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THE FOUNDATIONS OF NETWORK THEORY

by

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ABSTRACT

In these few pages, reader you will find
Complete a theory, from McMillan does it stem,
To satisfy the criticizing mind.

The topic network's pure foundation treats.
With definitions it begins, such is system;
Then n-port properties one here soon meets.

For general networks are such defined;
Then comes the distributional extension
For special type n-ports this holds refined.

The end has exponential variables to greet,
With generalized descriptions linear, the mention,
And paradoxes as an unsolved feat.

CONTENTS

ILLUSTRATIONS

I. INTRODUCTION

Recently much interest has centered on obtaining a rigorous theory of networks and their properties. Such interest seems to have been originated by McMillan and Raisbeck[1*] and has culminated in several detailed theories.[2] Of especial note are the results of Youla, Castriota and Carlin and the extensions of Zemanian. However, although these are quite rigorous, they are, we feel quite complicated for what is required. Further these treatments are strictly limited to linear, time-invariant, passive networks and even in this case are open to several interpretations which can lead to contradictory results.[3]

Here we present a basic theory of what we believe to be the most general types of conceivable networks. For this we abandon the operator concept, which we feel is more appropriate to control system studies, and revert to the concept of allowed pairs, whose origin can be traced to McMillan. The program is to define a system, a network, and an n-port in terms of infinitely differentiable functions of support bounded on the left. The major properties of such n-ports are then defined, such as linearity, time-invariance, passivity and solvability. Such physical networks are then extended to idealized networks and more generalized variables through the use of the theory of distributions. Once this is done, general descriptions of a large class of linear, time-invariant networks can be given by using exponential variables. At this point the (bilateral) Laplace transform approach can be used in the standard manner.

The reader is assumed to have a working knowledge of elementary matrix theory, as well as a small knowledge of the theory of distributions. The latter however, isn't used until Section IV, and it is available in a companion report.[4] Points which are of interest, but supplementary to the main portion of the theory are expanded upon in the appended notes.

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* The superscript numerals refer to the notes at the end of the report.

II. SYSTEMS, NETWORKS & n-PORTS

We begin with the notions of systems and networks. Consider a collection of physical elements which are connected together to form some device. At this stage the entities are not restricted, they may be electrical, mechanical, thermodynamical, etc. For the present it is sufficient to look at the device at the access points, called terminals, and thus essentially ignore the internal construction. In general the device will have k such terminals which can be consecutively labeled $\mathcal{T}_j$, $j = 1, \ldots, k$, where k may in fact be infinite. In order to obtain a mathematical model, we associate with the device two column vector variables, $\underline{f}(t)$ and $\underline{g}(t)$, of k entries, with the j^{th} entry being associated with $\mathcal{T}_j$. $\underline{f}$ and $\underline{g}$ are called conjugate variables and are taken such that $\tilde{\underline{f}}\,\underline{g}$ is the total instantaneous power entering the device, here the superscript tilde denotes matrix transposition.[5] Several ways are available for choosing the variables. In the electrical case, choosing $\underline{f}$ as voltage and $\underline{g}$ as current, two possibilities are shown in Fig. 1.

We have to delete the kth entry, say, in $\underline{f}$ and $\underline{g}$ for $\tilde{\underline{f}}\,\underline{g}$ = total power in, otherwise $\tilde{\underline{f}}\,\underline{g} = 0$

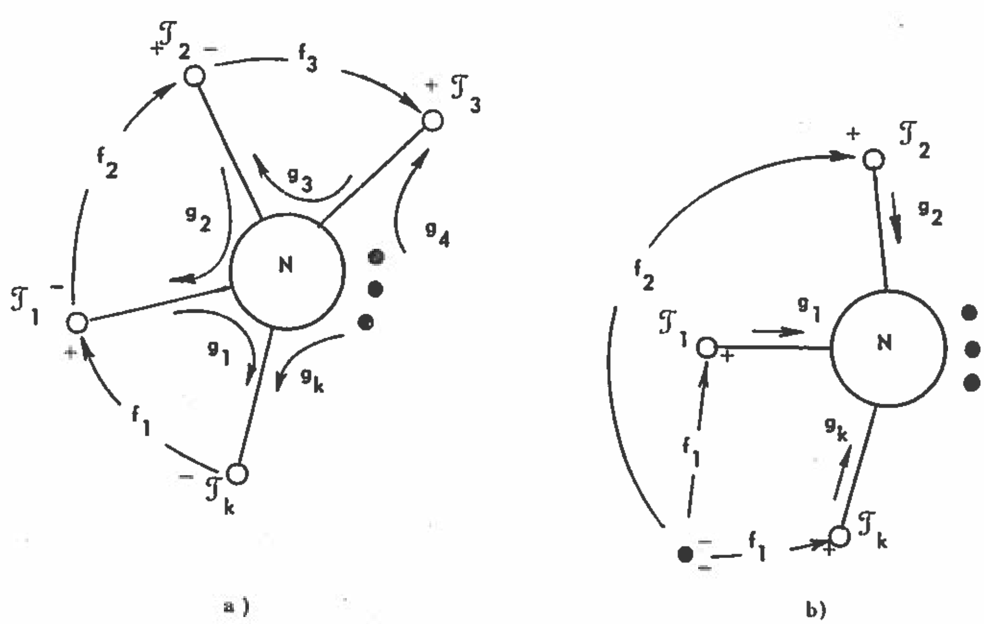


FIG. 1. POSSIBLE VARIABLE CHOICES.

We will let $\mathfrak{D}_{+k}$ denote the set of real-valued, k-vector, infinitely continuously differentiable functions of the real time variable, t, $-\infty \leq t \leq \infty$, with support bounded on the left.[6] Further, let $\mathfrak{D}_{Nk}$ be the same as $\mathfrak{D}_{+k}$, except that the support is bounded on the left by a fixed constant t_N, t_N may be infinite. For a given device we will then postulate that t_N exists and that[7] $\underline{f} \in \mathfrak{D}_{+k} \cap \mathfrak{D}_{Nk}$, $\underline{g} \in \mathfrak{D}_{+k} \cap \mathfrak{D}_{Nk}$; t_N is to be interpreted as the creation time of the device. This restriction on the variable is a physical one, since such quantities appear to be of the type actually measured.[8] However, by a limiting process we will later extend the domain of definition.

Now given a specific device, some constraint, C, is placed upon the $\underline{f}$ and the $\underline{g}$. We will incorporate in C the fact that $\underline{f}$ and $\underline{g}$ must be conjugate. For simplicity we will then denote these constraint requirements by $\underline{f}C\underline{g}$, which can be interpreted as follows. Given an $\underline{f}$, which, by virtue of the constraints, is allowed at the terminals, there is an accompanying set of allowed $\underline{g}$'s. Thus we can speak of allowed pairs of variables $\underline{f}$, $\underline{g}$. The set of all allowed pairs then serves to completely describe the device, which we define as a k-access system, $\mathcal{S}$. In more precise language, given t_N, a system is defined by[7]

$$\mathcal{S} = \{[\underline{f}, \underline{g}] \; \underline{f}, \underline{g} \in \mathfrak{D}_{+k} \cap \mathfrak{D}_{Nk}, \; \underline{f}C\underline{g}\} \tag{1}$$

Note that although a system may be created at $t_N = -\infty$, it can't be excited until some finite time, in this formulation. If the variables are electrical, we call the system a <u>network</u>. For networks we will assume $\underline{f}$ to be a voltage vector.[9] The remaining treatment will be limited to networks.

As an example consider a resistor $\mathcal{S}_r$, of resistance r, which was created with the universe, $t_N = -\infty$.[10] We can then apply any current, $g \in \mathfrak{D}_{+1}$, across the terminals. The voltage is then $f = rg$ and

$$\mathcal{S}_r = \{[f, g] | f \in \mathfrak{D}_{+1}, \; f = rg\}$$

The allowed pairs for the resistor are of the form $[rg, g]$ for any $g \in \mathfrak{D}_{+1}$.

In many cases the description given above yields too much information, and it is advantageous to delete the extraneous data. For many applications the terminals of a network are associated in n disjoint pairs with excitations being applied only between the two terminals of a pair. If such is the case, we take k even, and it is convenient to relabel the terminals, calling those for the j^{th} pair T_j and T'_j, $j = 1, \ldots, n = k/2$. In this situation, if the current entering T_j is equal to the current leaving T'_j, this pair is called a port. If all terminals occur as ports, the network is called an n-port, N.[11] For an n-port, only the n-vectors $\underline{v}(t)$ and $\underline{i}(t)$ of port voltages and currents are of interest. The variables for an n-port will be taken as shown in Fig. 2a) with Fig. 2b) being a convenient representation.

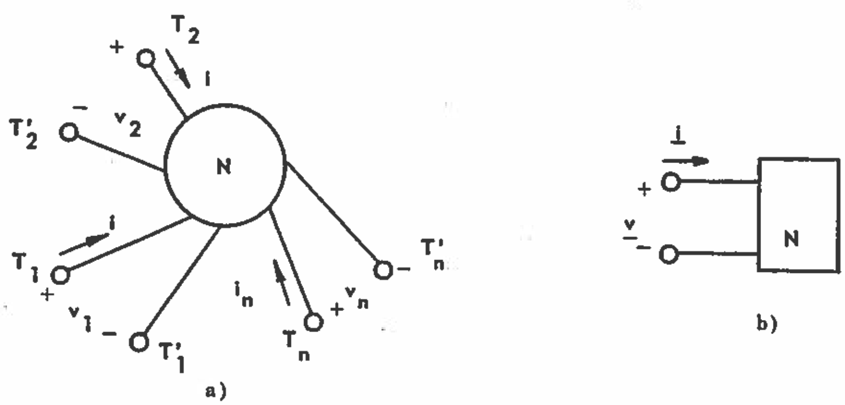


FIG. 2. n-PORT VARIABLES.

The network of Fig. 1a), for example, becomes an n-port if $k = 2n$ and we choose $v_j = f_{2j-1}$, $i_j = g_{2j-1}$, $g_{2j} = 0$, for $j = 1, \ldots, n$. Carrying over the previous language we characterize an n-port by

$$N = \{[\underline{v}, \underline{i}] | \underline{v}, \underline{i} \in \mathfrak{D}_{+n} \cap \mathfrak{D}_{Nn}, \underline{v}C\underline{i}\} \qquad (2)$$

The following examples serve to define the symbolism for some important and interesting 1-port and 2-port N.

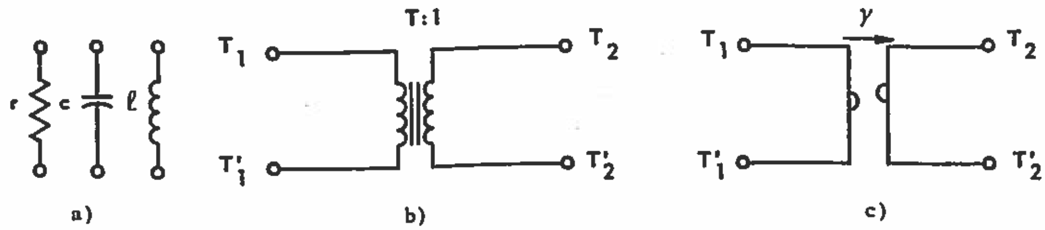


FIG. 3. REPRESENTATIONS FOR EXAMPLE 1.

