_v[v] \quad (2)$$

Although infinitely many network-function descriptions can be considered, here we concentrate on two: the admittance and the scattering matrix descriptions. In the admittance description it is assumed that the inverse of $L_i[\cdot]$ exists; using an admittance kernel $y(.,.)$, (2) becomes

$$i(t) = L_i^{-1}[L_v[v]] = \int_{-\infty}^{\infty} y(t,\tau)v(\tau)\,d\tau \quad (3)$$

For the scattering kernel $s(.,.)$, a transformation of port variables to incident v^{i} and reflected v^{r} variables is made:

$$\begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} 1_n & 1_n \\ 1_n & -1_n \end{bmatrix} \begin{bmatrix} v^{\mathrm{i}} \\ v^{\mathrm{r}} \end{bmatrix} \quad (4a)$$

where 1_n is the $n \times n$ identity, which when substituted in Eqn. (2) gives

$$v^{\mathrm{r}}(t) = (L_i + L_v)^{-1}(L_i - L_v)[v^{\mathrm{i}}] = \int_{-\infty}^{\infty} s(t,\tau)v^{\mathrm{i}}(\tau)\,d\tau \quad (4b)$$

Although time-variable synthesis is possible, we restrict the treatment here to time-invariant networks, for which

$$y(t,\tau) = y(t-\tau, 0), \qquad s(t,\tau) = s(t-\tau, 0) \quad (5)$$

that is, a shift in input time produces a corresponding shift in response time. Taking Laplace transforms $\mathscr{L}[\cdot]$, Eqn. (5) can be expressed in terms of the $n \times n$ admittance and scattering matrices:

$$Y(s) = \mathscr{L}[y(t,0)], \qquad S(s) = \mathscr{L}[s(t,0)] \quad (6)$$

respectively, with $s = \sigma + \mathrm{j}\omega$, $\mathrm{j} = \sqrt{-1}$, now the complex frequency variable. Given Y or S, one can switch to the other by using one of the equations

$$S = (1_n + Y)^{-1}(1_n - Y), \qquad Y = (1_n + S)^{-1}(1_n - S) \quad (7)$$

For a passive network, the existence of Y guarantees the existence of S, but not vice versa, thus showing the versatility of the scattering matrix description.

The basic realizability constraints are that $Y(s)$ is positive-real and $S(s)$ is bounded-real and are defined as follows.

$Y(s)$ is positive-real if and only if

(a) $Y(s)$ is analytic in $\sigma > 0$, and

(b) $Y^*(s) = Y(s^*)$ in $\sigma > 0$, and

(c) $Y(s) + Y^{\mathrm{T}}(s)$ is nonnegative in $\sigma > 0$.

$S(s)$ is bounded-real if and only if

(a) $S(s)$ is analytic in $\sigma > 0$, and

(b) $S^*(s) = S(s^*)$ in $\sigma > 0$, and

(c) $1_n - (S^{\mathrm{T}})^*(s)S(s)$ is nonnegative in $\sigma > 0$

where the asterisk denotes complex conjugation. In these definitions condition (a) is a stability requirement (following from the condition that $\mathscr{E}(t) \geq 0$). Condition (b) results from our physical assumption that the port variables are real valued, and can be dropped in a purely mathematical theory, as is sometimes appropriate for microwave circuits. Condition (c) is essentially the passivity constraint, which is reflected in the nonnegativity of the Hermitian forms of the stated matrices. When the matrix consists solely of rational functions of s, a positive-real Y or a bounded-real S can be used to synthesize a circuit yielding Y or S at its ports. And although functions other than rational functions are of interest, such as the positive-real $Y(s) = +\sqrt{s}$ (for an RC transmission line), techniques for synthesizing arbitrary positive-real or bounded-real matrices are not yet available.

Since Y and S can be looked upon as transfer function matrices, standard state-variable theory applies to rational functions. For a positive-real admittance, which has at most a simple pole at infinity, then

$$\dot{x} = Ax + Bv \tag{8a}$$

$$i = Cx + Dv + E\dot{v} \tag{8b}$$

$$Y(s) = sE + D + C[s1_k - A]^{-1}B \tag{8c}$$

with x the k-dimensional state. Realizability conditions on $\{A, B, C, D, E\}$ can also be given; for example, E must be symmetric and nonnegative-definite by virtue of the third positive-real constraint (for $S(s)$, E must be zero). If the network is lossless, $\mathscr{E}(\infty) = 0$ requires the Hermitian matrices of conditions (c) to be zero, which for rational functions is equivalent, for all s, to

$$Y(s) = -Y^{\mathrm{T}}(-s), \qquad S^{-1}(s) = S^{\mathrm{T}}(-s) \tag{9}$$

A matrix satisfying condition (9) on Y is called para-Hermitian, and one satisfying condition (9) on S paraunitary.

