The Foundations of Network Theory

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*(non-member)**

Prologue.

Beware of the past;
Within it lie
Dark haunted pools
That lure the eye.
James McAuley (Ref. 1).

1.—Introduction.

During the last ten years considerable interest has been shown in deciding just what networks are and what are their properties. This appears to have been prompted somewhat by the fact, as in my own case, that teachers have been bothered by fundamental questions concerning the types of networks being discussed. Thus, for instance, one is often pressed to determine if a network is linear in deciding if linear theories can be applied.

Previous work in giving precise formulation of network questions seems to begin with the 1952 work of McMillan (Ref. 2) who gave a formulation for linear, passive, time-invariant resistor-capacitor-inductor-transformer networks through an allowed pair formulation. This was followed by a Fourier transform approach to linear passive-time-invariant networks by Raisbeck (Ref. 3) in 1954. Following this König and Meixner in 1958 (Ref. 4) and Youla, Carlin and Castriota (Ref. 5) in 1959 gave theories based upon linear operators transforming inputs into outputs, again for time-invariant and passive networks. The theory of Youla *et al.*, which was extremely rigorous, being based on Hilbert space concepts, was then extended to distributional inputs and outputs by Zemanian in 1963 (Ref. 6). There are, however, devices which one would customarily consider as networks which fall outside the considerations of these theories. For instance the configuration of Fig. 1 defines two strange networks, for the choices $\gamma = \pm 1$, which were essentially introduced by Tellegen in 1953 (Ref. 7).

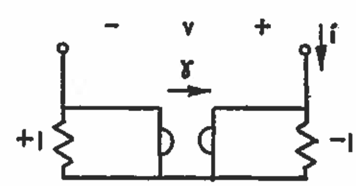

Fig. 1.—$\gamma = \ \ 1$ = *nullator.*
$\gamma = -1$ = *norator.*

If the gyrator in the middle is defined by the impedance matrix $\mathbf{Z} = \begin{bmatrix} 0 & \gamma \\ -\gamma & 0 \end{bmatrix}$, with port two at the right, then $\gamma = 1$ defines the nullator which has $v = i = 0$ while if $\gamma = -1$ we have the norator where v and i are completely independent and arbitrary. By observing Fig. 1, most people would call the configuration a network for these values of the gyration resistance γ, especially since the norator and nullator have been successfully used for synthesis by Carlin and Youla (Ref. 8).

As a consequence of this circuit and because of the limited nature of the above theories we will present here a theory which is based upon the ideas of McMillan (Ref. 2) but which will hold for more general networks, as well as non-linear and time-varying ones. Included in this will be definitions of the main ideas that occur in network theory. As a background we will assume familiarity with the common notions of circuit theory, say as contained in Newstead (Ref. 9). For conciseness, some results, such as those of the examples, will simply be stated. For many of these, through Section 3, a reader finding difficulty filling in the gaps may wish to refer to Ref. 10, but generally the steps should be clear. In several places we refer to the results and notation of Laurent Schwartz's theory of distributions (Ref. 11). The reader unfamiliar with this theory can probably still catch the physical significance of the results by ignoring the unfamiliar words.

The purpose of the paper is then to give a strengthened theory of networks on a rigorous basis. We, however, take this opportunity to announce the time-varying scattering matrix, giving conditions for existence and showing its apparent significance for synthesis in Section 4.

2.—Mathematical Model of Networks.

Consider a device as shown in Fig. 2 which is perhaps a connection of sub-devices to which there are connected $2n$ access points associated in pairs, each pair of which we will call a port.

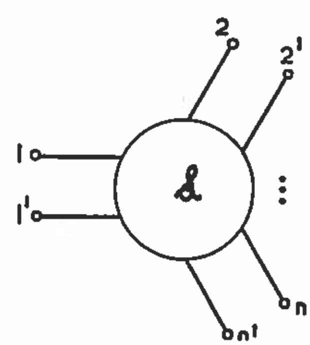

Fig. 2.—An n-Port System.

With the ith port we will associate two variables f_i and g_i which are formed into column n-vectors $\mathbf{f}$, $\mathbf{g}$. For example, with the superscript tilde, $\sim$, denoting matrix transposition $\tilde{\mathbf{f}} = [f_1, \ldots, f_n]$. We will place the physical restriction that each entry in $\mathbf{f}$ and $\mathbf{g}$ is a real valued, infinitely (continuously) differentiable function of time t which is zero until a finite time t, such as is illustrated in Fig. 3(*a*).

We denote this by $\mathbf{f} \epsilon \mathcal{D}_+$, $\mathbf{g} \epsilon \mathcal{D}_+$ where we will customarily drop the boldface in the 1-dimensional case. That is $\mathcal{D}_+$ is the space of real valued n-vectors which are infinitely differentiable and zero until a finite time. If an n-vector $\mathbf{f}$ in $\mathcal{D}_+$ is also zero after

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a finite time, as illustrated for $n = 1$ in Fig. 3(*b*), we say it belongs to $\mathcal{D}$, written $\mathbf{f}\epsilon\mathcal{D}$. We also denote the set of all real *n*-vector distributions, that is, impulses, doublets, ordinary functions, etc. by $\mathcal{D}'$, that is, $\mathbf{f}\epsilon\mathcal{D}'$ if $\mathbf{f}$ has all its entries as distributions. Then $\mathcal{D}\subset\mathcal{D}_+\subset\mathcal{D}'$.