Example 1:

a) The linear, time-invariant, resistor, capacitor, and inductor are illustrated in Fig. 3a). The resistor has allowed pairs $[ri, i]$ for any $i \in \mathcal{D}_{+1}$. The capacitor is described by $[v, d(cv)/dt]$ for any $v \in \mathcal{D}_{+1}$. An equally valid description, if $c \neq 0$ is $[\frac{1}{c}\int_{-\infty}^{t} i(x)dx, i]$ for any $i \in \mathcal{D}_{+1}$. Dually the inductor has $[d(\ell i)/dt, i]$ for any $i \in \mathcal{D}_{+1}$. Here r, ℓ, c are all real numbers.

b) The describing equations of the ideal transformer of Fig. 3b) are

$$v_1 = Tv_2$$

$$i_2 = -Ti_1$$

with the turns ratio T real. With $v_2, i_1 \in \mathcal{D}_{+1}$, but otherwise arbitrary, the allowable pairs are $[v_2\begin{bmatrix} T \\ 1 \end{bmatrix}, i_1\begin{bmatrix} 1 \\ -T \end{bmatrix}]$. Note that $\underline{v}$ and $\underline{i}$ are not related and that not all $\underline{v}$ are allowed; only those with $v_1 = Tv_2$ are acceptable. For each allowed $\underline{v}$ there are an infinite number of allowed $\underline{i}$. Since $\underline{\tilde{v}i} = 0$, the total power input is zero, and, when $T = 0$, port one is a short-circuit while port two is an open-circuit.

c) The ideal gyrator of Fig. 3c) is described by

$$v_1 = \gamma i_2$$

$$v_2 = -\gamma i_1$$

with the gyration resistance, γ, being real.[12]

$[\gamma \begin{bmatrix} i_2 \\ -i_1 \end{bmatrix}, \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}]$, for any $\underline{i} \in \mathcal{D}_{+2}$, are the allowed pairs. As with the transformer, the total power input is identically zero. If $\gamma = 0$, both ports are shorts, while, if we allow $\gamma = \infty$, both ports are opens.

The networks of Example 1 are all idealizations of actual physical elements. The inclusion of the ideal transformer is a convenience which allows a comparatively simple mathematical treatment. However, their use in physical structures is to be avoided since realizations require closely coupled coils with extremely large self and mutual inductances. Ideal gyrators have been practically realized only for microwave networks;[13] their inclusion is for completeness.

In contrast to the networks of Example 1 there is a whole class of degenerate networks which must be considered, in spite of the fact that they seem to bear no relationship to actual physical devices. Typical of these are the nullator, for which the only allowed pair is [0, 0], and the norator, which is described by [v, i] with v arbitrary and i arbitrary but independent of v.[14] These are, respectively, represented by the symbols of Fig. 4a) and b) where realizations in terms of the elements of Example 1 are also given. Such degenerate networks are often valuable for illustrating ideas, checking results, and investigating the generality of a concept.

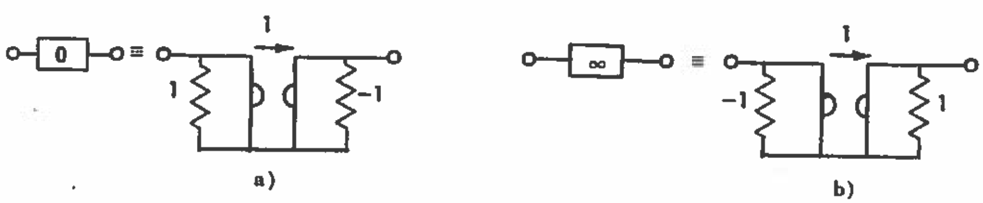


FIG. 4. NULLATOR AND NORATOR.

In general we will limit the treatment to n-ports, in contrast to k-terminal networks. Although this appears to put a restriction on the generality of the theory, such is not the case, since we are allowing the use of transformers. Thus, Cauer has shown how any k-terminal network can be made equivalent to a (k-1)-port.[15] The procedure is illustrated for k = 4 in Fig. 5 from which the extension to arbitrary k should be clear.

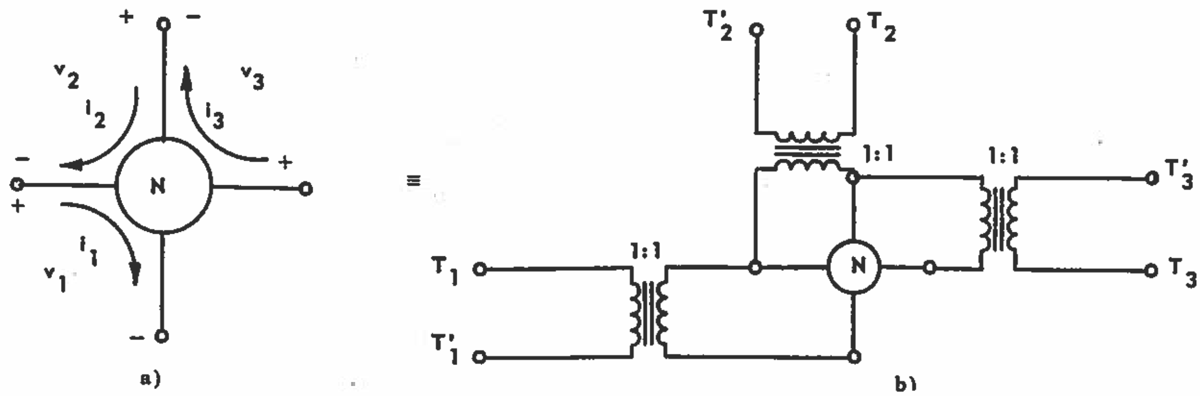


FIG. 5. CONVERSION OF 4 TERMINAL NETWORK TO A 3-PORT.

If, in Fig. 5, the variables are chosen as indicated, the network in a) is described by the same relationships as the 3-port in b).[16] The process is somewhat familiar from transistor theory where the transformers are omitted when treating the three-terminal transistor as a 2-port.

III. PROPERTIES OF n-PORTS

The description of N by the use of the allowable pairs is extremely general. Using this generality, we can define some specific properties in such a manner as to be applicable in the widest possible context. After this is accomplished we can extend the notion of an n-port and obtain more tractable descriptions for several important classes of n-ports.

We begin with the concept of linearity which is conveniently phrased in the $[\underline{v}, \underline{i}]$ language.[17]

Definition 1:

N is linear, if for every $[\underline{v}_1, \underline{i}_1]$, $[\underline{v}_2, \underline{i}_2] \epsilon N$ and all real constants, a,

(H) $[a\underline{v}_1, a\underline{i}_1] \epsilon N$ (homogeneity)

(A) $[\underline{v}_1+\underline{v}_2, \underline{i}_1+\underline{i}_2] \epsilon N$ (additivity)[18]

Physically the definition means that at the ports superposition holds. If N is not linear it will be called nonlinear. Apparently every physical N is actually nonlinear. At best any existing N will satisfy definition 1 for only some pairs and some a. Linearity is then an idealization which is strictly only valid in practice over some range of the variables. In spite of this, most of the theory will be limited to linear N. This is justified by the fact that a first approximation to a nonlinear N is a linear N. Further, very general results are available for linear N, while results for only specific nonlinear networks can be found.

The N of Example 1 are all linear while the network described by $[+\sqrt{i^2}, i]$ is not a linear N, if i is arbitrary. For control systems an important nonlinear network is the 1-port defined by the curve of Fig. 6a) and called the saturator. Another nonlinear N is the network of Fig. 6b) for which

$$[0, E/r] + [v, (v+E)/r] = [v, (v+2E)/r] \neq [v, (v+E)/r]$$

with E a fixed function in $\mathcal{D}_{+1}$. Here the fourth and the first two pairs are allowable while the third one is not.

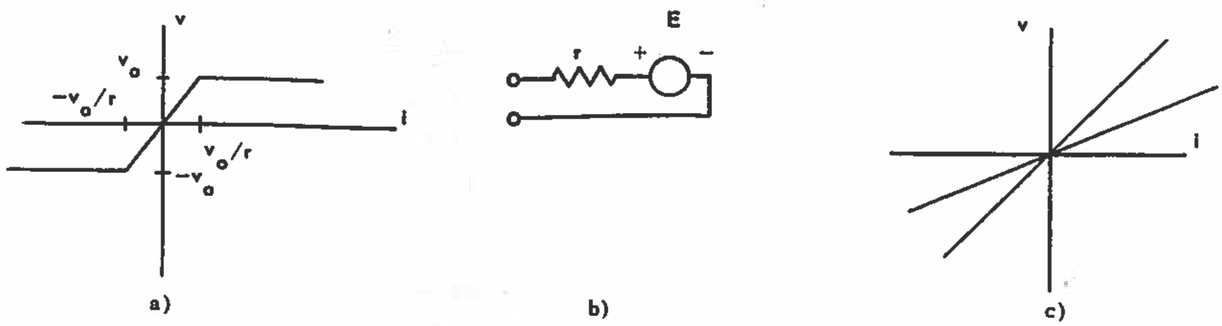


FIG. 6. NONLINEAR N.

Some interesting 1-ports show that homogeneity and additivity can be separately violated. Thus the N for which v and i take on only integer values for $t > t_o$, but for which v and i are in $\mathcal{D}_{+1}$ for all t, satisfies the additivity requirement but not the homogeneity one. The double-valued 1-port described by the curves of Fig. 6c) possesses homogeneity but not additivity.[19]

The ideal diode of Fig. 7a) is another useful nonlinear N. Using it in the linear network of Fig. 7b) shows that a network constructed from nonlinear elements need not be nonlinear.[20]

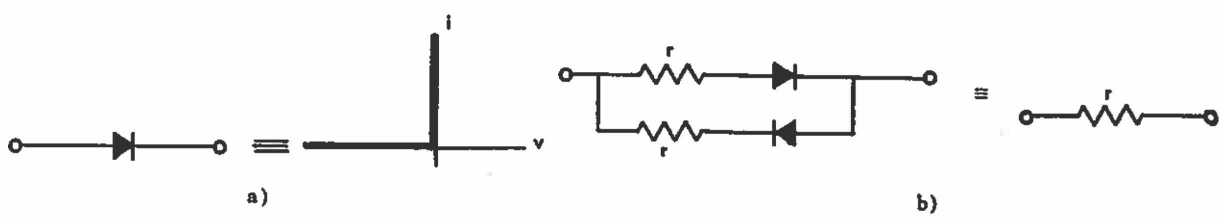


FIG. 7. THE IDEAL DIODE.

In passing, it is worth point out that the set of allowed pairs for a linear N forms a vector space over the field of real numbers, if we define[21]

$$a[\underline{v}, \underline{i}] = [a\underline{v}, a\underline{i}]$$

$$[\underline{v}_1, \underline{i}_1] + [\underline{v}_2, \underline{i}_2] = [\underline{v}_1+\underline{v}_2, \underline{i}_1+\underline{i}_2]$$

The second concept of interest is that of time-invariance. Physically we would like this to mean that no elements inside N have parameters which change in time. In terms of the terminal behavior this is best phrased in the following manner.[22]

Definition 2:

N is <u>time-invariant</u>, if for every $[\underline{v}, \underline{i}] \epsilon N$ there is a $[\underline{v}_o, \underline{i}_o] \epsilon N$ such that, for every real, finite, constant $\tau \geq 0$,

$$[\underline{v}(t), \underline{i}(t)] = [\underline{v}_o(t+\tau), \underline{i}_o(t+\tau)] \tag{3}$$

In this we are obliged to take $\tau \geq 0$ since, in general, $\underline{v}, \underline{i} \epsilon \mathcal{D}_{Nn}$. Pictorially v and v_o are related as shown in Fig. 8.

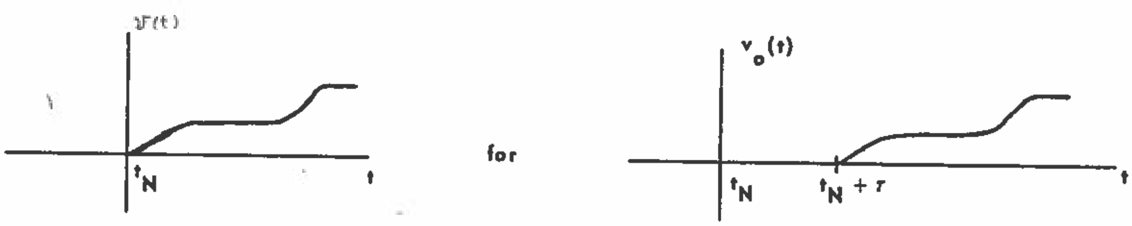


FIG. 8. VOLTAGES USED IN DEFINITION 2.

