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SCATTERING AND ADMITTANCE MATRICES OF SAW TRANSDUCERS*

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Abstract. The p-plane scattering and admittance matrices of SAW transducers consisting of n equal sections modeled through the hybrid equivalent circuit are explicitly calculated. The results are specialized to the in-line and crossed-field models, and the technique is developed for unequal section transducers.

1. Introduction

It is well known that circuit model representations can be the key for the development and design of surface acoustic wave, SAW, devices [1]. Especially, starting from Mason's characterizations [2], an equivalent circuit representation [1,3,4] has been found to be successful in characterizing the processing of surface waves. In these references the transfer characteristics and the input admittance at the electrical port of interdigital transducers, IDTs, have been determined [4,5]. More recently [6, 7, 8, 9] a formation to obtain the scattering matrices of SAW devices has been outlined. This paper applies these techniques to derive the scattering matrix of n section SAW transducers which can have equal or unequal sections represented by the hybrid equivalent circuit (sometimes called the variable model). This allows for representation through in-line or crossed-field models or anything in between, as well as extensions to more elaborate models.

In Section 2 a general mathematical theory is developed for finding the admittance matrix of a 3-port SAW network represented by n possibly unequal hybrid sections. Section 3 contains the main contribution as it develops explicit formulas for the admittance and the scattering matrix elements of a

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uniform SAW interdigital transducer based upon the hybrid model of one electroded section. These matrix elements are computed in terms of complex frequency and thus allow for transient response determinations.

2. General theory

A. SAW transducers. Figure 1(a) shows a top view of a typical n section

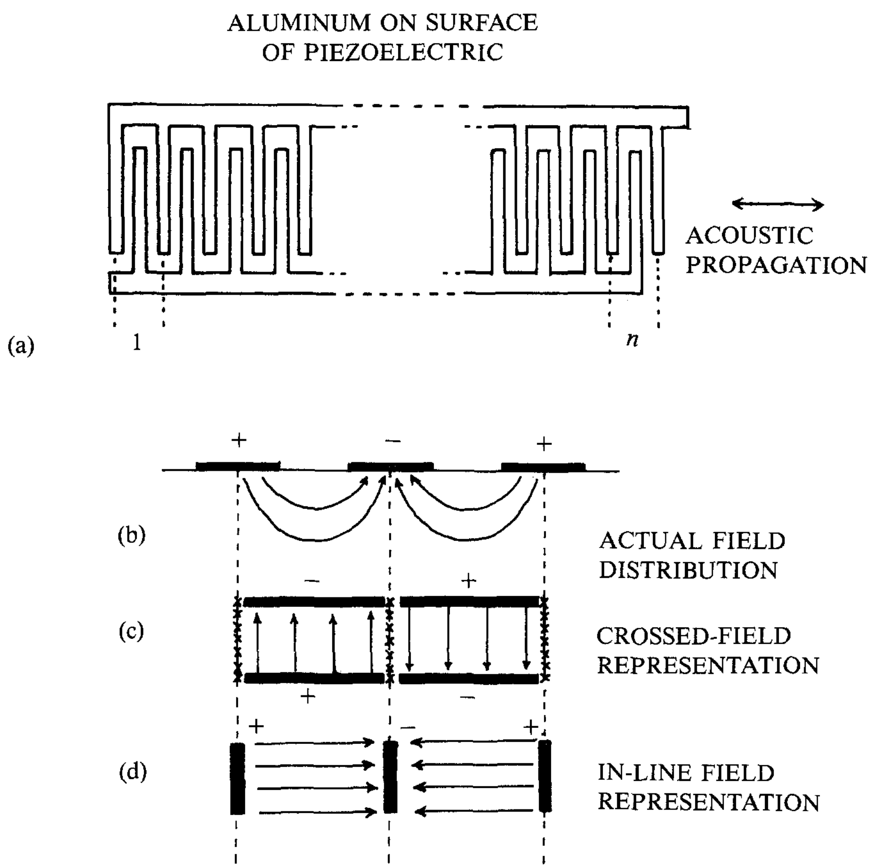

Figure 1. SAW transducer and field representations.
(a) Top view
(b) Side view, one section with field representations. Actual field distribution.
(c) Crossed field representation
(d) In-line field representation

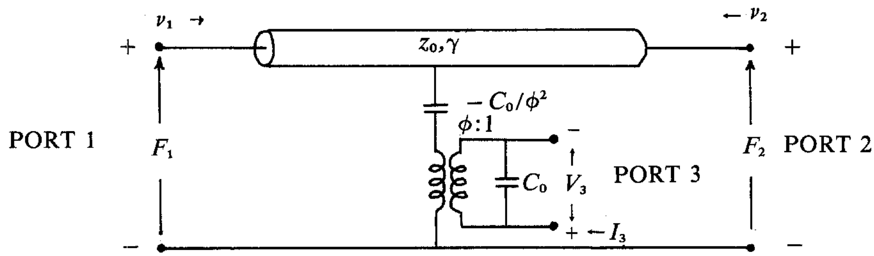

Figure 2. Hybrid equivalent circuit.

