Linearizing Transformation on Quadratic Systems

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Abstract

Previously it has been shown [1] that all unforced polynomic systems can be described within a nonassociative algebra by the canonic quadratic system $\dot{x} = x \cdot x$. Indeed many reasonable, even forced, systems can be approximated by such a quadratic system. Adapting some ideas of Takata [2] we here outline the technique to show that this canonical system can be further transformed into a linear system, albeit much augmented in dimension.

$$\dot{x} = a \cdot x$$

Review of Previous Results

Definition: Given a set of n real symmetric $n \times n$ matricies $A^i = [a^i_{jk}]$, the quadratic differential system S is defined by

$$\dot{x}^i = \bar{x} A^i x \qquad , i = 1, 2, \ldots n$$

Written in component form:

$$\frac{dx^i}{dt} = \sum_{j=1}^{n} \sum_{k=1}^{n} a^i_{jk} x^j x^k$$

A vector x in the algebra may be written as

$$x = \sum_{i=1}^{n} x^i u_i$$

where $u_1, u_2, \ldots u_n$ are independent basis vectors. Algebra is defined by the multiplication for basis vectors

$$u_j \cdot u_k = \sum_{i=1}^{n} a^i_{jk} u_i \equiv \Sigma A^i u_i$$

Example
Illustration of the notation

$$S: \quad \dot{x}^1 = x^2 x^2$$
$$\dot{x}^2 = -x^1 x^2$$

$$S: \quad \dot{x}^i = \bar{x} A^i x, \quad i = 1, 2$$

$$\dot{x}^1 = (x^1, x^2) \begin{bmatrix} a^1_{11} & a^1_{12} \\ a^1_{21} & a^1_{22} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

$$\dot{x}^2 = (x^1, x^2) \begin{bmatrix} a^2_{11} & a^2_{12} \\ a^2_{21} & a^2_{22} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

$$A^1 = \begin{bmatrix} a^1_{11} & a^1_{12} \\ a^1_{21} & a^1_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} a^2_{11} & a^2_{12} \\ a^2_{21} & a^2_{22} \end{bmatrix} = \begin{bmatrix} 0 & -1/2 \\ -1/2 & 0 \end{bmatrix}$$

$$S: \quad \dot{x}^1 = (x^1, x^2) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

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Example – Continued

Basis u_1, u_2.

The algebra for S is defined by

$$u_j u_k = \sum_{i=1}^{2} a^i_{jk} u_i \equiv \sum_{i=1}^{2} A^i u_i$$

$$\therefore \quad \begin{aligned} u_1 u_1 &= a^1_{11} u_1 + a^2_{11} u_2 \\ u_1 u_2 &= a^1_{12} u_1 + a^2_{12} u_2 \\ u_2 u_1 &= a^1_{21} u_1 + a^2_{21} u_2 \\ u_2 u_2 &= a^1_{22} u_1 + a^2_{22} u_2 \end{aligned}$$

Equivalently

$$\begin{bmatrix} u_1 u_1 & u_1 u_2 \\ u_2 u_1 & u_2 u_2 \end{bmatrix} = A^1 u_1 + A^2 u_2$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} u_1 + \begin{bmatrix} 0 & -1/2 \\ -1/2 & 0 \end{bmatrix} u_2$$

$$= \begin{bmatrix} 0 & -\frac{1}{2} u_2 \\ -\frac{1}{2} u_2 & u_1 \end{bmatrix}$$

This is a nonassociative algebra:

$$\begin{aligned} u_1 (u_1 u_2) &\stackrel{?}{=} (u_1 u_1) u_2 \\ u_1 \left(-\tfrac{1}{2} u_2\right) &\stackrel{?}{=} (0)\, u_2 \\ -\tfrac{1}{2} u_1 u_2 &\stackrel{?}{=} 0 \\ \tfrac{1}{4} u_2 &\neq 0 \end{aligned}$$

Example - Continued
Canonical System

By definition $x \in a$ may be written as

$$x = \sum_{i=1}^{n} x^i u_i$$

$\therefore$ we let $x = x^1 u_1 + x^2 u_2$

$$\dot{x} \triangleq x \cdot x$$
$$= (x^1 u_1 + x^2 u_2)(x^1 u_1 + x^2 u_2)$$
$$= x^1 x^1 (u_1 \cdot u_1) + x^1 x^2 (u_1 \cdot u_2)$$
$$+ x^2 x^1 u_2 u_1 + x^2 x^2 u_2 u_2$$
$$= x^2 x^2 u_1 - x^1 x^2 u_2.$$

Linearization

Takata's idea [2] :