If N is not time-invariant, it is only logical to call it <u>time-varying</u> or time-variable. The networks of Example 1 are all time-invariant, if the parameters are independent of time. Thus for the capacitor any $[v(t), d\{cv(t)\}/dt]$ has $[v(t-\tau), d(cv(t-\tau)/dt]$ allowed for any $v \epsilon \mathcal{D}_{+1}$ and any $\tau \geq 0$, when c is a constant. However, if c = t then

$$[v(t), d\{tv(t)\}/dt] = [v_o(t+\tau), i_o(t+\tau)]$$

implies by a simple change of variable

$$v_o(t) = v(t-\tau)$$

$$i_o(t) = d\{(t-\tau)v(t-\tau)\}/dt$$

But for $\tau > 0$ this i_o is not allowed since such a v_o requires a current of the form

$$d\{tv_o(t)\}/dt = d\{tv(t-\tau)\}/d\tau$$

Thus if $c = t$ the capacitor is time-varying, which lends some credibility to the definition.

As with linearity, time-invariance is an idealization which apparently can never occur in nature. This statement is justified by the fact that most networks must be built. Although it is possible to conceive of networks existing since $t = -\infty$, as Shelley's universe,[10] the most interesting devices are presently being conceived. However, a study of time-variable networks appears to be in its infancy. If N is time-invariant, it can be extended from $\mathcal{D}_{+n} \cap \mathcal{D}_{Nn}$ to $\mathcal{D}_{+n}$ by assuming 3) to hold for finite $\tau > 0$.

Another concept of importance is that of passivity. This notion is somehow tied to the presence of internal sources, however, the negative resistor shows that conceptually, at least, the source idea isn't quite what is desired. Passivity is best defined in terms of energy.[23]

Definition 3:

N is <u>passive</u> if for every $[\underline{v}, \underline{i}] \epsilon N$ and every finite t

$$\mathcal{E}(t) = \int_{-\infty}^{t} \tilde{\underline{v}}(\tau)\underline{i}(\tau)d\tau \geq 0 \qquad (4)$$

If N is not passive it will be called <u>active</u>. In contrast to linearity and time-invariance, passivity can actually occur in nature. The transformer and the gyrator, as well as the nullator, are passive networks for which $\mathcal{E}(t) = 0$.[24] If the describing parameters of the resistor, capacitor, and inductor are positive constants, these are passive N. For instance the capacitor has, recalling that $v(-\infty) = 0$,

$$\mathcal{E}(t) = \int_{-\infty}^{t} cv(\tau)\frac{dv(\tau)}{d\tau}d\tau = cv^2(t)/2$$

Note that if $c < 0$, the capacitor is an active network. The passivity of a given device will sometimes depend upon the point of view taken when defining the mathematical model. For instance, if only the small signal behavior of a transistor is considered, the transistor is active. However, if the bias sources are considered as being externally connected the transistor is passive.

Another property which has been greatly ignored, but whose importance has been pointed out by Youla, is that of solvability.[25]

Definition 4:

N is <u>solvable</u> if for every $\underline{e} \epsilon \mathcal{D}_{+n} \cap \mathcal{D}_{Nn}$ the equation

$$\underline{e} = \underline{v} + \underline{i}$$

is satisfied by a unique $[\underline{v}, \underline{i}] \epsilon N$

Here we can consider a new n-port, called the augmented network, N_a, assumed created at t_N, as shown in Fig. 9, with $[\underline{e}, \underline{i}] \epsilon N_a$.

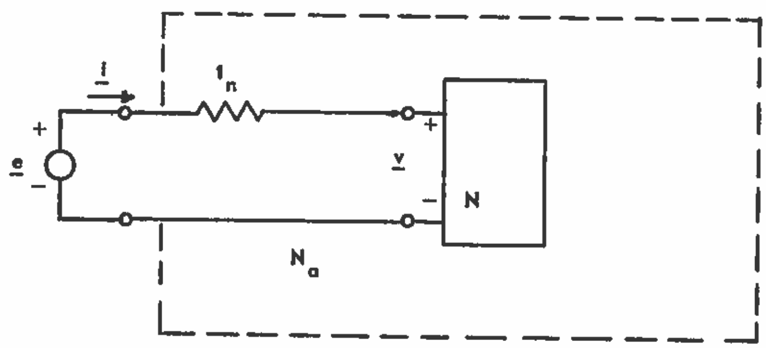


FIG. 9. AUGMENTED NETWORK.

Solvability doesn't seem to be an idealization,[26] assuming it, a matrix description will be guaranteed in the linear, time-invariant case. The open, the short, and the passive networks of Example 1 are all solvable while the nullaor, norator and negative unit resistor are not solvable.

If N has any of the properties of linearity, time-invariance or passivity then N_a has the same properties.[27] This is not the case for solvability as is seen by the original N being a minus two ohm resistor. However, if N is passive and solvable then N_a seems to also be solvable.

Although the notions of linearity, time-invariance, passivity and solvability are the most fundamental concepts with which we will work, there are several others of interest. Among these is that of reciprocity which stems directly from Maxwell's equations.[28] Physically a reciprocal network should have the voltage response due to a current excitation independent of an interchange of response and excitation points. Mathematically this is formulated in the following manner,[29] where the normal level asterisk denotes convolution, in this case of two vectors.

Definition 5:

N is <u>reciprocal</u> if for every $[\underline{v}_1, \underline{i}_1]$, $[\underline{v}_2, \underline{i}_2] \epsilon N$ we have

$$\tilde{\underline{v}}_1 * \underline{i}_2 = \tilde{\underline{v}}_2 * \underline{i}_1 \tag{6}$$

The difference between the power input to a network and the forms in (6) should be observed. In the latter, variables for the first measurement are convoluted by their conjugates for the second measurement. In contradistinction to the use of the words time-varying, customary usage requires that the class of nonreciprocal networks contains the class of reciprocal networks. It also appears that reciprocity is an idealization.

A transformer is a reciprocal 2-port, since

$$[Tv_{2_1}, v_{2_1}] * \begin{bmatrix} i_{1_2} \\ -Ti_{1_2} \end{bmatrix} = 0 = [Tv_{2_2}, v_{2_2}] * \begin{bmatrix} i_{1_1} \\ -Ti_{1_1} \end{bmatrix}$$

In contrast, the gyrator is a linear, passive 2-port which is not reciprocal, if $\gamma \neq 0$, since

$$[\gamma i_{2_1}, -\gamma i_{1_1}] * \begin{bmatrix} i_{1_2} \\ i_{2_2} \end{bmatrix} = \gamma(i_{1_2} i_{2_1} - i_{1_1} i_{2_2}) = -[\gamma i_{2_2}, -\gamma i_{1_2}] * \begin{bmatrix} i_{1_1} \\ i_{2_1} \end{bmatrix}$$

There are many linear 2-ports which are not reciprocal, but only one such 1-port, the norator. The ideal diode of Fig. 7 is a 1-port which is nonlinear, time-invariant, passive, solvable, but not reciprocal.

The notion of causality is of philosophical interest. Physically this means that the application of some variables at time t_o <u>causes</u> other variables to appear at time $t_1 \geq t_o$. We believe that by any mathematical theory it is impossible to tell what is a cause and what is an effect. Thus, we limit our ideas to the concept of antecedance which is sometimes confused with causality.[30]

Definition 6:

N is <u>antecedal</u> if for every $t_o > t_N$ there are $[\underline{0}, \underline{i}(t)]$, $[\underline{v}(t), \underline{0}] \epsilon N$ for $-\infty < t < t_o$, and for every such pair

a) $[\underline{0}, \underline{i}(t)] \epsilon N$ implies $\underline{i}(t) = \underline{0}$ for $-\infty < t < t_o$

b) $[\underline{v}(t), \underline{0}] \epsilon N$ implies $\underline{v}(t) = \underline{0}$ for $-\infty < t < t_o$

The resistor, inductor, capacitor and nullator are all antecedal, while the transformer, norator, opens and shorts are not. A concept somewhat related to that of antecedance is that of single-valuedness.

Definition 7:

N is <u>single-valued</u> if every $[\underline{v}, \underline{i}] \epsilon N$ has $\underline{i}$ uniquely determined by $\underline{v}$ and $\underline{v}$ uniquely determined by $\underline{i}$.

Typical single-valued networks are the resistor, inductor, nullator and gyrator. The saturator, transformer and norator are not single-valued. Clearly every antecedal N is single-valued but the converse need not be true, as is seen by Fig. 6b). As we shall see, the restriction of variables to $\mathfrak{D}_+$ rules out the non-antecedal and non-single-valuedness of some networks.

Almost every synthesis method is based upon the synthesis of lossless networks and hence the next definition is of some importance.[31]

Definition 8:

N is <u>lossless</u> if

S) N is solvable and

P) N is passive and

L) for every $\underline{e} \epsilon \mathfrak{D}_{+n} \cap \mathfrak{D}_{Nn} \cap L_{2n}$

$$\mathcal{E}(\infty) = \int_{-\infty}^{\infty} \tilde{\underline{v}}(\tau)\underline{i}(\tau)d\tau = 0 \tag{7}$$

Here $\underline{e}$ is $\underline{v} + \underline{i}$ as in (5). Relaxing the L_{2n} requirement causes trouble, since variables which don't vanish at infinity can give a finite, non-zero $\mathcal{E}(\infty)$. The passivity requirement is a convenience which will be used to assure that $\underline{v}$ and $\underline{i}$ will lie in L_{2n} if $\underline{e}$ does. Relaxing the passive requirement allows the active capacitor with c = v as a lossless network; however, the capacitor $c = v^2$ would still not be allowed since $\int_{-\infty}^{\infty} 3v^3 dv$ need not exist for $e \epsilon L_{21}$. The linear, passive inductor and capacitor, as well as the gyrator and transformer, are lossless, while the nullator and resistor are not. A network constructed only of lossless elements need not be lossless, as is seen by observing the inductor-capacitor equivalent circuit of an infinitely long transmission-line, which has a purely resistive input impedance.

IV. EXTENSIONS

Using the theory developed so far there are idealized situations which can't be considered. For instance we can't as yet excite with unit step functions or impulses, or handle initial conditions in the customary way, and a perfect DC battery doesn't qualify as a network. We therefore now relax some of the physical constraints and cover these and similar cases by taking suitable limits.

We will let $\mathcal{D}'_n$ denote the real-valued n-vector distributions and $\mathcal{D}'_{+n}$ be such distributions with support bounded on the left, $\mathcal{D}'_{Nn}$ is the same as $\mathcal{D}'_{+n}$ except the support is bounded at t_N. All the desired results will then follow from the fact that $\mathcal{D}_{+n}$ is dense in both $\mathcal{D}'_n$ and $\mathcal{D}'_{+n}$, that is, any distribution in $\mathcal{D}'_n$ or $\mathcal{D}'_{+n}$ is a limit of a sequence of infinitely differentiable functions in $\mathcal{D}_{+n}$.[32]

For example the idealized network of Fig. 10a) results by taking the limit of a sequence of networks of the type of Fig. 10b) where, as a distributional limit

$$\lim_{j\to\infty} u_j = u, \quad u_j \epsilon \ \mathcal{D}_{+1}, \quad u = \text{unit step}$$

A typical u_j is shown in Fig. 10c)

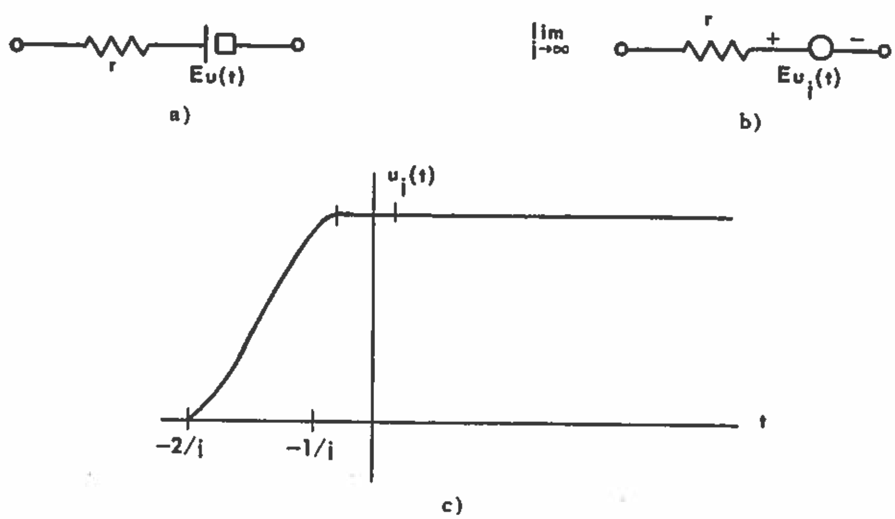


FIG. 10. LIMITING NETWORKS

If in Fig. 10c) we replace the points $-1/j$ and $-2/j$ by $-j$ and $-2j$, then when the limit is taken, $u(t)$ is replaced by the constant 1 for all time. This then allows consideration of perfect batteries. Since initial currents in inductors and voltages can be considered through the use of unit step functions, the above limiting process allows the standard procedure for incorporating initial conditions. The smooth functions of Fig. 10c) physically mean that such smooth variables, are the ones actually appearing at terminals when a finite network is constructed.