SAW transducer. Used as a transmitter, a voltage source is applied between the upper and lower fingers. These fingers generate mechanical motion, an acoustic wave, in the direction shown by virtue of being placed on piezoelectric material. The transducer may also be used as a receiver since the acoustic wave incident on the piezoelectric material generates an electric field which can be detected as a voltage between the upper and lower fingers. Figure 1(b) shows a cross section of the transducer, through the piezoelectric material by a plane in the direction of acoustic propagation, to illustrate the electric field pattern. The field is seen to have two components, one in-line with the propagation direction and one perpendicular to it, the cross-field. These representations are illustrated in Figures 1(c) and 1(d). The transducer or any section of it is therefore seen to be representable by a 3-port where one port is electrical (for the transducer fingers) and two ports are mechanical (for the two acoustical ends). Further, it has been found [1] that such a 3-port SAW section can be represented by the 'hybrid' equivalent circuit of Figure 2 for which Table 1 defines the parameters. The hybrid model reduces to the

Table 1. Hybrid Model Parameters [1, p. 88]

ℓ = single section length

C_o = single section capacitance

Z_o = acoustic characteristic impedance = $1/Yo$

v_a = acoustic propagation velocity

k = piezoelectric coupling coefficient

$$\phi = \left[\frac{k^2 C_o Z_o v_a}{\ell}\right]^{1/2} = \text{transformer turns ratio}$$

$-C_S = -C_o/\phi^2$ = negative series capacitance

α = cross-field to in-line field coupling constant, $0 \le \alpha \le 1$

$\gamma = \ell/v_a$

$p = \sigma + j\omega$, $\omega = 2\pi f$, f = frequency, $j = \sqrt{-1}$

crossed-field model when $\alpha = 0$ and to the in-line model when $\alpha = 1$, with intermediate values of α giving different weights to the crossed and in-line fields present in the actual field distribution. Other equivalent circuits exist [3], the most important of which [10] fits within our theory (see Figure 4 below).

For the hybrid equivalent circuit or any other 3-port SAW model we define for the two mechanical ports

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \text{input force applied} \\ \text{output force applied} \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \text{material velocities}$$

while the current and voltage I_3 and V_3 are for the electrical port. Then we can form the admittance description, $i = Yv$. For the hybrid model

$$\begin{bmatrix} v_1 \\ v_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{12} & y_{11} & y_{13} \\ y_{13} & y_{13} & y_{33} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ V_3 \end{bmatrix} \tag{2.1}$$

where $y_{22} = y_{11}$ and $y_{23} = y_{13}$ by symmetry while Y is symmetric by reciprocity.

B. Hybrid reverse chain matrix. We now connect n of these 3-port sections in cascade at the mechanical ports and in parallel at the electrical ports. Initially we do this in Figure 3, with every other one reversed. The reversal

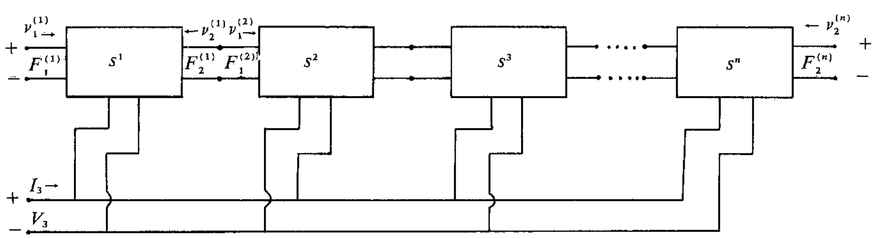

Figure 3. Cascaded single-electroded sections for an SAW transducer.

occurs since most often we consider alternate sections to contain fingers attached to alternate terminals of the electrical source (in order to generate a distance-directed field between sections). If we have a connection of the type of Figure 3 but without reversal of every other port, then the $(-1)^{i-1}$'s occurring below will be absent. Indeed we could use this nonreversal of connections and put the reversal of the true connections in the turns ratios ϕ which appear, though we choose not to do so. We allow all sections to be described by different parameters, and thus for the i^{th} section we write $i^{(i)} = Y^{(i)}v^{(i)}$.

For mechanical ports the interconnection laws are with $i > 1$

$$F_1^{(i)} = F_2^{(i-1)}, \qquad \nu_1^{(i)} = -\nu_2^{(i-1)} \tag{2.2a}$$

where

$$F_1^{(1)} = F_{\text{in}} = F_{\text{left boundary}}, \qquad \nu_1^{(1)} = \nu_{\text{in}} \tag{2.2b}$$

$$F_2^{(n)} = F_{\text{out}} = F_{\text{right boundary}}, \qquad \nu_2^{(n)} = \nu_{\text{out}}, \tag{2.2c}$$

while for electrical ports with $i \geq 1$ the interconnection laws are

$$V_3^{(i)} = (-1)^{i-1}V_3, \qquad I_3 = \sum_{i=1}^{n} (-1)^{i-1}I_3^{(i)} \tag{2.2d}$$

In order to find the interconnected network 3-port descriptions, we convert to a hybrid reverse chain matrix, $\mathfrak{y}_R$, type of description. From Equation (2.2), this hybrid description can be found as

$$\begin{bmatrix} F_2^{(i)} \\ -\nu_2^{(i)} \\ \hline I_3^{(i)} \end{bmatrix} = \begin{bmatrix} \mathfrak{y}_R^{(i)} \\ \hline y_1^{(i)},\ w^{(i)} \end{bmatrix} \begin{bmatrix} F_1^{(i)} \\ \nu_1^{(i)} \end{bmatrix} + \begin{bmatrix} A^{(i)} \\ \hline y_3^{(i)} \end{bmatrix} V_3 \tag{2.3a}$$

where

$$\mathfrak{y}_R^{(i)} = \frac{1}{-y_{12}^{(i)}} \begin{bmatrix} y_{11}^{(i)} & -1 \\ [(y_{12}^{(i)})^2 - (y_{11}^{(i)})^2] & y_{11}^{(i)} \end{bmatrix} \tag{2.3b}$$

$$A^{(i)} = \frac{-(-1)^{i-1}y_{13}^{(i)}}{y_{12}^{(i)}} \begin{bmatrix} 1 \\ y_{12}^{(i)} - y_{11}^{(i)} \end{bmatrix} \tag{2.3c}$$

$$[y_1^{(i)}, w^{(i)}] = -\frac{y_{13}^{(i)}}{y_{12}^{(i)}}[y_{11}^{(i)} - y_{12}^{(i)}, 1] \tag{2.3d}$$

$$y_3^{(i)} = \frac{(-1)^{i-1}}{y_{12}^{(i)}}[y_{33}^{(i)}y_{12}^{(i)} - (y_{13}^{(i)})^2] \tag{2.3e}$$

