1) According to steps a) and b),

$$p_0 = 0.5825, \qquad \Delta v_a = 0.153, \qquad \omega_{12}^2 = 0.567,$$

$$\omega_{23}^2 = 0.106, \qquad A = 0.627 \qquad \text{and} \quad B = 0.0705.$$

2) Eqs. (14c) and (14d) give $h = -0.14$ and $k = 0.373$.

3) As shown in Fig. 3, let this center be point P and the intersection between $y = x^2/2$ and $x = 0.5$ be point Q. Then by reading the x scale of point R, the other intersection, we obtain $2(p_0 - p_c) = -0.835$. Thus $p_c = 1$ and $\Delta v_c = 0.0825$.

d) By step c), p_a and p_b can be determined as

$$\left.\begin{matrix} p_a \\ p_b \end{matrix}\right\} = (p_0 - \Delta v_c) \pm j\omega_{ab} = 0.5 \pm j0.86.$$

Clearly this is the case of Butterworth response and the pole pattern is shown in Fig. 2.

Loci of Pole Migration

Fig. 3 also shows the loci of the center point of the same example for (a) when $\omega_{12}^2(=h_{12}^2 - 0.01)$ is fixed at 0.567 and $\omega_{23}^2(=h_{23}^2 - 0.164)$ is varied, and (b) the converse case. These loci are straight lines.

The effect that a variable ω_{23}^2 has on p_a, p_b and p_c when ω_{12}^2 is fixed is shown in Fig. 4(a). The converse case is shown in Fig. 4(b). In both diagrams, the lower half of the complex plane is omitted.

In Fig. 4(a) h_{12} is fixed at 0.76, and h_{23} is increased from zero to 0.855 ($\omega_{23} = \omega_{12}$) and then to ∞. The point p_c moves from p_3 to $p_2(p_2' - p_0 = p_0 - p_2)$ and then asymptotically to p_1, while σ_{ab} proceeds from σ_{12} to p_0 and then closely to σ_{23}. During this time ω_{ab}^2 increases monotonically from ω_{12}^2 to ∞. As a result, the locus of the coupled pole p_a moves from the fixed auxiliary pole p_{12} and proceeds toward the vertical asymptote through σ_{23}, the locus of the coupled pole p_{23}. Similarly, in Fig. 4(b), h_{23} is fixed at 0.855, and h_{12} is increased from $0 \to 0.76$ ($\omega_{12} = \omega_{23}$) $\to \infty$. The point p_c moves from $p_1 \to p_2' \to p_3$, while p_a starts from p_{23} and moves toward the vertical asymptotes through σ_{12}, the locus of the coupled poles p_{12}.

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On the n-Port Brune Resistance Extraction*

In Belevitch's paper on a nonreciprocal n-port Brune synthesis[1] a discussion is given comparing the two resistance extractions credited to McMillan[2] and Tellegen.[3] Besides pointing out that Tellegen's extraction can be found in an earlier paper of Oono,[4] we show here that these two types of extractions are special cases of a general extraction, and, hence, not as different as one first concludes.[5] We state the result as a theorem. By a PR matrix we mean a positive-real one[6] which is rational. By $\tilde{A}$, A^*, A_* we mean, respectively, the transpose, the complex conjugate, and the Hurwitz conjugate (replacement of p by $-p$). By A_H we mean the Hermitian part, $2A_H = A + \tilde{A}^*$; however, $2A_H(j\omega)$ we form by letting $p = j\omega$ in $A + \tilde{A}_*$, in which case, if A is PR, $A_H(j\omega)$ is bounded and continuously differentiable for each ω. We will write "det" for determinant and $\dot{+}$ for the direct sum of two matrices.

Theorem:

If $A(p)$ is an $n \times n$ PR matrix and if A_0 is any real, constant, positive semi-definite (symmetric) matrix, then there exists a unique constant $r \geq 0$ such that

$$A_m(p) = A(p) - rA_0 \tag{1}$$

is PR and has det $A_{mH}(p_0) = 0$ for some $p_0 = j\omega_0$, $0 \leq \omega_0 \leq \infty$.