In general it is hard to state exactly when extensions to distributional variables can be made. For instance the nonlinear resistor $r = i$ can not be extended to impulsive currents $i = \delta$, since δ^2 is undefined.[33] However, in the linear, time-invariant and solvable case, to which most the remainder of this work is restricted, we can always extend N_a, and hence N, to distributional $\underline{e}$ and $\underline{i}$ in the following way.

By the solvability of N, if a sequence $\{\underline{e}_j\}$ has $\underline{e}_j \to 0$ then $\underline{v}_j \to -\underline{i}_j$ and these tend to a unique value. This value can only be zero, since by linearity $[\underline{0}, \underline{0}] \in N$. Consequently, if $\underline{e}_j \to \underline{0}$ then $\underline{i}_j \to \underline{0}$ and N_a is said to define a continuous mapping, defined as

$$\underline{i} = \mathcal{Y}_a[\underline{e}]$$

Now, exciting with $\{\underline{e}(t+\tau)-\underline{e}(t)\}/\tau$, the current becomes by linearity

$$\mathcal{Y}_a\left[\frac{e(t+\tau)-e(t)}{\tau}\right] = \{\mathcal{Y}_a[\underline{e}(t+\tau)] - \mathcal{Y}_a[\underline{e}(t)]\}/\tau$$

But by time-invariance, for $\tau > 0$, we can write this as $\{\underline{i}(t+\tau)-\underline{i}(t)\}/\tau$. Taking the limit as $\tau \to 0$ we see that

$$d\underline{i}/dt = \mathcal{Y}_a[d\underline{e}/dt]$$

In other words the continuous mapping $\mathcal{Y}_a$ commutes with the operation of differentiation. Considering any $\underline{e} \in \mathcal{D}'_{+n} \cap \mathcal{D}'_{Nn}$, by the denseness of $\mathcal{D}_{+n} \cap \mathcal{D}_{Nn}$ in $\mathcal{D}'_{+n} \cap \mathcal{D}'_{Nn}$ we can find a sequence of $\underline{e}_j \in \mathcal{D}_{+n} \cap \mathcal{D}_{Nn}$ such that $\underline{e}_j \to \underline{e}$. Applying these $\underline{e}_j$ to N_a yields $\underline{i}_j \in \mathcal{D}_{+n} \cap \mathcal{D}_{Nn}$, and, by the continuity of the mapping, $\underline{i}_j \to \underline{i}$ with $\underline{i} \in \mathcal{D}'_{Nn}$. By first restricting $\underline{e}$ to $\underline{e} \in \mathcal{D}'_{Nn} \cap \mathcal{E}'_n$, where $\mathcal{E}'_n$ is the set of n-vector distributions of compact support, we see with Schwartz, since $\mathcal{Y}_a$ is continuous

and commutes with differentiation, that $\mathcal{Y}_a$ can be represented by an nxn matrix, convolution operator, y_a.[34]
That is

$$\underline{i} = y_a * \underline{e} \quad (8a)$$

$$\underline{v} = [\delta 1_n - y_a] * \underline{e} \quad (8b)$$

where δ is the unit impulse and 1_n is the nxn identity matrix. By choosing the entries of $\underline{e}$ to approach impulses, we see that the entries of y_a are in $\mathcal{D}'_{Nn}$ and, therefore, have support bounded on the left, if $t_N > -\infty$. We then extend (8a) from $\underline{e} \epsilon \mathcal{D}'_{Nn} \cap \mathcal{E}'_n$ to any $\underline{e}$ for which the convolution is defined. If the columns of y_a are in $\mathcal{D}'_{+n}$, which is always the case if $t_N > -\infty$ or, as we will soon see, if N is passive, then (8a) can be used for any $\underline{e} \epsilon \mathcal{D}'_{+n}$, in which case $\underline{i} \epsilon \mathcal{D}'_{+n}$ results. Of course this extension coinsides with the one obtained by letting $\tau < 0$ in Eq. (3) when $\underline{e} \epsilon \mathcal{D}_{+n}$.[35]

To illustrate this procedure we consider a familiar example.

Example 2:

Consider a linear, time-invariant inductor. The augmented network is described by

$$e = i + \ell di/dt$$

We can extend N_a from $\mathcal{D}_{+1}$ to unit step functions by either solving this for e of the form of Fig. 10c) and taking a limit or by inserting unit step functions directly in the differential equation.

But

$$\delta(t) = \lim_{j\to\infty} \{ju(t) - ju(t-1/j)\}$$

and thus we apply

$$e = e_j = ju(t) - ju(t-1/j)$$

to the differential equation. This gives

$$i_j(t) = j\{u(t) - u(t-1/j) - e^{-t/\ell}u(t) + e^{(-t+1/j)/\ell}u(t-1/j)\}$$

By a power series expansion

$$e^{(-t+1/j)/\ell} = e^{-t/\ell} + (1/j\ell)e^{-t/\ell} + \text{ order } (1/j^2)$$

Consequently, since $e^{-t/\ell}\delta(t) = \delta(t)$,

$$\lim_{j\to\infty} i_j = (1/\ell)e^{-t/\ell}u(t) = y_a(t)$$

Using this with (8a) gives i for any $e \in \mathcal{D}'_{+1}$. Note that the above differential equation for i allows the solution $i = -(1/\ell)e^{-t/\ell}u(-t)$ when $e = \delta$. This solution is ruled out, however, by our extension procedure.

If $t_N = -\infty$, then apparently $\underline{e} \in \mathcal{D}'_{+n}$ need not imply that $\underline{i} \in \mathcal{D}'_{+n}$. For instance the "network" for which i_j takes the form of Fig. 11, when $e_j \in \mathcal{D}_{+1}$ and $e_j \to \delta$, has $e = \delta$ accompained by $i = u(-t) = y_a$.

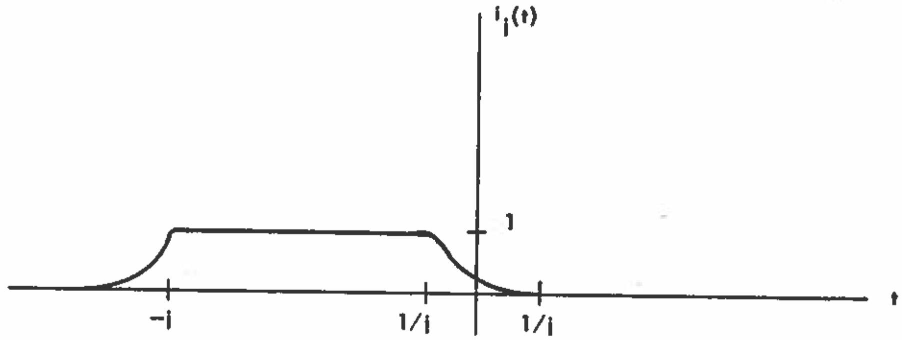


FIG. 11. FUNCTIONS CONVERGING TO u(-t).

Such a network, if it is actually a network, presents somewhat of a paradox, since inserting suitable $e_j \to \delta$ into (8a) shows that the extended network has $i_j(-\infty) \neq 0$. That is, the original network and the extended network wouldn't coincide in this case; it appears than that solvability essentially doesn't hold here.

Let $\mathcal{D}'_{L_{2n}}$ be the set of real, n-vector distributions which are finite sums of derivatives of L_{2n} vectors. If N is linear, time-invariant, solvable and passive, then, as we now show, $\underline{e} \in \mathcal{D}'_{L_{2n}}$ implies $\underline{i}, \underline{v} \in \mathcal{D}'_{L_{2n}}$. This is seen by noting that if $\underline{e} \in \mathcal{D}'_{L_{2n}}$ then $\alpha * \underline{e} \in L_{2n}$ for α any scalar in the set $\mathcal{D}$ of infinitely differentiable functions of compact support.[36] Now $\alpha * \underline{e} = \alpha * \underline{v} + \alpha * \underline{i}$ and $\alpha * \underline{e}$ can by approximated arbitrarily well by $\underline{e}_j \epsilon \mathcal{D}_{+n}$ and hence so will be $\alpha * \underline{v}$ and $\alpha * \underline{i}$ by solvability and the continuity of a linear, time-invariant mapping. But

$$\int_{-\infty}^{t} (\alpha * \underline{\tilde{e}})(\alpha * \underline{e}) d\tau = \int_{-\infty}^{t} (\alpha * \underline{\tilde{v}})(\alpha * \underline{v}) d\tau + \int_{-\infty}^{t} (\alpha * \underline{\tilde{i}})(\alpha * \underline{i}) d\tau + 2 \int_{-\infty}^{t} (\alpha * \underline{\tilde{v}})(\alpha * \underline{i}) d\tau \tag{9}$$

By assumption the term on the left is finite and non-negative. The same holds for the last term on the right, by passivity and the fact that the terms under the integral are approached arbitrarily closely by $\mathcal{D}_{+n}$ vectors. Consequently the other terms, which must be non-negative, are finite and $\underline{e} \epsilon \mathcal{D}'_{L_{2n}}$ does imply $\underline{i}$, $v \epsilon \mathcal{D}'_{L_{2n}}$. This result will show that the Laplace transform of y_a is analytic in the right-half plane. We now turn to such considerations.

Consider a linear, time-invariant, solvable N with its domain of definition extended as far into $\mathcal{D}'_n$ as possible by (8). Let the voltage of N_a be

$$\underline{e}_k = \mathrm{Re}\underline{E} e^{pt} u(t+k)$$

for k real, $p = \sigma + j\omega$ a fixed complex number, $j = \sqrt{-1}$, and $\underline{E}$ an arbitrary complex constant n-vector; of course Re denotes the "real part of". The resulting current is given by the convolution of (8a) as

$$\underline{i}_k(t) = \mathrm{Re}\{[\int_{-\infty}^{\infty} y_a(\tau) e^{-p\tau} u(t-\tau+k) d\tau] \underline{E} e^{pt}\}$$

By linearity the real part can be taken at any stage. For convenience we then drop the Re and assume it to be inserted at the end of any calculation. Taking the limit as $k \to \infty$ we get

$$\underline{i}(t) = [\int_{-\infty}^{\infty} y_a(\tau) e^{-p\tau} d\tau] \underline{e}(t) = Y_a(p) \underline{e}(t) \tag{10a}$$

$$\underline{e}(t) = \underline{E} e^{pt} \tag{10b}$$

This process can only be carried out if the term in brackets, $Y_a(p)$, which is the bilateral Laplace transform, $\mathcal{L}_b[y_a]$, exists for some p. In general there will be some $\sigma_\ell \leq \sigma_r$ such that $Y_a(p)$ exists for

$\sigma_\ell \leq \sigma \leq \sigma_r$. Equation (8b) gives a similar result for $\underline{v}$; such a representation holds for $\underline{v}$ if and only if it holds for $\underline{i}$ by (8b), where $\mathcal{L}_b[\delta 1_n] = 1_n$. If a linear, time-invariant N can be extended by this, or any other, method such that $\underline{v} = \underline{V}e^{pt}$ and $\underline{i} = \underline{I}e^{pt}$ are paired, then we will call N, $\mathcal{L}_b$-representable. The above shows that most such solvable N are $\mathcal{L}_b$-representable, however, some nonsolvable N, such as the nullator, are $\mathcal{L}_b$-representable.[37]