Equation (2.3a) is the key for our analysis. Since for n sections the reverse chain matrices multiply in reverse order, we obtain

$$\begin{bmatrix} F_{\text{out}} \\ -\nu_{\text{out}} \end{bmatrix} = Ч_R \begin{bmatrix} F_{\text{in}} \\ \nu_{\text{in}} \end{bmatrix} + A V_3 \tag{2.4a}$$

where

$$Ч_R = \prod_{i=1}^{n}{}_R Ч_R{}^{(i)}, \quad A = \sum_{i=1}^{n} \left(\prod_{k=i}^{n-1}{}_R Ч_R{}^{(k+1)} \right) A^{(i)} \tag{2.4b}$$

with the notation Π_R meaning to multiply by lower indexed terms on the right ($\prod_{k=n}^{n-1}{}_R \cdot$ = identity). Since the currents $I_3{}^{(i)}$ in Figure 3 add (within a sign), I_3 is also readily determined.

As an example we determine the description of Equation (2.3a) for a SAW section characterized by the Milsom-Redwood equivalent circuit [10] shown in Figure 4. For this we first calculate the Y matrix of the hybrid model

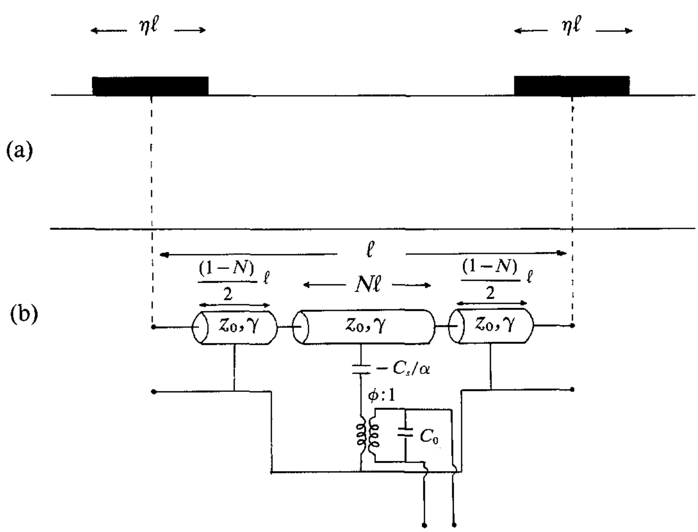

Figure 4. Milsom-Redwood equivalent circuit.

equivalent circuit of Figure 2. Using a p-plane, $p = \sigma + j\omega$, description of the transmission line [11, p. 55, with $\gamma = \sqrt{LC}\,\ell = \ell/\nu_0$] and a considerable amount of manipulation, we find for Figure 2, with $Y_0 = 1/Z_0$,

$$Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_0 p \end{bmatrix} \tag{2.5a}$$

$$+\frac{Y_0}{1-\frac{2\alpha Y_0}{C_s p}\tanh\frac{\gamma p}{2}}\begin{bmatrix} -\frac{\alpha Y_0}{C_s p}+\operatorname{ctnh}\gamma p & \frac{\alpha Y_0}{C_s p}-\operatorname{csch}\gamma p & \phi\tanh\frac{\gamma p}{2} \\ \frac{\alpha Y_0}{C_s p}-\operatorname{csch}\gamma p & -\frac{\alpha Y_0}{C_s p}+\operatorname{ctnh}\gamma p & \phi\tanh\frac{\gamma p}{2} \\ \phi\tanh\frac{\gamma p}{2} & \phi\tanh\frac{\gamma p}{2} & 2\phi^2\tanh\frac{\gamma p}{2} \end{bmatrix}$$

The corresponding hybrid reverse chain matrix equation is for Figure 2

$$\begin{bmatrix} F_2 \\ -v_2 \\ \hline I_3 \end{bmatrix} = \frac{1}{\operatorname{csch}\gamma p-\frac{\alpha Y_0}{C_s p}}\left\{\begin{bmatrix} \operatorname{ctnh}\gamma\, p-\frac{\alpha Y_0}{C_s p} & -Z_0(1-\frac{2\alpha Y_0}{C_s p}\tanh\frac{\gamma p}{2}) \\ -Y_0 & \operatorname{ctnh}\gamma p-\frac{\alpha Y_0}{C_s p} \\ \hline \phi Y_0 & -\tanh\frac{\gamma p}{2} \end{bmatrix}\begin{bmatrix} F_1 \\ v_1 \end{bmatrix} + \phi\begin{bmatrix} \tanh\frac{\gamma p}{2} \\ -Y_0 \\ \hline \phi Y_0 \end{bmatrix} V_3\right\} + \begin{bmatrix} 0 \\ 0 \\ \hline C_0 p \end{bmatrix} V_3 \qquad \textbf{(2.5b)}$$