2. Circuit Realizations

In circuit realizations, one particularly fruitful idea is to consider the circuit elements as being partitioned into two classes, one forming an $(n+m)$-port coupling network in cascade, with an m-port load forming the other class. Frequently one class is resistive (resistors, transformers and gyrators) and the other class lossless (capacitors, inductors, transformers and gyrators). For example, in Darlington synthesis the resistive class is the load on the lossless elements, but vice versa in state-variable synthesis. By using the additive property of energy the passivity constraints can be placed individually on the coupling network and the load. From Kirchhoff's laws, for the interconnection we have

$$S = S_{11} + S_{12}[1_m - S_l S_{22}]^{-1} S_l S_{21} \tag{10a}$$

$$Y = Y_{11} - Y_{12}(Y_l + Y_{22})^{-1} Y_{21} \tag{10b}$$

Here the $(n+m)$-port coupling network is described by $S_c = [S_{ij}]$ or $Y_c = [Y_{ij}]$, where S_{ij} and Y_{ij}, $i, j = 1, 2$, are appropriately sized submatrices (S_{11} and Y_{11} are $n \times n$; S_{12}, S_{21}^{T}, and Y_{21}^{T} are $n \times m$; and S_{22} and Y_{22} are $m \times m$) and S_l and Y_l are $m \times m$ descriptions of the load.

Consider first Darlington synthesis. For the load we take m uncoupled $1\,\Omega$ resistors. Using

$$r = \operatorname{rank}[1_n - S^{\mathrm{T}}(-s)S(s)] \tag{11}$$

any $m \geq r$ will work. This choice of load fixes the scattering matrix S_l, which is calculated from Eqn. (7) as $S_l = 0_m$, the $m \times m$ zero matrix. Next the coupling scattering matrix S_c is formed for which, from Eqn. (10a), the (1, 1) submatrix is immediately found to be $S_{11} = S$. To find the remainder of S_c the lossless constraint of Eqn. (9) is used, giving

$$1_n - S^{\mathrm{T}}(-s)S(s) = S_{21}^{\mathrm{T}}(-s)S_{21}(s) \tag{12a}$$

$$1_n - S(s)S^{\mathrm{T}}(-s) = S_{12}(s)S_{12}^{\mathrm{T}}(-s) \tag{12b}$$

$$S_{22}(s)S_{12}^{\mathrm{T}}(-s) = -S_{21}(s)S^{\mathrm{T}}(-s) \tag{12c}$$

By appropriate factorizations, Eqns. (12a, b) yield S_{21} and S_{12}, and by the use of a proper pseudo-inverse for S_{12}, Eqn. (12c) yields S_{22}, and the resulting (lossless) S_c can be chosen bounded-real when S is. Once formed, the coupling scattering matrix is converted to a coupling admittance matrix Y_c via Eqn. (7), after perhaps a port reduction (corresponding to the extraction of short circuits) to obtain the nonsingularity of $1_n + S_c$. The synthesis of Y_c is direct, though the description is lengthy since it is a positive-real lossless matrix. From Eqn. (9) and condition (a) in Sect. 1, all poles are on the $j\omega$ axis and, by condition (c), these poles must be simple with nonnegative residue matrices; by condition (b) poles occur in conjugate pairs as do the associated residue matrices. Hence the partial-fraction expansion takes the form

$$Y_c(s) = sA_\infty + B_\infty + \frac{1}{s}A_0 + \sum_{k=1}^{K}\left(\frac{H_k}{s + j\omega_k} + \frac{H_k^*}{s - j\omega_k}\right) \tag{13a}$$

$$= sA_\infty + B_\infty + \frac{1}{s}A_0 + \sum_{k=1}^{K}\frac{sA_k + B_k}{s^2 + \omega_k^2} \tag{13b}$$

where all the A_k's are real, symmetric and nonnegative, while the B_k's are real and skew-symmetric. Y_c can be constructed as a parallel connection of $(n \times m)$-ports, one for each term in Eqn. (13b). By writing, for $k = 0, \ldots, K$,

$$sA_k + B_k = \sum_{p=1}^{P} T_p^{\mathrm{T}}\begin{bmatrix} s & \omega_k \\ -\omega_k & s \end{bmatrix} T_p + sT_k^{\mathrm{T}}T_k, \quad 2P = \operatorname{rank} B_k \tag{13c}$$

$$sA_\infty + B_0 = sT_\infty^{\mathrm{T}}T_\infty + \sum_{p=1}^{g} T_{p\infty}^{\mathrm{T}}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} T_{p\infty}, \quad 2g = \operatorname{rank} B_\infty \tag{13d}$$

each of the individual $(n+m)$-port circuits can be designed as inductor–capacitor (or just capacitor) one-port loading gyrators, which are coupled through multiport transformers of turns-ratio (nonsquare) matrices (the various T's). Termination of the final m ports of this lossless network for S_c by the m unit resistors synthesizes S to be seen at the first n ports. There are other ways of synthesizing the lossless S_c, including factorization of S_c into factors of degree one or two with a cascade