An important class of $\mathcal{L}_b$ representable N are the linear, time-invariant, solvable, passive N. For such N, $\sigma_\ell \leq 0$, $\sigma_r = \infty$, and consequently, in this case $Y_a(p)$ is analytic in $\sigma > 0$. This follows from the fact, as we saw above, that $\underline{e} \epsilon \mathcal{D}'_{L_{2n}}$ yields $\underline{i} \epsilon \mathcal{D}'_{L_{2n}}$. Choosing the entries of $\underline{e}$ all zero except for one impulsive entry, we can isolate any column of y_a by $\underline{i} = y_a * \underline{e}$. The entries for y_a are all zero for $t < 0$ since otherwise, convoluting with $\alpha \epsilon \mathcal{D}$, $\alpha = 0$ for $t < 0$, we would have $\alpha * \underline{i} = y_a * (\alpha * \underline{e})$ nonzero for $t < 0$ which would contradict (9), all for the above impulsive type of $\underline{e}$.[38] Thus, for the type of N under consideration, we know that the columns of y_a are in $\mathcal{D}'_{L_{2n}}$ and zero for $t < 0$. By the very definition of $\mathcal{D}'_{L_{2n}}$ we can write, where k is finite,

$$y_a = \sum_{j=1}^{k} d^j(f_{aj})/dt^j$$

with the columns of the nxn matrices f_{aj} in L_{2n} and zero for $t < 0$. Taking the bilateral Laplace transform gives

$$\mathcal{L}_b[y_a] = Y_a(p) = \sum_{j=1}^{k} p^j \mathcal{L}_b[f_{aj}]$$

By a well-known theorem of Laplace transform theory, each $\mathcal{L}_b[f_{aj}]$ is analytic in $\sigma > 0$, and almost everywhere on $\sigma = 0$, and by this last formula for $Y_a(p)$, the same result holds for $Y_a(p)$.[39] That is $Y_a(p)$, exists and is analytic in $\sigma > 0$ and almost everywhere on $\sigma = 0$. For networks with rational $Y_a(p)$ this gives a stability result, since it says that no poles of Y_a can lie in the right half-plane.

The network for which $e = \delta$ yields $i = \exp[t^2]u(t)$ is not

$\mathcal{L}_b$-representable. From the above comments on the existence of $Y_a(p)$, this network can't be passive, but a direct determination of this through (4) seems hopeless.

By exciting N_a with exponential $\underline{e}(t)$ we obtain, by (10) and (8), exponential responses $\underline{i}$ and $\underline{v}$ for $\mathcal{L}_b$-representable N. That is

$$\underline{e}(t) = \underline{E}e^{pt} \text{ yields} \tag{11a}$$

$$\underline{i}(t) = \underline{I}e^{pt} \tag{11b}$$

$$\underline{v}(t) = \underline{V}e^{pt} \tag{11c}$$

For such N we therefore extend the domain of definition further by allowing $[\underline{V}e^{pt}, \underline{I}e^{pt}]\epsilon N$. We can then work entirely with Laplace transform quantities by defining the frequency domain network, N(p), as

$$N(p) = \{[\underline{V}(p), \underline{I}(p)] \mid [\underline{V}e^{pt}, \underline{I}e^{pt}]\epsilon N\} \tag{12}$$

For the frequency domain network, time no longer appears and we work with functions of the complex variable p. As is well-known, for constraints which are represented by ordinary differential equations with constant coefficients, N(p) is simpler than N to work with, since the constraints become albegraic. The main advantage of such a description probably shows up when synthesis is considered.

Example 3:

If we consider the network of Fig. 12, we see that N_a is described by[40]

$$e+\frac{de}{dt} = 2i+3\frac{di}{dt}+2\frac{d^2i}{dt^2}$$

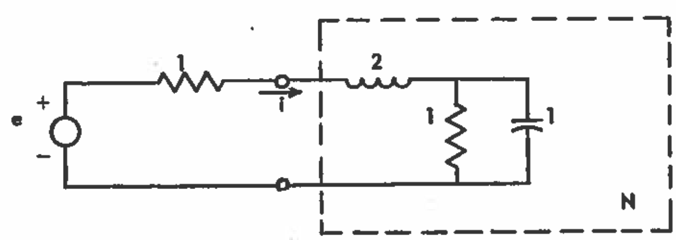


FIG. 12. NETWORK FOR EXAMPLE 3.

Letting $e(t) = Ee^{pt}$, $i(t) = Ie^{pt}$ gives

$$(1+p)E = (2+3p+2p^2)I$$

Since

$$Y_a(p) = (p+1)/(2p^2+3p+2)$$

we see from the Laplace transform interpretation, that $\sigma > -3/4$ since the singularity furthest to the right is on the $\sigma = -3/4$ line. Since $e = v+i$, $N(p)$ is described by

$$[\{(2p^2+2p+1)/(p+1)\}I,\ I]\epsilon N(p)$$

for any complex I and $\sigma > \sigma_\ell = -3/4$

Note that if a physical N is given, defined by (2), then N(p) is unique, when it exists. Further, if N(p) is given such that $Y_a(p)$ exists, then by finding the inverse Laplace transform, $\mathcal{L}_b^{-1}[Y_a(p)]$, and using (8a), a physical network, N, is well-determined.

Limiting ourselves to the linear, time-invariant case, if $\sigma_\ell < \sigma_r$, N(p) is seen to form a vector space over the field of functions of a complex variable defined in $\sigma_\ell < \sigma < \sigma_r$, if we define

$$a[\underline{V},\ \underline{I}] \triangleq [a\underline{V},\ a\underline{I}]$$

and

$$[\underline{V}_1,\ \underline{I}_1] + [\underline{V}_2,\ \underline{I}_2] \triangleq [\underline{V}_1+\underline{V}_2,\ \underline{I}_1+\underline{I}_2].$$ [41]

As a result of the contraint defining N, we see that the vectors $[\underline{V},\ \underline{I}]$ in N(p) form an r-dimensional sub-space of all possible vectors $[\underline{V},\ \underline{I}]$. If we arrange $\underline{V}$ and $\underline{I}$ in a 2nx1 column matrix, $\begin{bmatrix}\underline{V}\\ \underline{I}\end{bmatrix}$, then we can find some matrix M(p) of complex functions defined in $\sigma_\ell < \sigma < \sigma_r$ and of rank 2n-r and order mx2n, where $m \geq 2n-r$, such that $M(p)\begin{bmatrix}\underline{V}\\ \underline{I}\end{bmatrix} = 0$ constrains $\underline{V}$ and $\underline{I}$ to N(p). If $2n-r \geq n$, we choose $m = 2n-r$ while if $2n-r < n$, we choose $m = n$ such that M is nx2n. Partioning M into $M = [A, -B]$ then yields the fundamental equations

$$A(p)\underline{V}(p) = B(p)\underline{I}(p) \tag{13}$$

These equations constrain $\underline{V}$ and $\underline{I}$ to N(p) and thus describe N. Of course the A and B are not unique, since M(p) can be premultiplied by any nonsingular mxm matrix, with a description of the form of (13) still resulting. As is seen by the nullator and norator, these equations can be used to describe other than solvable networks. Further, for a given network (13) can be obtained in many ways. For instance, as seen through (10a), A(p) and B(p) are essentially Laplace transformed quantities and hence (13) can sometimes be obtained by taking Laplace transforms of describing equations.

A more satisfying proof of (13) can be given if N is linear, time-invariant, and solvable, as then

$$\underline{i} = y_a * e \tag{8a}$$

$$\underline{v} = [\delta 1_n - y_a] * \underline{e} \tag{8b}$$

Convoluting (8a) by $[\delta 1_n - y_a]$ on the left gives

$$[\delta 1_n - y_a]*\underline{i} = [\delta 1_n - y_a]*y_a*\underline{e} = [y_a - y_a*y_a]*\underline{e} = y_a*[\delta 1_n - y_a]*\underline{e} = y_a*\underline{v}$$

or

$$\boxed{y_a*\underline{v} = [\delta 1_n - y_a]*\underline{i}} \tag{14a}$$

Taking bilateral Laplace transforms

$$Y_a(p)\underline{V} = [1_n - Y_a(p)]\underline{I} \tag{14b}$$

This is again (13) with A and B explicitely evaluated; in fact A and B are here nxn.

Example 4:

a) Consider the network of Fig. 13

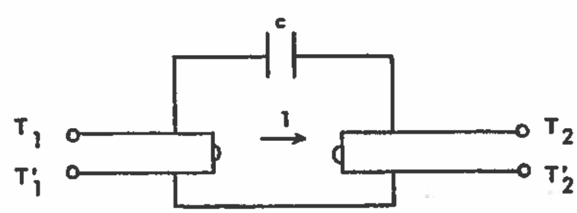


FIG. 13. NETWORK FOR EXAMPLE 4a).

The current flowing to the right through the capacitor is

$$i_c = cd(v_1-v_2)/dt$$

The current i_1 is

$$i_1 = -v_2+cd(v_1-v_2)/dt$$

while the current i_2 is

$$i_2 = v_1-cd(v_1-v_2)/dt$$

Letting $\underline{v}$ and $\underline{i}$ be of exponential form gives

$$\begin{bmatrix} pc & -pc-1 \\ -pc+1 & pc \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

This is (13) with A(p) taken as the admittance matrix. Note however that

$$\begin{bmatrix} pc & -pc-1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

also describes the network since we have merely premultiplied by

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

b) The nullator is described by

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} [V] = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [I]$$

for which A and B are not square.

c) The norator is described by

$$[0][V] = [0][I]$$

d) The transformer is described by

$$\begin{bmatrix} -T & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & T \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Note that, as with the nullator and norator no inverses exist for A and B.

e) The voltage inversion negative impedance converter is defined by

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

By choosing A and B in different ways one can define new and perhaps interesing linear, time-invariant networks. Pre-multiplying (13) by a non-singular matrix C(p) we get

$$CA\underline{V} = CB\underline{I} \tag{15}$$

Properly choosing C gives the familiar network descriptions. For instance the impedance, admittance, and (normalized) scattering matrices can be, respectively, defined by

$$Z = A^{-1}B; \; C = A^{-1} \tag{16a}$$

$$Y = B^{-1}A; \; C = B^{-1} \tag{16b}$$

$$S = (B+A)^{-1}(B-A); \; CA = 1_n-S, \; CB = 1_n+S \tag{16c}$$

At this point we omit any physical interpretation of these and merely point out that, since

$$\underline{V}-\underline{I} = (1_n-2Y_a)\underline{E} = S(\underline{V}+\underline{I}) = S\underline{E}$$

by (16c) we have

$$S(p) = 1_n-2Y_a(p) \tag{17}$$

As was shown before (11), every linear, time-invariant, solvable and passive N possess a $Y_a(p)$ which is in fact analytic in $\sigma > 0$ and exists for almost all $p = j\omega$. Consequently, by (17), we conclude the fundamental result that every linear, time-invariant, solvable, passive N has a scattering matrix which is analytic in $\sigma > 0$ and exists for almost all $p = j\omega$.[42]

Equation (13) is generally only defined for a certain region in the p plane, that for which the appropriate Laplace transform converges. Whenever possible, we extend this region by analytically continuing N(p) to the left and right. Since N(p) may then become multiple-valued, whenever we meet a branch point $\alpha+j\beta$ we make a linear branch cut to $-\infty+j\beta$, if on the left, or $+\infty+j\beta$ if on the right, and thus make N(p) single-valued. For instance a branch point is met at $p = 0$ when considering the infinitely long R-L cable, with $Z(p) = +\sqrt{p}$ originally defined only in the right-half plane. This analytically continued $N(p)$ is identified with the original, since they both coincide in the original domain of definition.