Proof:

$A_H(j\omega)$ and A_0 are positive semi-definite, since A is PR and A_0 is assumed to have this property. The two matrices in $A_H(j\omega) - \lambda A_0$, where λ is a parameter, can then be simultaneously diagonalized.[7] That is, for each fixed ω, there exists a complex $T(\omega)$ such that

$$\begin{aligned} A_H(j\omega) - \lambda A_0 &= \tilde{T}^*(\omega)\{[\alpha_1(\omega) - \lambda] \dot{+} \cdots \\ &\quad \dot{+} [\alpha_k(\omega) - \lambda] \dot{+} \alpha_{k+1}(\omega) \dot{+} \cdots \\ &\quad \dot{+} \alpha_n(\omega)\}T(\omega) \end{aligned} \tag{2}$$

where k is the rank of A_0; $\alpha_i(\omega) \geq 0$ for all i and ω. We then choose

$$r = 0 \text{ if } \min_{1 \leq i \leq n} \{\min_{0 \leq \omega \leq \infty} \alpha_i(\omega)\} = 0, \tag{3a}$$

otherwise we choose

$$r = \min_{1 \leq i \leq k} \{\min_{0 \leq \omega \leq \infty} \alpha_i(\omega)\}. \tag{3b}$$

By letting $\lambda = r$ in (2), we see that $A_{mH}(j\omega)$ is positive semi-definite with its rank falling below n at the ω_0 for which the minimum in (3) occurs. Noting that the poles and residues for A and A_m are identical, and, applying a known PR test,[8] we see that A_m is PR. As the α_i are unique, r is unique. *Q. E. D.*

The most interesting situations arise when $A_H(j\omega)$ is positive definite for all ω, in which case $r > 0$. In such cases McMillan requires A_0 to be positive definite, while Belevitch, Oono and Tellegen require $A_0 = 1 \dot{+} 0_{n-1}$. For this latter choice, use of the Gauss diagonalization, adding the later rows and columns to the first, allows T real and gives[9]

$$r = \min_{0 \leq \omega \leq \infty} \Delta(\omega)/\Delta_{11}(\omega) \tag{3c}$$

where Δ and Δ_{11} are the determinant and (1, 1) minor of $A_H(j\omega)$, respectively. In this case $A_{mH}(j\omega)$ has rank $n - 1$. As Belevitch points out, the choice of A_0 leading to (3c) is probably the most desirable for synthesis purposes, since other choices are wasteful of resistors.[1]

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[1] V. Belevitch, "On the Brune process for n-ports," IRE Trans. on Circuit Theory, vol. CT-7, pp. 280–296; September, 1960. See pp. 280–281.

[2] B. McMillan, "Introduction to formal realizability theory—II," *Bell Sys. Tech. J.*, vol. 31, pp. 541–600; May, 1952. See pp. 556–558.

[3] B. D. H. Tellegen, "Synthesis of $2n$-poles by networks containing the minimum number of elements," *J. Math. Phys.*, vol. 32, pp. 1–18; April, 1953. See pp. 4–5.

[4] Y. Oono, "Synthesis of a finite $2n$-terminal network as the extension of Brune's two-terminal network theory," *J. Inst. Elec. Commun. Eng. (Japan)*, vol. 31, pp. 163–181; August, 1948 (in Japanese). See pp. 168–169.

[5] Belevitch, *op. cit.*, see p. 281. Line 8 states "Moreover, Tellegen's method is not even included as a particular case in McMillan's, ... ," which is certainly true.

[6] D. Youla, L. Castriota, and H. Carlin, "Bounded real scattering matrices and the foundations of linear passive network theory," IRE Trans. on Circuit Theory, vol. CT-6, pp. 102–124; March, 1959. See p. 122.

[7] R. W. Newcomb, "On the simultaneous diagonalization of two semi-definite matrices," *Quart. Appl. Math.*, vol. 19, pp. 144–146; July, 1961.

[8] R. W. Newcomb, "On network realizability conditions," Proc. IRE (*Correspondence*), vol. 50, p. 1995, September, 1962.

[9] F. R. Gantmacher, "The Theory of Matrices," Chelsea, New York, N. Y., vol. 1, p. 26 eq. (13); 1959.

Topological Formulas for Passive Transformerless 3-Terminal Networks Constrained by One Operational Amplifier*

Nathan [1] shows how the determinants required in the conventional analysis of networks constrained by operational amplifiers may be obtained from the determinants of the networks without the constraints. The same idea could be used in the analysis of networks using the topological concepts, as will be shown presently. We shall confine ourselves to 3-terminal networks con-

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