As a check the upper left 2×2 matrix, $Ч_R$, has $\det Ч_R = 1$, confirming the reciprocity of the circuit. For the overall Milsom-Redwood equivalent circuit we first replace γ by $\gamma' = (1-N)\gamma/2$ for the outer sections and by $\gamma'' = N\gamma$ for the inner section where N is the proportional length of the inner section and, as before $\gamma = \ell/v_0$ is analogous to $\sqrt{LC}\,\ell$. Then, by setting $\phi = 0 = C_0 = 1/C_S$ in (2.5b), for the two outer sections of the Milsom-Redwood circuit we get,

$$\begin{bmatrix} F_2 \\ -v_2 \\ \hline I_3 \end{bmatrix} = \frac{1}{\operatorname{csch}\gamma' p}\begin{bmatrix} \operatorname{ctnh}\gamma' p & -Z_0 \\ -Y_0 & \operatorname{ctnh}\gamma' p \\ \hline 0 & 0 \end{bmatrix}\begin{bmatrix} F_1 \\ v_1 \end{bmatrix} \qquad \textbf{(2.5c)}$$

while for the center section we use Equation (2.5b) with γ replaced by $\gamma'' = N\gamma$. Consequently, as per Equation (2.4),

$$\text{Ч}_R = \text{Ч}_R^{(3)}\text{Ч}_R^{(2)}\text{Ч}_R^{(1)}$$

$$= \frac{1}{(\cosh\gamma' p)^2(\operatorname{csch}\gamma'' p - \frac{\alpha Y_0}{C_s p})} \begin{bmatrix} \operatorname{ctnh}\gamma' p & -Z_0 \\ -Y_0 & \operatorname{ctnh}\gamma' p \end{bmatrix} x$$

$$\begin{bmatrix} \operatorname{ctnh}\gamma'' p - \frac{\alpha Y_0}{C_s p} & -Z_0(1 - \frac{2\alpha Y_0}{C_s p}\tanh\frac{\gamma'' p}{2}) \\ -Y_0 & \operatorname{ctnh}\gamma'' p - \frac{\alpha Y_0}{C_s p} \end{bmatrix} \begin{bmatrix} \operatorname{ctnh}\gamma' p & -Z_0 \\ -Y_0 & \operatorname{ctnh}\gamma' p \end{bmatrix} \quad \textbf{(2.5d)}$$

$$A = a^{(3)} + \text{Ч}_R^{(3)}A^{(2)} + [\text{Ч}_R^{(2)} \cdot \text{Ч}_R^{(3)}]A^{(1)} = \text{Ч}_R^{(3)}A^{(2)}$$

$$= \frac{\phi^2}{\cosh\gamma' p(\cosh\gamma'' p - \frac{\alpha Y_0}{C_s p})} \begin{bmatrix} \operatorname{ctnh}\gamma' p & -Z_0 \\ -Y_0 & \operatorname{ctnh}\gamma' p \end{bmatrix} \begin{bmatrix} \tanh\frac{\gamma'' p}{2} \\ -Y_0 \end{bmatrix} \quad \textbf{(2.5e)}$$

Also, the current is readily found as

$$I_3 = \frac{1}{(\operatorname{csch}\gamma'' p - \frac{\alpha Y_0}{C_s p})\cosh\gamma' p} [\phi Y_0, -\phi\tanh\frac{\gamma'' p}{2}] \begin{bmatrix} \operatorname{ctnh}\gamma' p & -Z_0 \\ -Y_0 & \operatorname{ctnh}\gamma' p \end{bmatrix} \begin{bmatrix} F_1 \\ \nu_1 \end{bmatrix}$$

$$+ \ (C_0 p + \frac{\phi^2 Y_0}{\operatorname{csch}\gamma'' p - \frac{\alpha Y_0}{C_s p}}) V_3 \quad \textbf{(2.5f)}$$

From this example we note that we have achieved a hybrid reverse chain matrix type of description for the Milsom-Redwood model, but that it is rather messy to work with, compared to the description (2.5b) for the hybrid model of the same physical device. Nevertheless, its added parameters may give the improved accuracy needed for highly accurate designs, and toward this it fits well within the theory developed in this paper.

C. Admittance matrix. The general Equations (2.4) will be used to derive the admittance matrix for the transducer consisting of n possibly nonuniform basic sections [9].

Explicit expansion of the matrix equation (2.4) yields the upper two rows of the admittance matrix

$$\begin{bmatrix} \nu_{\text{in}} \\ \nu_{\text{out}} \end{bmatrix} = \begin{bmatrix} -P_{12}^{-1} P_{11} & P_{12}^{-1} & -P_{12}^{-1} A_{11} \\ -P_{21} + P_{22} P_{12}^{-1} P_{11} & -P_{22}P_{12}^{-1} & P_{22}P_{12}^{-1}A_{11} - A_{21} \end{bmatrix} \begin{bmatrix} F_{\text{in}} \\ F_{\text{out}} \\ V_3 \end{bmatrix} \quad \textbf{(2.6)}$$

where P_{11}, P_{12}, P_{22}, etc., represent the corresponding elements of the final (product) reverse chain matrix $Ч_R$ for the entire cascaded network.