Given a nonlinear system

$$\dot{x}(t) = f(x(t)) \qquad , x(t_0) = x_0$$

$x \in R^n$ is $n \times 1$

$f : R^n \to R^n$ is a vector valued function

Introduce a sequence of linearly independent functions of real values

$$\{\varphi_r(x) : r = 0, 1, 2, \ldots\} \quad , \quad \varphi_0(x) \triangleq 1$$

Now $\dot{\varphi}_m(x) = \dfrac{\partial \varphi_m(x)}{\partial x^T} \dot{x} = \dfrac{\partial \varphi_m(x)}{\partial x^T} f(x) \triangleq f_m(x)$

Expand $f_m(x) \longleftrightarrow \sum_{r=1} a_{mr} \varphi_r(x) + a_{m0}$

$$\Rightarrow \quad \dot{\varphi}(x) = A \varphi(x) + b$$

where $\varphi(\cdot) = [\varphi_1(\cdot), \ldots \ldots \varphi_r(\cdot) \ldots]^T$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} & \cdots \\ a_{21} & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$b = [a_{10}, a_{20} \ldots \ldots]$$

Since $\phi_0(x) = 1$

$$\Rightarrow \quad \dot{\phi}_0(x) = 0$$

$$\Rightarrow \quad \dot{z} = Fz$$

$$F = \begin{bmatrix} & 0^T \\ b_1 & A \end{bmatrix}$$

$$z = \begin{bmatrix} \phi_0 \\ \phi \end{bmatrix}$$

$$0^T = [0, 0, \ldots]$$

Hermite Polynomials

Given a set H of complete orthonormal polynomials with weight $W(x)$ for $x \in R$.
An n-dimensional complete orthonormal set of polynomials for $x_i \in R$ for $i = 1, 2, \dots n$ is

$$H_{m_1, m_2, \dots m_n}(x) = \prod_{i=1}^{n} H_{m_i}(x_i)$$

with respect to $W(x) = \prod_{i=1}^{n} w_i(x_i)$.

In two dimensions, let $H_{m_1}(x_1)$ be the polynomials for all $x_1 \in R$ with weight function

$$\phi_1(x_1) = e^{-x_1^2/2}$$

Let $H_{m_2}(x_2)$ be the polynomials for all $x_2 \in R$ with weight function

$$\phi_2(x_2) = e^{-x_2^2/2}$$

$H_{m_1, m_2}(x) = H_{m_1}(x_1) H_{m_2}(x_2)$ is the set of orthogonal polynomials with weight function

$$\phi(x) = e^{-x_1^2/2} \, e^{-x_2^2/2} \quad \forall \; x_1, x_2 \in R$$

Properties of Hermite Polynomials

We use Hermite polynomials to expand $f_{m_1 m_2}(x)$

weight function $\phi(x) = e^{-x^2/2}$, $x \in R$

$H_0(x^1) = 1$ $\qquad$ $H_0(x^2) = 1$

$H_1(x^1) = x^1$ $\qquad$ $H_1(x^2) = x^2$

$H_2(x^1) = 4x^1x^1 - 2$ $\qquad$ $H_2(x^2) = 4x^2x^2 - 2$

$H_3(x^1) = 8x^1x^1x^1 - 12x^1$ $\qquad$ $H_3(x^2) = 8x^2x^2x^2 - 12x^2$

$\vdots$ $\qquad$ $\vdots$

Some Hermite Polynomials formulii

$$\frac{\partial H_{m_j}}{\partial x^j} = m_j H_{m_j - 1}(x^j)$$

$$= x^j H_{m_j}(x^j) - H_{m_j+1}(x^j)$$

$$x H_n(x) = H_{n+1}(x) + n H_{n-1}(x)$$

$$x^2 H_n(x) = x H_{n+1}(x) + n x H_{n-1}$$

$$= H_{n+2}(x) + (n+1) H_n + n\left[H_n(x) + (n-1) H_{n-2}(x)\right]$$

Main Results

Using these as the sequence of functions we may reduce a 2-dimensional quadratic differential system to a linear differential system

$$\dot{H} = A H$$

Previous Example:

Let S: $\dot{x}^1 = x^2 x^2$, $x^1(t_0)$, $x^2(t_0)$ given

$$\dot{x}^2 = -x^1 x^2$$

Linearization:

$$H_{m_1,m_2}(x) \triangleq H_{m_1}(x^1)\, H_{m_2}(x^2)$$

$$\dot{H}_{m_1,m_2}(x) = H'_{m_1}(x^1)\, H_{m_2}(x^2)\, \dot{x}^1 + H_{m_1}(x^1)\, H'_{m_2}(x^2)\, \dot{x}^2$$

$$= H'_{m_1}(x^1)\, H_{m_2}(x^2)\, x^2 x^2 - H_{m_1}(x^1)\, H'_{m_2}(x^2)\, x^1 x^2$$