Natural frequencies are conveniently defined in terms of the analytically continued $N(p)$. p_o is called a <u>short-circuit natural frequency</u> if $[\underline{0}, \underline{I}(p_o)]\epsilon N(p_o)$, $\underline{I}(p_o)\neq\underline{0}$. Physically this means that some nonzero exponential current can be forced into the (extended) network when zero voltage appears across the terminals. For instance a short-circuit has every p_o as a short-circuit natural frequency. From $A\underline{V} = B\underline{I}$ we see that $B(p_o)$ must be singular at a short-circuit natural frequency and hence, if Z exists, (15a) shows that p_o is a zero of the determinant of Z. The definition then corresponds to the calculations normally made and generalizes that of "frequencies present due to only initial conditions."[43] The definition also holds for non-meromorphic Z matrices, for instance, for the infinitely long R-L cable mentioned above, $p = 0$ is a short-circuit natural frequency. Dually p_o is an <u>open-circuit natural frequency</u> if $[\underline{V}, \underline{0}]\epsilon N(p_o)$ for $\underline{V}(p_o)\neq\underline{0}$.

In synthesis it is easiest to work with passivity conditions in terms of $N(p)$ in place of the N of definition 3. Since, in the linear, time-invariant, solvable case, after extending from $\mathcal{D}_{Nn}$ to $\mathcal{D}_{+n}$, we can approximate any $[\text{Re } \underline{V}e^{pt}, \text{Re } \underline{I}e^{pt}]\epsilon N$ arbitrarily closely by a sequence of allowed pairs $[\underline{v}_j(t), \underline{i}_j(t)]$ for the original N, passivity necessarily requires[44]

$$\mathcal{E}_p(t) = \int_{-\infty}^{t} (\text{Re } \underline{\tilde{V}}e^{p\tau})(\text{Re } \underline{I}e^{p\tau})d\tau \geq 0 \qquad (18)$$

for all t and every $[\underline{V}, \underline{I}]\epsilon N(p)$ for every p in $\text{Re } p > 0$.

In many situations it is desirable to investigate the internal structure of a network. For this we define the notion of a circuit.

Definition 9:

A <u>circuit</u> C is an interconnection of networks N_i, $i = 1, \ldots, k$. By an interconnection is meant that the n_1-vector of N_1, the

n_2-vectors of N_2, etc., are interrelated by Kirchhoff's laws. Here k is an integer or infinity. We can connect external leads to a circuit such that looking in from the outside C looks like an n-port. Of course C must be constructed at a creation time greater or equal to the creation time of any of the interconnected networks. It should be pointed out that a circuit can be closed upon itself, that is have no terminals, as is seen by the circuit of Fig. 14.

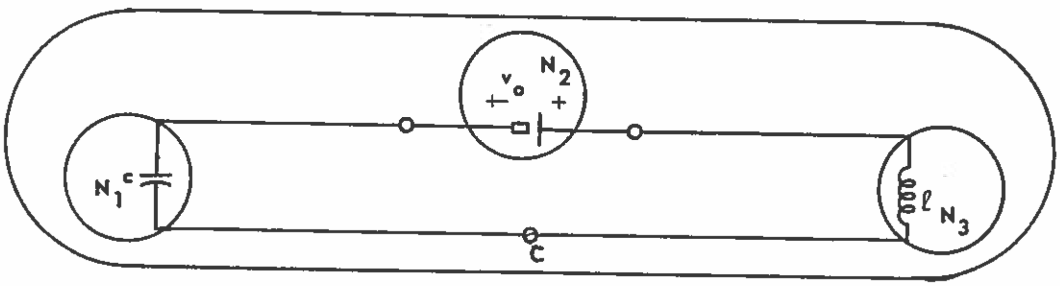


FIG. 14. A CLOSED CIRCUIT.

The concept of a circuit allows us to conveniently look at finite networks.

Definition 10:

N is <u>finite</u> if it can be constructed as a circuit consisting of a finite number of the networks of Example 1 (resistors, inductors, capacitors, transformers, gyrators).

In essence, a finite network is defined by an equivalence class of circuits.[45] Thus, if circuits, C_1 and C_2 have external leads connected such that they define networks N_{C_1} and N_{C_2}, then we will call C_1 and C_2 <u>equivalent</u> (at the terminals), written $C_1 \approx C_2$, if $N_{C_1} = N_{C_2}$. For example, the circuits of Fig. 15a), assumed to have $t_N = -\infty$, are equivalent. However, the circuits of Fig. 15b), which are assumed created at $t_N = 0$ with different initial voltages, are not equivalent. This latter result only seems reasonable, since their responses to different excitations will be different.

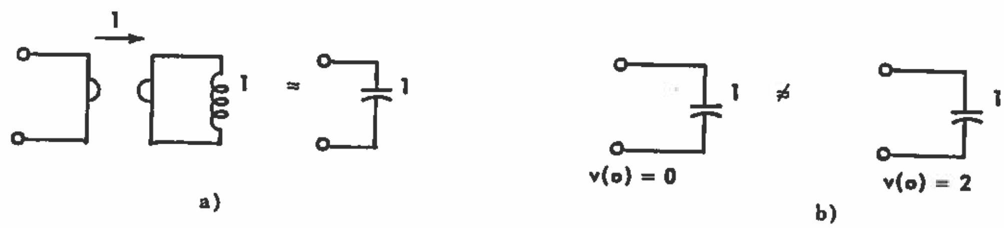


FIG. 15. ILLUSTRATION OF EQUIVALENT CIRCUITS.

V. CONCLUSIONS

Using what appear to be the most basic ideas, an n-port has been defined as well as the various types of properties that can be possessed. In contrast to other such theories in the literature, such an n-port can be nonlinear, time-varying or active. In the linear, time-invariant, solvable cases the types of variables allowed can be extended from infinitely differentiable vectors of support bounded on the left to distributional n-vectors. The variables in many such cases can also be extended to be of exponential form, this always being true in the passive case.

The definition of a network is different than commonly found. That is, we don't define a network in terms of operators but as a collection of pairs of variables. Our justification for this is that in the operator description, $L[\underline{f}] = \underline{g}$, the variables $\underline{f}$ are best thought of as inputs while the $\underline{g}$ are to be considered as outputs. However, for a network it seems that sometimes a variable will be used as an input while at others times the same variable should be considered as an output. For instance if a voltage source is applied to a resistor, one naturally considers the voltage as the input and the current as the output. But one can equally well apply a current source to the same resistor in which case the current would be considered as the input. It seems that the operator description is most appropriate in studying control systems since, in $L[\underline{f}] = \underline{g}$, one here would wish to control an output $\underline{g}$ by the use of the input $\underline{f}$. Of course we did use the operator idea in the extension to distributional variables. This was done, since we worked with the augmented network where we always considered $\underline{e}$ as the excitation and either $\underline{i}$ or $\underline{v}$ as the response.

The definitions of network properties must be given in terms of the original variables, in place of the distributional or exponential variables, since otherwise meaningless results occur. For instance, the energy integral Eq. (4) can't be evaluated for an impulse of voltage on a capacitor. Further, an inductor would not look reciprocal, since

$$i_1 = u(t),\ v_1 = \ell\delta(t)$$

and

$$i_2 = I = \text{constant},\ v_2 = 0$$

are allowed with distributional and exponential extensions, but

$$\ell\delta * I = \ell I \neq 0 = 0 * u$$

if

$$I \neq 0,\ \ell \neq 0$$

In several works in the literature it is postulated that a network is something possessing the various properties of linearity, time-invariance, passivity and sometimes solvability. However, it seems to us that this is somewhat a misuse of the concept of a postulate. It seems that what should be postulated is the existence of physical devices subject to a mathematical description of the form of Eq. (2), in the theory presented here. The various properties of such a description can then be checked to see if the given network possesses such a property, as passivity, say.

From working with the various definitions of part b), it seems that most of these definitions are as desired. However, it may be that some future considerations would require some sort of modifications. For instance, the definition of solvable is important in the theory of linear networks. Perhaps, though, there is some better concept which contains that of solvability, which is needed for the study of nonlinear networks. Similarly, reciprocity is defined for general networks, but perhaps with more study it would become necessary to restrict it to purely linear networks. Like-wise the definition of lossless, which hasn't been worked with much, may not be the most appropriate one.

The "network" associated with Fig. 11 raises interesting questions. For instance by observing this, one wonders if perhaps every linear, time-invariant, solvable N can't be made passive by augmenting (as in Fig. 9) with sufficient resistance.[46] If so, then it would appear that even for active networks, the Laplace transform representation should converge in a right half-plane.

NOTES

1. See McMillan, [Ref. 1], and Raisbeck, [Ref. 2]. McMillan's approach is closer to the work presented here than any other. Raisbeck, although not quite so rigorous, is the starting point of most recent studies. McMillan essentially assumes linearity, time-invariance and reciprocity while Raisbeck assumes the existence of an impedance matrix; both assume passivity.
2. See Youla, [Ref. 3], Zemanian, [Ref. 4], as well as König, [Ref. 5]. These papers are all based upon the concept of a linear transformation. [Ref. 3] works in Hilbert space and gives an excellent but abstract theory for the scattering matrix. [Ref. 4] works with distributions with some interesting time domain results. [Ref. 5] gives a rather complicated theory but has some interesting impedance representations.
3. See Newcomb, [Ref. 6]. This shows that by a suitable choice of the operator domain of [Ref. 3], networks which one would normally consider as passive, causal and single-valued, need not have these properties.
4. See the second section of [Ref. 35].
5. Typical conjugate variables would be force and velocity, voltage and current, temperature and entropy change. One wonders if such a choice can be made for biological systems.
6. Intiuitively, by the support of a function of time, h(t), is meant the set of t for which h(t) is non-zero. More precisely, the support is the closure of the set of points in t for which $h(t)\neq 0$, [Ref. 7, p. 17]. The support of a vector being bounded means that the support of each entry of the vector is bounded.
7. We use the standard set theory symbols. That is, ϵ denotes "contained in", $\cap$ represents "intersection", and $A = \{x|P(x)\}$ means that A is the set of all x such that the proposition P(x) is satisfied.

   We comment that $\underline{f}\epsilon\mathcal{D}_{+k}\cap\mathcal{D}_{Nn}$ can not be replaced by $\underline{f}\epsilon\,\mathcal{D}_{Nn}$ if $t_N = -\infty$, since then $\underline{f}$ would not need to have support bounded on the left.

8. If a measured entity were not infinitely differentiable, infinite values would be obtained by the use of differentiators. This argument is not too solid, however, since ideal differentiators don't exist. One feels that impulses aren't allowed since many non-linear devices can't tolerate them.

9. If $\underline{g}$ is given as the voltage vector we merely relabel.

10. We assume for simplicity and with Shelley, [Ref. 8, p. 813], that the universe has existed from eternity.

11. In german an n-port is called a 2n-pole with this nomenclature being taken over into English by McMillan, [Ref. 1, p. 225]. McMillan's definition is the only precise one we know of in English, but it holds only for finite networks. Certainly the original use of the word port is imprecise, [Ref. 9]. See also the IRE standard, [Ref. 10], [Ref. 11].

12. An interesting analog of the gyrator results from plasma physics. For a charged particle of mass m and charge q moving in a magnetic field $\vec{B}$ with velocity $\vec{v}$, we have $\frac{m d\vec{v}}{dt} = q\vec{v}x\vec{B}$. If $\vec{B}$ is directed only in the z direction with magnitude B_z, this gives

$$\begin{bmatrix} \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} 0 & qB_z/m \\ -qB_z/m & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

where the cyclotron frequency $\omega_c = qB_z/m$ is analogous to γ and represents the rate at which the particle gyrates around the B field.

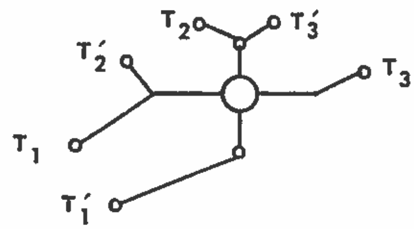
13. For microwave realizations of the gyrator see Lax, [Ref. 12, p. 544]. For low frequency realizations see Bogert, [Ref. 13], where active circuits are used.