To obtain the third row of Y is much more complicated, though straightforward. Toward this we introduce the matrices $C^{(i)}$ and $D^{(i)}$ through the equations

$$\begin{bmatrix} F_2{}^{(i)} \\ -\nu_2{}^{(i)} \end{bmatrix} = C^{(i)} \begin{bmatrix} F_{\text{in}} \\ \nu_{\text{in}} \end{bmatrix} + D^{(i)} V_3 \tag{2.7}$$

That is, $C^{(i)}$ and $D^{(i)}$ are evaluated through (2.4b) as the reverse chain matrix description from input through the i^{th} section. Then we find

$$\begin{aligned} I_3 = \sum_{i=1}^{n} (-1)^{i-1} y_{13}{}^{(i)} [\{[C_{11}{}^{(i)} + C_{11}{}^{(i-1)}] - [C_{12}{}^{(i)} + C_{12}{}^{(i-1)}] P_{11} P_{12}^{-1}\} F_{\text{in}} \\ + [C_{12}{}^{(i)} + C_{12}{}^{(i-1)}] P_{12}^{-1} F_{\text{out}} + \{[D_{11}{}^{(i)} + D_{11}{}^{(i-1)}] \\ - [C_{12}{}^{(i)} + C_{12}{}^{(i-1)}] A_{11} P_{12}^{-1}\} V_3] + \sum_{i=1}^{n} y_{33}{}^{(i)} V_3 \end{aligned} \tag{2.8}$$

Equations (2.6) and (2.8) give all the admittance matrix elements for n sections, possibly of unequal type. Since the resulting 3-port is reciprocal, as an interconnection of reciprocal subnetworks, we know that $Y_{12} = Y_{21}$, $Y_{13} = Y_{31}$, $Y_{23} = Y_{32}$.

D. Equal section matrices. The situation where all sections are equal is customary and mathematically much more tractable with somewhat simple expression resulting. From the previous equations, (2.1), (2.3), (2.4), we have

$$Ч_R = (Ч_R{}^{(1)})^n = \frac{(-1)^n}{y_{12}{}^n} \begin{bmatrix} y_{11} & -1 \\ y_{12}{}^2 - y_{11}{}^2 & y_{11} \end{bmatrix}^n \tag{2.9a}$$

and, on letting $n - i = k$,

$$A = \sum_{i=1}^{n} (Ч_R{}^{(1)})^{n-i} \{(-1)^{i-1} A^{(1)}\} = \sum_{k=0}^{n-1} (Ч_R{}^{(1)})^k (-1)^{n-k} \frac{y_{13}}{y_{12}} \begin{bmatrix} 1 \\ y_{12} - y_{11} \end{bmatrix} \tag{2.9b}$$

By the use of induction, an explicit evaluation of matrix elements in these expressions gives

$$Y_{13} = y_{13}, \; Y_{23} = (-1)^{n-1} y_{13} \tag{2.9c}$$

Here Y_{ij} are the matrix elements of the final admittance matrix Y for the n cascaded sections, and y_{ij} are the matrix elements for the single electrode section. It remains to calculate I_3 to complete the admittance matrix since Equation (2.6) yields the first two rows. But by reciprocity $Y_{31} = y_{13}$, $Y_{32} = Y_{23}$, which can also be checked by a direct but long calculation. In I_3 the coefficient of V_3 is the sum of the $y_{33}^{(i)}$, as per Equation (2.8); in the equal section case this is ny_{33}.

Therefore, the admittance matrix Y for the n equal section case is given by

$$Y = \begin{bmatrix} -P_{12}^{-1}P_{11} & P_{12}^{-1} & y_{13} \\ P_{12}^{-1} & -P_{12}^{-1}P_{11} & (-1)^{n-1}y_{13} \\ y_{13} & (-1)^{n-1}y_{13} & ny_{33} \end{bmatrix} \tag{2.10}$$

This is the admittance matrix for the "n cascaded" network shown in Figure 3 where the n sections are identical, the P_{ij} being the entries of $Ч_R$ as given in Equation (2.9a).

E. The scattering matrix. The scattering matrix S of a 3-port network characterized by its admittance matrix Y is given by [11, p. 51]

$$S = l_3 - 2Y(l_3 + Y)^{-1} \tag{2.11}$$

where l_3 is the 3×3 identity matrix. Here $Y(l_3 + Y)^{-1}$ can be physically interpreted as the augmented admittance matrix. As can be appreciated, evaluation of the scattering matrix for an n section SAW transducer is tedious in view of the expressions obtained above for the admittance matrix.

3. Scattering matrix of n equal sections

We next find explicit expressions for the admittance and scattering matrices of an IDT consisting of n identical sections represented by the hybrid model. These are then reduced to the in-line and crossed-field models by specification of α.

At Equation (2.5a) we have already given the admittance matrix of a single section to which Equation (2.9a) can be applied to get the reverse chain matrix after considerable manipulation (see Appendix) as

$$Ч_R = \sinh(n\delta(p)) \begin{bmatrix} \operatorname{ctnh}(n\delta(p)) & \dfrac{-Z_0 \sinh(\delta(p))[1 - \frac{K^2}{\gamma p}\sinh\gamma p]}{\sinh\gamma p} \\ \dfrac{-Y_0 \sinh\gamma p}{\sinh(\delta(p))(1 - \frac{K^2}{\gamma p}\sinh\gamma p)} & \operatorname{ctnh}(n\delta(p)) \end{bmatrix} \tag{3.1a}$$

where

$$\sinh(\delta(p)) = -\frac{\{[\sinh\gamma p + \frac{2K^2}{\gamma p}(1-\cosh\gamma p)][\sinh\gamma p]\}^{1/2}}{[1-\frac{K^2}{\gamma p}\sinh\gamma p]} \tag{3.1b}$$

$$K^2 = \frac{\alpha Y_o \gamma}{C_S} = \alpha k^2 \tag{3.1c}$$

Using (3.1a) in Equation (2.10) yields the admittance matrix of n equal sections as

$$Y(p) = \frac{Y_o}{D(p)} \times \tag{3.2a}$$

$$\begin{bmatrix} D^{1/2}(p)\cdot\text{ctnh}(n\delta(p)) & -D^{1/2}(p)\cdot\text{csch}(n\delta(p)) & \phi\tanh\left(\frac{\gamma p}{2}\right) \\ -D^{1/2}(p)\cdot\text{csch}(n\delta(p)) & D^{1/2}(p)\cdot\text{ctnh}(n\delta(p)) & (-1)^{n-1}\phi\tanh\left(\frac{\gamma p}{2}\right) \\ \phi\tanh\left(\frac{\gamma p}{2}\right) & (-1)^{n-1}\phi\tanh\left(\frac{\gamma p}{2}\right) & Z_0 p C_T[D(p)+\frac{2k^2}{p}\tanh\left(\frac{\gamma p}{2}\right)] \end{bmatrix}$$

where

$$C_T = nC_0, \quad D(p) = 1-\frac{2K^2}{\gamma p}\tanh\left(\frac{\gamma p}{2}\right) \tag{3.2b}$$