$$= f_{m_1,m_2}(x)$$

Example - Continued
Linearization

$$\therefore f_{m_1 m_2} = m_1 H_{m_1-1}(x^1)\Big[H_{m_2+2}(x^2) + (2m_2+1) H_{m_2}(x^2) + m_2(m_2-1) H_{m_2-2}(x^2)\Big]$$
$$- m_2\Big[H_{m_1+1}(x^1) + m_1 H_{m_1-1}(x^1)\Big]\Big[H_{m_2}(x^2) + (m_2-1) H_{m_2-2}(x^2)\Big]$$

after multiplication:

$$\Longrightarrow f_{m_1 m_2}(x) = m_1(m_2+1) H_{m_1-1}(x^1) H_{m_2}(x^2)$$
$$- m_2(m_2-1) H_{m_1+1}(x^1) H_{m_2-2}(x^2)$$
$$+ m_1 H_{m_1-1}(x^1) H_{m_2+2}(x^2)$$
$$- m_2 H_{m_1+1}(x^1) H_{m_2}(x^2)$$
$$= \dot{H}_{m_1 m_2}(x)$$

Using $H_{m_1}(x^1) H_{m_2}(x^2) = H_{m_1 m_2}(x)$

$$\dot{H}_{m_1 m_2}(x) = m_1(m_2+1) H_{m_1-1, m_2}(x) - m_2(m_2-1) H_{m_1+1, m_2-2}(x)$$
$$+ m_1 H_{m_1-1, m_2+2}(x) - m_2 H_{m_1+1, m_2}(x)$$

Example – Continued

Using

$$k = \frac{1}{2}\left[(m_1+m_2)^2 + 3m_1 + m_2\right],$$

map from the set of ordered pairs of natural numbers onto the natural numbers according to:

m_1 \ m_2	0	1	2	3
0	0	1	3	6
1	2	4	7	11
2	5	8	12	
3	9	13		

$$\therefore \quad \dot{H}_{0,0}(x) \longrightarrow \dot{H}_0(x)$$

$$\dot{H}_{01}(x) \longrightarrow \dot{H}_1(x) = -H_{m_1+1, m_2} = -H_{1,1} = -H_4$$

$$\dot{H}_{1,0}(x) \longrightarrow \dot{H}_2(x) = H_{0,0} + H_{0,2} = H_0 + H_3$$

$$\dot{H}_{0,2}(x) \longrightarrow \dot{H}_3(x) = 2H_{1,0} - 2H_{1,2} = -2H_2 - 2H_7$$

$$\dot{H}_{1,1}(x) \longrightarrow \dot{H}_4(x) = 2H_{01} + H_{03} - H_{21} = 2H_1 + H_6 - H_8$$

$$\vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots$$

∴ Final linearized differential equation equivalent to the 2. dimensional quadratic system S is: in matrix form

$$\dot{H} = A H$$

where

$$\begin{bmatrix} \dot{H}_0 \\ \dot{H}_1 \\ \dot{H}_2 \\ \dot{H}_3 \\ \dot{H}_4 \\ H_5 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} H_0 \\ H_1 \\ H_2 \\ H_3 \\ H_4 \\ \vdots \\ \vdots \end{bmatrix}$$

and

$$H_0(0) = 1 = H_{00}(0)$$

$$H_1(0) = x^2(0) = H_{01}(x^2(0))$$

$$H_2(0) = x^1(0) = H_{10}(x^1(0))$$

with other initial conditions from Hermite polynomial relations, i.e.

$$H_4(0) = H_{1,1} = x^1 \cdot x^2 \Big|_0 = x^1(0) \cdot x^2(0) = H_2(0) \cdot H_1(0)$$

For $\dot{\underline{h}} = \underline{a}\cdot\underline{h}$

set $\underline{a} = \underline{v}_0$

$$\underline{h} = H_0\underline{v}_0 + H_1\underline{v}_1 + H_2\underline{v}_2 + \cdots$$

and equate on basis vectors $\underline{v}_0, \underline{v}_1, \cdots$

This gives

$$\underline{v}_0\underline{v}_0 = \underline{v}_2 + \cdots$$
$$\underline{v}_0\underline{v}_1 = 2\underline{v}_4 + \cdots$$
$$\underline{v}_0\underline{v}_2 = -2\underline{v}_3 + 2\underline{v}_5 + \cdots$$
$$\underline{v}_0\underline{v}_3 = \underline{v}_2 + 3\underline{v}_7 + \cdots$$
$$\vdots$$

with $\underline{h}(0)$ given.

If triangularize A, or 1st row, can get power-associative algebra and $\dot{\underline{h}} = \underline{a}\cdot\underline{h} \Longrightarrow \underline{h}(t) = e^{\underline{a}t}\,\underline{h}(0)$ after adjoin identity to algebra.