14. See Carlin, [Ref. 14], where these degenerate networks form a basis for a synthesis technique.

15. See Cauer, [Ref. 15, p. 161]. Cauer calls the k-terminal network a "complete 2k-pole".

16. Note that the variables for the lower right terminal pair of Fig. 5a) are uniquely determined from the other variables by

the use of Kirchhoff's laws. If sources are only applied at the ports of Fig. 5b) the transformers can be omitted and the network redrawn as



17. The standard definition is in terms of operators, see Papoulis, [Ref. 16, p. 82], or Youla, [Ref. 3, p. 106]. An alternate, but somewhat complicated, definition in terms of the state has been given by Zadeh, [Ref. 17]. This was slightly improved at a later date, [Ref. 18], to include L-C networks. Essentially Zadeh's definitions attempt to distinguish between linear equations and linear systems. Thus in his terminology one system could have different initial conditions, while in the theory we present, the same "elements" with different initial conditions define different systems. This latter viewpoint seems to be the only consistent one when just terminal behavior is considered. It does, however, have drawbacks when considering internal construction.

18. If $[\underline{v}_j, \underline{i}_j] \epsilon N$ then for real a_j and finite m,

$$[\sum_{j=1}^{m} a_j \underline{v}_j, \sum_{j=1}^{m} a_j \underline{i}_j] \epsilon N.$$ However, for a linear N, this result apparently need not hold for infinite m. We could postulate that it does hold for $m = \infty$, but none of the theory seems to be changed by such an assumption.

19. The fact that homogeneity and additivity could be separately violated was essentially proven by Gerald Alonzo in the Stanford graduate course EE 235.

20. Of course anyone would be silly to build Fig. 7b) using diodes. Practically the diodes wouldn't be ideal anyway and the resistor equivalence would only hold as an approximation. The example does illustrate the fact stated, which would be taken as a weakness of the definitions. Another example is given in [Ref. 17].

21. For the concept of a vector space see Birkhoff, [Ref. 19, p. 162].

22. The standard definition for time-invariance can be found in Papoulis, [Ref. 16, p. 83]. This essentially reads: A system is time-invariant if $g(t)$ is the response to $f(t)$ then $g(t-t_1)$ is the response to $f(t-t_1)$. Although the physical meaning of such a definition is clear the mathematics is rather imprecise since, by simply changing the variable, almost any system is (mathematically) time-invariant).

23. Besides the definition given, others can be found. Shekel, [Ref. 20, p. 269], defines a network as being passive if the average power input in the sinusoidal steady state is positive. This, however, allows a resistive network containing a negative capacitor as passive. Raisbeck, [Ref. 2, p. 1511], requires only

$$\mathcal{E}(\infty) = \int_{-\infty}^{\infty} \underline{\tilde{v}}(\tau)\underline{i}(\tau)d\tau \geq 0$$

This, however, seems to restrict $\underline{v}$ and $\underline{i}$ and since $\mathcal{E}(t)$ could be negative for some $t < \infty$ but still $\mathcal{E}(\infty) \geq 0$, this doesn't seem reasonable. Youla, [Ref. 3, p. 110], allows complex excitations and responses and then defines

$$\mathcal{E}(t) = \mathrm{Re}\int_{-\infty}^{t} \underline{\tilde{v}}^{*}(\tau)\underline{i}(\tau)d\tau$$

with a superscript asterisk the complex conjugate. We believe that a more meaningful definition for complex valued variables would be

$$\mathcal{E}(t) = \int_{-\infty}^{t} (\mathrm{Re}\ \underline{\tilde{v}}(\tau)(\mathrm{Re}\ \underline{i}(\tau))d\tau$$

Both of the latter $\mathcal{E}(t)$ agree with $\mathcal{E}(t)$ of (4) when the variables are real.

24. Youla, [Ref. 21], has shown that any realization of the nullator using a finite number of the elements of example 1 must contain at least one negative resistor and one gyrator. For this reason one hesitates to call the nullator passive. In fact in early lectures on this material the author inserted an extra clause in the passivity definition to rule out nullators at any port. Such a theory seems too ad hoc and we rely upon only the solvability concept to rule the nullator out, when so desired.

25. See [Ref. 3, pp. 103 & 113]. Youla refers to the concept appearing in two earlier works which we haven't seen as yet, those of Wu and Toll. Essentially, solvability is assumed as a postulate for a network in Youla's theory.

26. It seems to us that any truly physical device is solvable. This is true for multiple-valued or hysteresis type characteristic curves such as obtained for tunnel diodes or iron-core inductors. If, for such devices, one knows the entire past history the device will be in a unique "state" at any given instant.

27. These are easily established.

Linearity: For N_a we have $[a\underline{e}, a\underline{i}] = [a\underline{v}+a\underline{i}]\epsilon N_a$ since $[a\underline{v}, a\underline{i}]\epsilon N$ for every a. Likewise $[\underline{e}_1+\underline{e}_2, \underline{i}_1+\underline{i}_2] = [\underline{v}_1+\underline{v}_2+\underline{i}_1+\underline{i}_2, \underline{i}_1+\underline{i}_2]\epsilon N_a$ since $[\underline{v}_1+\underline{v}_2, \underline{i}_1+\underline{i}_2]\epsilon N$ by linearity of N.

Time-invariance: Consider $[\underline{e}, \underline{i}] = [\underline{v}+\underline{i}, \underline{i}]\epsilon N_a$, then $[\underline{v}, \underline{i}]\epsilon N$. By the time-invariance of N, for $\tau \geq 0$, $[\underline{v}(t), \underline{i}(t)] = [\underline{v}_o(t+\tau), \underline{i}_o(t+\tau)]$ for $[\underline{v}_o, \underline{i}_o]\epsilon N$. Therefore $[\underline{e}(t), \underline{i}(t)] = [\underline{v}_o(t+\tau)+\underline{i}_o(t+\tau), \underline{i}_o(t+\tau)] = [\underline{e}_o(t+\tau), \underline{i}_o(t+\tau)]$ where $[\underline{e}_o, \underline{i}_o] = [\underline{v}_o+\underline{i}_o, \underline{i}_o]\epsilon N_a$ since $[\underline{v}_o, \underline{i}_o]\epsilon N$.

Passivity: We have for N_a

$$\mathcal{E}_a(t) = \int_{-\infty}^{t} (\underline{\tilde{v}}+\underline{\tilde{i}})\underline{i}d\tau = \int_{-\infty}^{t} \underline{\tilde{v}}\underline{i}d\tau + \int_{-\infty}^{t} \underline{\tilde{i}}\underline{i}d\tau$$

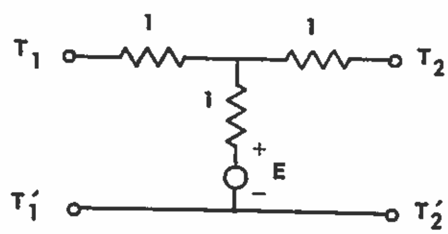
The last term is always non-negative while the next to the last term is, if N is passive.

28. See Ramo and Whinnery, [Ref. 22, p. 454], where Maxwell's equations are shown to yield Lorentz reciprocity. The result of [Ref. 22] is in terms of phasor quantities and thus we have replaced multi-

plication by convolution to convert to the time domain. The first real use of this result in a rigorous theory seems to be in McMillan, [Ref. 1, p. 236].

29. Matrix convolution is defined by replacing multiplication by convolution in the definition. Thus $A*B$ is defined by $\sum_k a_{ik}*b_{kj}$. For finite, linear, passive networks this agrees with the usual statement that for a reciprocal network the ratio of response to excitation is invariant to an interchange of the points of excitation and observation, [Ref. 23, p. 148]. The definition we give is valid for nonlinear, time-variable or active networks but it is not clear if it is meaningful then. The definition does make the following network nonreciprocal, since

$$\tilde{\underline{v}}_1*\underline{i}_2-\tilde{\underline{v}}_2*\underline{i}_1 = E*(i_{1_2}+i_{2_2}-i_{1_1}-i_{2_1})$$



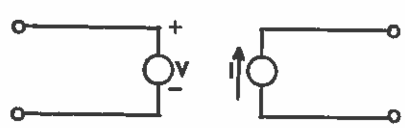
Note that if the current "excitations" for measurements one and two are the same, this network would look reciprocal (for these excitations). In fact any reciprocal, solvable 1-port is linear since, with $\underline{i} = \underline{e}-\underline{v}$, $\tilde{\underline{v}}_1*\underline{i}_2 = \underline{v}_1*(\underline{e}_2-\underline{v}_2) = \tilde{\underline{v}}_2*(\underline{e}_1-\underline{v}_1)$ requires with $\underline{e}_2 = a\underline{e}_1$, $[\tilde{\underline{v}}_2-a\tilde{\underline{v}}_1]*\underline{e}_1 = 0$ or for the 1-port $\underline{v}_2 = a\underline{v}_1$ and $\underline{i}_2 = a\underline{i}_1$ are in N. Similarly with $\underline{e}_3 = \underline{e}_1+\underline{e}_2$, $\tilde{\underline{v}}_1*\underline{i}_3 = \tilde{\underline{v}}_1*(\underline{e}_3-\underline{v}_3) = \tilde{\underline{v}}_3*(\underline{e}_1-\underline{v}_1)$ or $\tilde{\underline{v}}_1*\underline{e}_3 = \underline{v}_3*\underline{e}_1$ and similarly $\tilde{\underline{v}}_2*\underline{e}_3 = \tilde{\underline{v}}_3*\underline{e}_2$ adding gives $(\tilde{\underline{v}}_1+\tilde{\underline{v}}_2-\tilde{\underline{v}}_3)*(\underline{e}_1+\underline{e}_2) = 0$ or for the 1-port $[\underline{v}_1+\underline{v}_2, \underline{i}_1+\underline{i}_2] \in N$ and hence N is linear. It is not clear if such is the case for n-ports.

Sometimes a reciprocal network is called "bilateral," [Ref. 24, p. 875]. Although we have never seen a precise technical definition of the word bilateral, we would prefer to apply it exclusively to 1-ports meaning that current passes in the same manner into or out of a device. The ideal diode would not be bilateral and the concept couldn't be meaningfully applied to the gyrator.

30. Here $\underline{0}$ is the n-vector of zeros.

A typical definition of causal is given in Youla, [Ref. 3, p. 111]. The definition so given allows the open circuit as causal but not the dual network, a short-circuit. For this reason our definition has two clauses. In [Ref. 3, p. 111] it is "proven" that "except in pathological cases, linearity and passivity imply causality". However, as shown in [Ref. 6] we believe this only to be the case when a proper interpretation of the concepts is used. Thus we essentially assume some sort of "causality" by restricting N to $\underline{v}$ and $\underline{i}$ in $\mathfrak{D}_{+n}$. An excellent discussion of causality versus antecedance is contained in Bunge, [Ref. 25].

A definition differing from that of Youla, but somewhat more common, is that given by Papoulis, [Ref. 16, p. 85]. This requires the output to be zero until t_o if the input is zero until t_o. Youla's requires that if $\underline{v}_1 = \underline{v}_2$ for $t < t_o$ then $\underline{i}_1 = \underline{i}_2$ for $t < t_o$. In definition 6 the existence of $[\underline{0}, \underline{i}]$ and $[\underline{v}, \underline{0}]$ is needed to be able to consider a) and b). Without such a clause one couldn't consider the non-antecedal 2-port.



31. Here L_{2n} is the set of real-valued square integrable n-vectors. That is $\underline{f} \epsilon L_{2n}$ if

$$\int_{-\infty}^{\infty} \tilde{\underline{f}}(\tau)\underline{f}(\tau)d\tau < \infty$$

It seems that an alternate definition in terms of conservative systems, as studied in mechanics, [Ref. 26, pp. 3 & 347], could be given. However, it also seems that such existing theories don't have the complete generality contained in the concept of a network. Definition 8 is essentially that given by Youla, [Ref. 3, p. 119].