Substitution of this admittance matrix into the expression for S in terms of Y, Equation (2.11), yields the scattering matrix. After extensive calculations the results are

$$S_{11} = S_{22} = \frac{1}{M}\Big[\left(1-\frac{Y_o^2}{D}\right)\{1+pC_T[1+\frac{2k^2}{\gamma pD}\tanh\left(\frac{\gamma p}{2}\right)]\}$$
$$+\frac{2\phi^2 Y_0^3}{D^{5/2}}\tanh^2\left(\frac{\gamma p}{2}\right)\{\text{ctnh}(n\delta)+(-1)^{n+1}\text{csch}(n\delta)\}\Big] \tag{3.3a}$$

$$S_{12} = S_{21} = \frac{2}{M}\Big[\frac{Y_0}{D^{1/2}}\text{csch}(n\delta)\{1+pC_T[1+\frac{2k^2}{\gamma pD}\tanh\left(\frac{\gamma p}{2}\right)]\} \tag{3.3b}$$
$$+(-1)^{n+1}\frac{\phi^2 Y_0^2}{D^2}\tanh^2\left(\frac{\gamma p}{2}\right)\Big]$$

$$S_{13} = S_{31} = \frac{2Y_0}{M}\left[\frac{\phi}{D}\tanh\left(\frac{\gamma p}{2}\right) + \frac{\phi Y_0}{D^{3/2}}\tanh\left(\frac{\gamma p}{2}\right)\cdot \right. \tag{3.3c}$$

$$\left.\{\operatorname{ctnh}(n\phi) + (-1)^{n+1}\operatorname{csch}(n\phi)\}\right]$$

$$S_{23} = S_{32} = (-1)^{n-1}S_{13} \tag{3.3d}$$

$$S_{33} = \frac{1}{M}\left[\{1 - pC_T[1 + \frac{2k^2}{\gamma pD}\tanh\left(\frac{\gamma p}{2}\right)]\}\{\left(1 + \frac{Y_0^2}{D}\right)\right.$$

$$+\frac{2Y_0}{D^{1/2}}\operatorname{ctnh}(n\delta)\} + \frac{2\phi^2 Y_0^2}{D^2}\tanh^2\left(\frac{\gamma p}{2}\right) + \frac{2\phi^2 Y_0^3}{D^{5/2}}\tanh^2\left(\frac{\gamma p}{2}\right)\cdot \tag{3.3e}$$

$$\left.\{\operatorname{ctnh}(n\delta) + (-1)^{n+1}\operatorname{csch}(n\delta)\}\right]$$

where

$$M = \left[\ \{1 + pC_T[1 + \frac{2k^2}{\gamma pD}\tanh\left(\frac{\gamma p}{2}\right)]\}\{(1 + \frac{Y_0^2}{D} + \frac{2Y_0}{D^{1/2}}\operatorname{ctnh}(n\delta)\}\right. \tag{3.3f}$$

$$\left.-\frac{2\phi^2 Y_0^2}{D^2}\tanh^2\left(\frac{\gamma p}{2}\right) - \frac{2\phi^2 Y_0^3}{D^{5/2}}\tanh^2\left(\frac{\gamma p}{2}\right)\{\operatorname{ctnh}(n\delta) + (-1)^{n+1}\operatorname{csch}(n\delta)\}\right]$$

and D is given by Equation (3.2b).

For the hybrid model the matrix entries S_{ij} are given by Equations (3.3). It is also clear that these results are valid for both the in-line model, $\alpha = 1$, and the crossed-field model, $\alpha = 0$, of a basic section. For the in-line model there is essentially no simplification in the results since $K^2 = \alpha k^2$ is just replaced by k^2. But in the crossed-field situation considerable simplification results.

Setting $\alpha = 0$ for the crossed-field case gives $K = 0$ and hence, by (3.1b), $\delta(p) = \gamma p$. Hence (3.1a) becomes

$$\text{Ч}_R = \sinh\ (n\gamma p)\begin{bmatrix}\operatorname{ctnh}(n\gamma p) & -Z_o \\ -Y_o & \operatorname{ctnh}(n\gamma p)\end{bmatrix} \tag{3.4}$$

while from (2.13a) the admittance matrix is

$$Y = Y_o \begin{bmatrix} \operatorname{ctnh}(n\gamma p) & -\operatorname{csch}(n\gamma p) & \phi\tanh\left(\frac{\gamma p}{2}\right) \\ -\operatorname{csch}(n\gamma p) & \operatorname{ctnh}(n\gamma p) & (-1)^{n-1}\phi\tanh\left(\frac{\gamma p}{2}\right) \\ \phi\tanh\left(\frac{\gamma p}{2}\right) & (-1)^{n-1}\phi\tanh\left(\frac{\gamma p}{2}\right) & [Z_o pC_T + 2n\phi^2 \\ & & \tanh\left(\frac{\gamma p}{2}\right)] \end{bmatrix} \quad \textbf{(3.5)}$$