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3. *State-Variable*

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3. *State-Variable Circuit Synthesis*

For the load we take m uncoupled $1/F$ capacitors, where the minimum value of m is known to be

$$\delta = \text{degree}\,[Y(s)] = \text{degree}\,[S(s)] \qquad (14)$$

and any $m \geq \delta$ will work; here "degree" is the McMillen degree of a matrix. $Y_l = s1_m$ is consequently fixed and, by choosing the state x as the voltage on the load capacitors, Eqns. (8) and (10b) show that $Y_{11} = sE + D$, $Y_{12} = -C$, $Y_{21} = B$, and $Y_{22} = -A$ can be identified. However, the resulting coupling admittance may not be positive-real, so that first one considers a change of the state, $x = T\hat{x}$, with T constant and nonsingular, for which

$$Y_c = \begin{bmatrix} 1_n & 0 \\ 0 & T^{-1} \end{bmatrix} \begin{bmatrix} sE+D & -C \\ B & -A \end{bmatrix} \begin{bmatrix} 1_n & 0 \\ 0 & T \end{bmatrix} \qquad (15)$$

is chosen. The similarity transformation T is chosen so that Y_c is positive-real, which is always possible when $Y(s)$ is positive-real. A circuit is constructed from the final Y_c by writing Y_c as the sum of its symmetric part (which includes sE) and its skew-symmetric part; these are then diagonalized and skew-diagonalized, respectively, with sE treated separately, to give ($\dotplus$ denoting matrix direct sum)

$$Y_c = sT_E^{\mathrm{T}} T_E + T_r^{\mathrm{T}} T_r + T_g^{\mathrm{T}} \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) T_g \qquad (16)$$

the number of skew-diagonal terms being $\frac{1}{2}$ rank $(Y_c - Y_c^{\mathrm{T}})$.

Each of the three terms on the right-hand side of Eqn. (16) represents an $(n+m)$-port, and Y_c results from their parallel connection. The first term on the right-hand side represents capacitors loading a multiport transformer of turns-ratio matrix T_E with the combination placed in shunt with the input ports. The second term represents unit resistors seen through a transformer of turns-ratio matrix T_r, while the last term represents unit uncoupled gyrators seen through a transformer T_g. When the m unit capacitors are placed on the final m ports, $Y(s)$ is seen at the input n ports. By proper choice of T in Eqn. (15), this method will allow synthesis to be carried out with the minimum number of gyrators, this number being $g = \frac{1}{2}\,\text{rank}\,[Y(s) - Y^{\mathrm{T}}(s)]$.

For the same input–output behavior there are many different circuits that can be synthesized. Some of these circuits are more practical than others for various reasons, such as element value spread, circuit configuration and sensitivity to parameter variations. Consequently, it is desirable to be able to obtain equivalent circuits, circuits having the same specified network function, from a given circuit. There are several ways of doing this, but here only the concepts behind the Darlington and state-variable syntheses are outlined. In the Darlington synthesis one forms the lossless coupling scattering matrix

$$S_c = \begin{bmatrix} S & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \qquad (17)$$

Given this $(n+m) \times (n+m)$ S_c, we can form a larger matrix $\hat{S}_c$ in which S_c is embedded and then transformed:

$$\hat{S}_c = [1_n \dotplus U][S_c \dotplus W][1_n \dotplus V] \qquad (18)$$

where $U(s)$, $V(s)$ and $W(s)$ are arbitrary para-unitary matrices. If W is $q \times q$, then U and V are of size $(m+q) \times (m+q)$ and the termination of a lossless circuit for $\hat{S}_c$ in $m+q$ unit resistors yields S at the input n ports. For equivalents from the state-variable synthesis one proceeds along similar lines by introducing noncontrollable and nonobservable parts into a given A, B, C and then transforming, as in Eqn. (15), the augmented matrices with a similarity transformation T to obtain the (constant in s) coupling admittance.

The recent semistate theory (Lewis and Newcomb 1986), which modifies the state-variable equations (8a, b) by placing a singular matrix coefficient on $\dot{x}$ and setting $D = E = 0$, is opening up new methods for synthesis as well as analysis.

See also: Abstract Realization Theory; Approximate Realization Theory: Comparison of Optimal Methods

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R. W. Newcomb

Linear Programming: An Introduction

Linear programming was originated in the late 1940s by George Dantzig and has since become a major foundation for optimization theory. Today, large-scale problems in a wide variety of application areas are routinely formulated and solved as linear programming problems. Linear programming continues to be an active area of

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