32. Except for the subscript n, the notation is the standard one found in Schwartz, [Ref. 7, p. 25]. The fact that $\mathcal{D}_{+n}$ is dense in $\mathcal{D}'_{+n}$ follows in a manner similar to Schwartz, [Ref. 7, p. 75]. A simpler proof follows from Schwartz, [Ref. 27, p. 22]. Thus let a sequence of infinitely differentiable functions, φ_j, of $\mathcal{D}_1$ converge to δ, $\varphi_j \to \delta$, where δ is the unit impulse. Then, for any distribution $f \epsilon \mathcal{D}'_{+1}$ we can form $\varphi_j * f$ and $\varphi_j * f \to \delta * f = f$. Since $\varphi_j * f$ is in $\mathcal{D}_{+1}$, this gives the desired result, after extending to n-vectors in an obvious manner (component by component).

33. The impossibility of defining δ^2 as a distribution is shown in Schwartz, [Ref. 7, p. 117]. König has shown how the concept of a distribution can be extended, such that δ^2 is defined, [Ref. 28, p. 448]. However, $\delta . u \neq u . \delta$ under this definition.

34. See Schwartz, [Ref. 27, p. 18 theorem X and p. 20], which states: Every linear continuous operation from either $\mathcal{E}'$ or $\mathcal{D}$ into $\mathcal{D}'$, commuting with differentiation, is the convolution $\mathcal{L}(T) = S*T$ with a fixed distribution $S \epsilon \mathcal{D}'$ and reciprocally. This gives a rigorous justification of the physical result that the response of a linear, time-invariant system is found by convoluting the impulse response with the actual input.
Schwartz's result is not for the matrix case, but by considering separate components of $\underline{e}$ and $\underline{i}$, it is seen to yield the matrix y_a.

35. At this point, at least when n = 1, it is possible to set up a Mikusiński's algebra, [Ref. 29], since the algebra $\mathcal{D}'_{+1}$ has no divisors of zero, [Ref. 27, p. 29]. This allows us to work with more general networks than those later called $\mathcal{L}_b$-representable. Of course (8) yields the same $[\underline{v}, \underline{i}]$ pairs which originally defined the network, by the definition of $\mathcal{Y}_a$.

36. See Schwartz, [Ref. 27, p. 57, theorem XXV], where it is shown that: In order for a distribution T to belong to $\mathcal{D}'_{L_p}$ it is

necessary and sufficient that, for any $\alpha \epsilon \mathcal{D}$, $\alpha^* T^* \epsilon L_p$, recall that $f \epsilon L_p$ if $\int_{-\infty}^{\infty} |f(t)|^p dt < \infty$

37. We could have simply obtained $Y_a(p)$ by taking the Laplace transform of (8a). We feel that the physical insight of $Y_a(p)$ being the response coefficient in (10a) for exponential excitations lends motivation to the heavy use of the Laplace transform in engineering work. The approach used follows that of Mason and Zimmerman. [Ref. 30, p. 340].

38. Choosing $t < 0$ in Eq. (1-9) would give

$$0 = \int_{-\infty}^{t} (\alpha^*\tilde{\underline{v}})(\alpha^*\underline{v})d\tau + \int_{-\infty}^{t} (\alpha^*\tilde{\underline{i}})(\alpha^*\underline{i})d\tau + 2\int_{-\infty}^{t} (\alpha^*\tilde{\underline{v}})(\alpha^*\underline{i})d\tau$$

But each term on the right must be non-negative, by passivity or the sum of squares, and hence zero.

39. See Widder, [Ref. 31, p. 80, theorem 10]. The theorem reads: If $f(t)$ belongs to L_2 in $(0, \infty)$ then

$$\underset{R\to\infty}{\overset{(2)}{\ell.i.m.}} \int_{0}^{R} f(t)e^{-pt}dt$$

exists for $\sigma \geq 0$ and defines a function $F(p)$ which is analytic for $\sigma > 0$. Moreover

$$F(p) = \int_{0}^{\infty} f(t)e^{-pt}dt (\sigma > 0)$$

the integral converging absolutely for $\sigma > 0$ and

$\underset{\sigma\to 0+}{\ell.i.m.} F(\sigma+j\omega) = F(j\omega)$

40. This is obtained as follows. Letting v_c be the voltage across the capacitor, plus at the top, Kirchhoff's voltage law gives $e = i+2(di/dt)+v_e$. Letting i_c and i_r be the current down through the right hand capacitor and resistor gives, by Kirchhoff's current law,

$$i = i_r+i_c, \; dv_c/dt = i_c, \; v_c = i_r$$

Therefore $i = v_c+dv_c/dt$. Solving the equation in e for v_c, adding this to itself differentiated and equating to i gives the desired result.

41. For the concept of a vector space see Halmos, [Ref. 32, p. 3], or Birkhoff and MacLane, [Ref. 19, p. 162].

42. It is also true that for $\sigma > 0$, $S^*(p) = S(p^*)$. This results from exciting N_a with $\underline{e}_1 = \underline{E}e^{pt}$ and $\underline{e}_2 = \underline{E}e^{p^*t}$ with $\underline{E}$ real. Then $\underline{i}_1(t) = Y_a(p)\underline{E}e^{pt}$ and $\underline{i}_2(t) = Y_a(p^*)\underline{E}e^{p^*t}$ are the corresponding currents. Conjugating $i_2(t)$ we get

$$\underline{i}_2^*(t) = Y_a^*(p^*)\underline{E}e^{pt}$$

but this must be $\underline{i}_1(t)$ as it is the response to $\underline{e}_1$. Thus, since $\underline{E}$ was arbitrary

$$Y_a(p) = Y_a^*(p^*)$$

which gives the above stated result for $S(p)$. Z and Y are also analytic in $\sigma > 0$ whenever they exist, since for instance

$$Z = (1_n+S)(1_n-S)^{-1}$$

and Z becomes non-analytic in $\sigma > 0$ only where (1_n-S) is singular, but (1_n-S) is non-singular everywhere in $\sigma > 0$ if Z exists (for a passive network), [Ref. 3, p. 119].

43. We comment that in the extension to exponential excitations, terms of the forms $e^{p_1t}+e^{p_2t}$, $p_1 \neq p_2$, haven't been allowed. This means that the separate natural frequencies can be individually considered. For instance in working with (17) we don't have to kill off natural frequency terms by a suitable choice of initial conditions, as is sometimes done, [Ref. 33, p. 418].

44. To see this form

$$\mathcal{E}_p(t)-\mathcal{E}_j(t) = \int_{-\infty}^{t}[\mathrm{Re}(\underline{\tilde{V}}e^{p\tau})\mathrm{Re}(\underline{I}e^{p\tau})-\underline{\tilde{v}}_j(\tau)]d\tau$$
$$= \int_{t_j}^{t}[\mathrm{Re}(\underline{\tilde{V}}e^{p\tau})\mathrm{Re}(\underline{I}e^{p\tau})-\underline{\tilde{v}}_j(\tau)\underline{i}_j(\tau)]d\tau + \int_{-\infty}^{t_j} \mathrm{Re}(\underline{\tilde{V}}e^{p\tau})\mathrm{Re}(\underline{I}e^{p\tau})d\tau$$

where t_j is the left bound on the support of $\underline{v}_j$ and $\underline{i}_j$. By choosing $t_j \to -\infty$ with $\sigma > 0$, by time-invariance, the second integral tends to zero, while the first integral does also, since $\underline{v}_j$, $\underline{i}_j$ approximate the exponentials. Thus

$$\lim_{j\to\infty} \mathcal{E}_p(t)-\mathcal{E}_j(t) = 0$$

for every finite t and hence $\mathcal{E}_p(t) \geq 0$ as $\mathcal{E}_j(t) \geq 0$. Note that the second integral on the right won't tend to zero if $\sigma < 0$. Letting $V_i = |V_i|e^{j\measuredangle V_i}$, $I_i = |I_i|e^{j\measuredangle I_i}$, (18) becomes

$$\mathcal{E}_p(t) = \int_{-\infty}^{t} \sum_{i=1}^{n} \frac{|V_i|\ |I_i| e^{2\sigma\tau}}{2} [\cos(2\omega\tau + \measuredangle V_i + \measuredangle I_i) + \cos(\measuredangle V_i - \measuredangle I_i)]d\tau$$

$$= \text{Re} \int^{t} [\frac{\underline{\tilde{V}}^* \underline{I}}{2} e^{2\sigma\tau} + \frac{\underline{\tilde{V}}\underline{I}}{2} e^{2p\tau}]d\tau$$

$$= \frac{1}{4} \text{Re}\ [\frac{\underline{\tilde{V}}^* \underline{I}}{\sigma} + \frac{\underline{\tilde{V}}\underline{I}}{p} e^{j2\omega t}]e^{2\sigma t}$$

Letting $t\to\infty$ through instants when the second term assumes its "minimum" requires, as $\sigma > 0$

$$\text{Re}\ (\underline{\tilde{V}}^* \underline{I}) - \frac{\sigma}{|p|}|\underline{\tilde{V}}\underline{I}| \geq 0 \text{ if } \omega \neq 0$$

or

$$(\text{Re}\ \underline{\tilde{V}})(\text{Re}\ \underline{I}) \geq 0 \text{ if } \omega = 0$$

By choosing real $\underline{V}$ and $\underline{I}$ and combining terms, this latter requires Re $\underline{\tilde{V}}^* \underline{I} \geq 0$ for $\omega = 0$ and thus a passive network necessarily has $Q(p) \geq 0$ for $\sigma > 0$ where

$$Q(p) = \begin{cases} \text{Re}\ (\underline{\tilde{V}}^* \underline{I}) - \frac{\sigma}{p}|\underline{\tilde{V}}\underline{I}| & \text{if } \omega = 0 \\ \text{Re}\ \underline{\tilde{V}}^* \underline{I} \text{ if } \omega = 0 \end{cases}$$

Clearly this requires Re $\underline{\tilde{V}}^* \underline{I} \geq 0$ in $\sigma > 0$ or if Y exists Re$[\underline{\tilde{V}}^* Y\underline{V}] \geq 0$ in $\sigma > 0$ (for any complex $\underline{V}$), which is the essential positive real condition. Writing

$$\text{Re}\ \underline{\tilde{V}}^* \underline{I} = \frac{1}{2}[\underline{\tilde{V}}^* \underline{I} + \underline{\tilde{I}}^* \underline{V}] = \frac{1}{4}[\underline{\tilde{V}}^* \underline{I} + \underline{\tilde{I}}^* I + \underline{\tilde{V}}^* \underline{V} + \underline{\tilde{I}}^* \underline{V}]$$

$$+ \frac{1}{4}[\underline{\tilde{V}}^* \underline{I} - \underline{\tilde{I}}^* \underline{I} - \underline{\tilde{V}}^* \underline{V} + \underline{\tilde{I}}^* \underline{V}]$$

$$= \frac{1}{4}[\underline{\tilde{V}} + \underline{\tilde{I}}]^* [\underline{V} + \underline{I}] - \frac{1}{4}[\underline{\tilde{V}} - \underline{\tilde{I}}]^* [\underline{V} - \underline{I}]$$

$$= \frac{1}{4}[\underline{\tilde{V}} + \underline{\tilde{I}}]^* [1_n - \tilde{S}^* S][\underline{V} + \underline{I}] \geq 0$$

by the reasoning above (17). Thus we know that every linear, time-invariant, solvable N has a scattering matrix $S(p)$ and

1) $S(p)$ is analytic in $\sigma > 0$

2) $S^*(p) = S(p^*)$ in $\sigma > 0$

3) $1_n - \tilde{S}^*(p)S(p)$ in $\sigma > 0$

If S is rational and satisfies these conditions it can be synthesized by known methods, [Ref. 34].

45. The relation $C_1 \approx C_2$ is a true equivalence relation, see [Ref. 19, p. 155]. That is

$C_1 \approx C_1$

$C_1 \approx C_2$ implies $C_2 \approx C_1$

$C_1 \approx C_2$, $C_2 \approx C_3$ implies $C_1 \approx C_3$

46. (Added in proof) This is not the case as the following counter-examples, which we previously exhibited, show, [Ref. 36, p. 36]. For this let $z(p) = p^2$ or $Z(p) = \begin{bmatrix} 0 & 0 \\ -p & 0 \end{bmatrix}$.

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