Either by direct substitution of Y in the formula for S or by specializing Equations (3.3) we have the scattering matrix elements of the crossed-field model as (recall that $C_T = nC_o$):

$$S_{11} = S_{22} = \frac{1}{M}\left[\{(1 - Y_o^2)(1 + pC_T + \frac{2n\phi^2}{Z_0}\tanh\left(\frac{\gamma p}{2}\right))\}\right.$$

$$\left. + \frac{2\phi^2}{Z_0^3}\tanh^2\left(\frac{\gamma p}{2}\right)\{\operatorname{ctnh}(n\gamma p) + (-1)^{n+1}\operatorname{csch}(n\gamma p)\right] \quad \textbf{(3.6a)}$$

$$S_{12} = S_{21} = \frac{2}{M}\left[Y_o\operatorname{csch}(n\gamma p)\{1 + pC_T + 2nY_o\phi^2\right.$$

$$\left.\tanh\left(\frac{\gamma p}{2}\right)\} + (-1)^{n+1}\phi^2 Y_o^2 \tanh^2\left(\frac{\gamma p}{2}\right)\right] \quad \textbf{(3.6b)}$$

$$S_{13} = S_{31} = \frac{2}{M}\left[\phi Y_o \tanh\left(\frac{\gamma p}{2}\right)\{1 + Y_o\operatorname{ctnh}(n\gamma p) + \right.$$

$$\left.(-1)^{n+1}Y_o\operatorname{csch}(n\gamma p)\}\right] = (-1)^{n-1}S_{23} = (-1)^{n-1}S_{32} \quad \textbf{(3.6c)}$$

$$S_{33} = \frac{1}{M}\left[\{1 - pC_T - 2nY_o\phi^2\tanh\left(\frac{\gamma p}{2}\right)\}\{1 + Y_o^2 + 2Y_o\operatorname{ctnh}(n\gamma p)\right.$$

$$\left. + 2\phi^2 Y_o^2\tanh^2\left(\frac{\gamma p}{2}\right)\{1 + Y_o[\operatorname{ctnh}(n\gamma p) + (-1)^{n+1}\operatorname{csch}(n\gamma p)]\}\right] \quad \textbf{(3.6d)}$$

where

$$M = \left[\{1 + pC_T + 2nY_o\phi^2\tanh\left(\frac{\gamma p}{2}\right)\}\{1 + Y_o^2 + 2Y_o\text{ctnh}(n\gamma p)\}\right]$$

$$- 2\phi^2 Y_o^2\tanh^2\left(\frac{\gamma p}{2}\right)\left[1 + Y_o\{\text{ctnh}(n\gamma p) + (-1)^{n+1}\text{csch}(n\gamma p)\}\right] \quad (3.6e)$$

4. Discussion

Since scattering descriptions are natural for wave propagating systems, we have developed here the scattering matrix of SAW devices. For n equal sections represented by the hybrid equivalent circuit we have been able to explicitly calculate the scattering matrix which can henceforth be used for analytic studies of such SAW devices. The explicit expressions for the admittance and scattering matrices of an interdigital SAW transducer consisting of n nonequal sections can also be obtained with the help of general expressions derived in the previous sections. However, the reverse chain matrix Ч_R in the nonuniform case is not easy to calculate in a straightforward analytical way and becomes almost impossible for a large number of sections without the use of a digital computer. But the general form presented for the matrix elements are suitable for the computer-aided design of SAW transducer, especially for the analysis of dispersive delay lines and pulse compression filters. Therefore, the existence of the present analysis paves the way for more extensive design formalisms of surface acoustic wave devices. Considering the nature of the results, it would also be appropriate to develop the transfer scattering matrix description of SAW devices.

Because the results have been developed as functions of complex frequency p, they can be used for transient analysis of SAW devices, a topic which could also use some development.

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Appendix

Derivation of the Reverse Chain Matrix Ч_R for n Equal Sections of the Hybrid Model

Application of the similarity transformation for diagonalization of $\text{Ч}_R^{(i)}$ given by (2.5b) gives the reverse chain matrix Ч_R for n cascaded sections in a very convenient and useful form. The expression for $Ч_R^{(i)}$ can be rewritten as:

$$Ч_R^{(i)} = -\frac{1}{y_{12}}\begin{bmatrix} y_{11} & -1 \\ (y_{12}^2 - y_{11}^2) & y_{11}\end{bmatrix} = \begin{bmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d}\end{bmatrix} \tag{A.1}$$

where

$$\left.\begin{aligned} \underline{a} &= \underline{d} = -y_{11}y_{12}^{-1} \\ \underline{b} &= y_{12}^{-1} \\ \underline{c} &= -y_{12} + y_{11}^2 y_{12}^{-1} \end{aligned}\right\} \tag{A.2}$$

The matrix eigenvalue problem for $Ч_R^{(i)}$ can be written in the usual form

$$\begin{bmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{a}\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix} = \lambda \begin{bmatrix} x \\ y\end{bmatrix} \tag{A.3}$$

where $\begin{bmatrix} x \\ y\end{bmatrix}$ is the eigenfunction and λ is the eigenvalue of the matrix $Ч_R^{(i)}$. The eigenvalues of the equation (A.3) can easily be found to be

$$\left.\begin{aligned} \lambda_1 &= \underline{a} - \sqrt{\underline{b}\,\underline{c}} \\ \lambda_2 &= \underline{a} + \sqrt{\underline{b}\,\underline{c}} \end{aligned}\right\} \tag{A.4}$$

Using (A.2) and (A.4), one can write

$$\lambda = -y_{11}y_{12}^{-1} \pm \sqrt{(y_{11}^2 y_{12}^{-2} - 1)} \tag{A.5}$$

Also simple calculations lead to the eigenfunctions corresponding to two different eigenvalues in the form

$$\begin{bmatrix} x \\ y\end{bmatrix}_1 = \frac{1}{\sqrt{\underline{b}(\underline{b}+\underline{c})}}\begin{bmatrix} \underline{b} \\ -\sqrt{\underline{b}\,\underline{c}}\end{bmatrix} \tag{A.6}$$

$$\begin{bmatrix} x \\ y\end{bmatrix}_2 = \frac{1}{\sqrt{\underline{b}(\underline{b}+\underline{c})}}\begin{bmatrix} \underline{b} \\ \sqrt{\underline{b}\,\underline{c}}\end{bmatrix} \tag{A.7}$$

where the subscripts 1 and 2 refer to the eigenfunctions corresponding to $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively. From these eigenfunctions one can construct a new matrix T such that $T^{-1}Ч_R^{(i)}T$ would be a diagonal matrix having the characteristic roots of $Ч_R^{(i)}$ as diagonal elements. The use of Equations (A.6) and (A.7) gives

$$T = \frac{1}{\sqrt{\underline{b}(\underline{b}+\underline{c})}} \begin{bmatrix} \underline{b} & \underline{b} \\ -\sqrt{\underline{b}\,\underline{c}} & \sqrt{\underline{b}\,\underline{c}} \end{bmatrix} \tag{A.8}$$

Therefore

$$T^{-1}\text{Ч}_R^{(i)}T = \frac{1}{2\underline{b}\sqrt{\underline{b}\,\underline{c}}} \begin{bmatrix} \sqrt{\underline{b}\,\underline{c}} & -\underline{b} \\ \sqrt{\underline{b}\,\underline{c}} & \underline{b} \end{bmatrix} \begin{bmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{a} \end{bmatrix} \begin{bmatrix} \underline{b} & \underline{b} \\ -\sqrt{\underline{b}\,\underline{c}} & \sqrt{\underline{b}\,\underline{c}} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{a}-\sqrt{\underline{b}\,\underline{c}} & 0 \\ 0 & \underline{a}+\sqrt{\underline{b}\,\underline{c}} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \equiv \lambda \tag{A.9}$$

Thus it follows from (A.9) that

$$(\text{Ч}_R^{(i)})^n = T\lambda^n T^{-1} \tag{A.10}$$

Finally, using the results (A.2), (A.8), and (A.9) in equation (A.10), one obtains a more convenient form of the product reverse chain matrix as

$$(\text{Ч}_R^{(i)})^n = \frac{1}{2} \begin{bmatrix} (\lambda_1^n+\lambda_2^n) & (y_{11}^2-y_{12}^2)^{-1/2}(\lambda_2^n-\lambda_1^n) \\ (y_{11}^2-y_{12}^2)^{1/2}(\lambda_2^n-\lambda_1^n) & (\lambda_1^n+\lambda_2^n) \end{bmatrix} \tag{A.11}$$

This general result (A.11) is now utilized to calculate the reverse chain matrix Ч_R for n equal cascaded sections of the hybrid model.

The Equation (A.5) together with the Equation (2.5a) gives

$$\lambda = \frac{(\operatorname{ctnh}\gamma p - \frac{\alpha}{C_s Z_o p})}{(\operatorname{csch}\gamma p - \frac{\alpha}{C_s Z_o p})} \pm j\,\frac{[-1+\frac{2\alpha}{C_s Z_o p}(\operatorname{ctnh}\gamma p - \operatorname{csch}\gamma p)]^{1/2}}{(\operatorname{csch}\gamma p - \frac{\alpha}{C_s Z_o p})}$$
$$= a \pm jb \tag{A.12}$$

λ_1 and λ_2 can be written in the form

$$a - jb = \lambda_1 = |\lambda_1|e^{j\theta}; \quad j\theta = -\delta(p) = j\arctan(-b/a)$$
$$\lambda_2 = |\lambda_2|e^{-j\theta} \tag{A.13}$$

where $|\lambda_1| = |\lambda_2| = 1$. The result leads to

$$Ч_R = (Ч_R^{(i)})^n = \sinh(n\delta(p)) \begin{bmatrix} \text{ctnh}(n\delta(p)) & -(y_{11}^2 - y_{12}^2)^{-1/2} \\ -(y_{11}^2 - y_{12}^2)^{1/2} & \text{ctnh}(n\delta(p)) \end{bmatrix} \quad \textbf{(A.14)}$$

$\delta(p)$ would, according to Equation (A.12), be given by (since $j\tan\theta = \tanh\delta(p)$)

$$\tanh(\delta(p)) = \frac{\{[\sinh\gamma p + \frac{2K^2}{\gamma p}(1 - \cosh\gamma p)][\sinh\gamma p]\}^{1/2}}{(\cosh\gamma p - \frac{K^2}{\gamma p}\sinh\gamma p)} \quad \textbf{(A.15)}$$

with K^2 defined by Equation (3.1c). Equation (3.1b), for $\sinh\delta(p)$, is found by using $\tanh^2\delta = \sinh^2\delta/[1 + \sinh^2\delta]$. The sign of $\sinh\delta$ is chosen to give the correct $y(p)$ for $n = 1$.

Finally, direct substitution of y_{11} and y_{12} from Equation (2.5a) into (A.14) yields (3.1a) as the final form of the reverse chain matrix $Ч_R$ for n equal sections of the hybrid model.

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