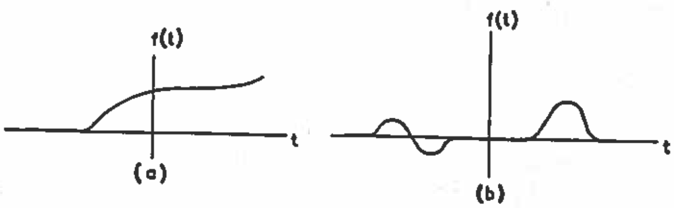

Fig. 3.—(a) Typical $f\epsilon\mathcal{D}_+$
(b) Typical $f\epsilon\mathcal{D}$.

By the conjugate relation we will mean that we will require $\mathbf{f}$ and $\mathbf{g}$ to be chosen such that the total power input $p(t)$ is given by:

$$p(t) = \tilde{\mathbf{f}}(t)\mathbf{g}(t) \quad \text{.................................(1)}$$

The given device places constraints C_s on the variables $\mathbf{f}$, $\mathbf{g}$. These constraints essentially define a *system* $\mathcal{S}$. A system can then be mathematically represented as a set of allowed pairs $[\mathbf{f}, \mathbf{g}]$ of variables satisfying the system constraints C_s, which we will take to include the $\mathcal{D}_+$ as well as the conjugate relation.

This can be precisely formulated by Eq. (2):

$$\mathcal{S} = \{[\mathbf{f}, \mathbf{g}] | \mathbf{f}C_s\mathbf{g}\} \quad \text{.................................(2)}$$

which states: a system $\mathcal{S}$ is the set of pairs of variables $[\mathbf{f}, \mathbf{g}]$ such that $\mathbf{f}$ and $\mathbf{g}$ satisfy the defining constraints C_s. If $[\mathbf{f}, \mathbf{g}]$ is a member of this set, $[\mathbf{f}, \mathbf{g}]\epsilon\mathcal{S}$, it is called an allowed pair. For a *control system* some of $\mathbf{f}$ and $\mathbf{g}$ are inputs, some are outputs, and a transformation maps inputs into outputs. However, for a network some variables might be considered as inputs in one context but not in another. We are then led to:

if

$$\mathbf{f} = \mathbf{v} \text{ [voltages]} \quad \text{.......................................(3a)}$$
$$\mathbf{g} = \mathbf{i} \text{ [currents]} \quad \text{.......................................(3b)}$$

then

$$\mathcal{S} = \mathbf{N} \text{ [\textit{network} (n-port)]} \quad \text{........................(3c)}$$

That is, an *n*-port network $\mathbf{N}$ is defined by:

$$\mathbf{N} = \{[\mathbf{v}, \mathbf{i}] | \mathbf{v}C_n\mathbf{i}\} \quad \text{............................(3d)}$$

The *n*-port is conveniently represented by the symbolism of Fig. 4.

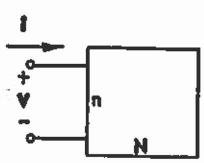

Fig. 4.—n-Port Representation.

We remark that general networks can be defined for which access is at terminals which need not be associated in pairs (Ref. 10), but for our purposes here this is unnecessary. A similar situation holds for the creation of networks at some finite time instead of at $t = -\infty$ as is implicit in the above (Ref. 10).

As an example consider a resistor N_r of resistance r. If the current is prescribed the voltage is given by $v = ir$ thus:

$$N_r = \{[v, i] | v = ir\}$$

The allowed pairs for the resistor are of the form $[ir, i]$ for any $i\epsilon\mathcal{D}_+$. Another example is the $(l + m)$-port transformer $\mathbf{N}_T$ defined by

$$\mathbf{N}_T = \{[\mathbf{v}, \mathbf{i}] | \mathbf{v} = \begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{bmatrix}, \ \mathbf{i} = \begin{bmatrix}\mathbf{i}_1\\ \mathbf{i}_2\end{bmatrix}, \ \mathbf{v}_1 = \tilde{\mathbf{T}}\mathbf{v}_2, \ \mathbf{i}_2 = -\mathbf{T}\mathbf{i}_1\}$$

Here the *mxl* turns ratio matrix $\mathbf{T}(t) = [t_{ij}(t)]$ is assumed independent of the variables, and the variables are partitioned into *l*-vectors $\mathbf{v}_1$, $\mathbf{i}_1$ and *m*-vectors $\mathbf{v}_2$, $\mathbf{i}_2$. The transformer is illustrated in Fig. 5, where Belevitch's representation is given in (*a*) and a compact symbolism is given in (*b*). We note that for the transformer the $\mathbf{i}$'s are completely independent of the $\mathbf{v}$'s; thus there is no transformation which maps the $\mathbf{i}$'s into the $\mathbf{v}$'s or vice versa.

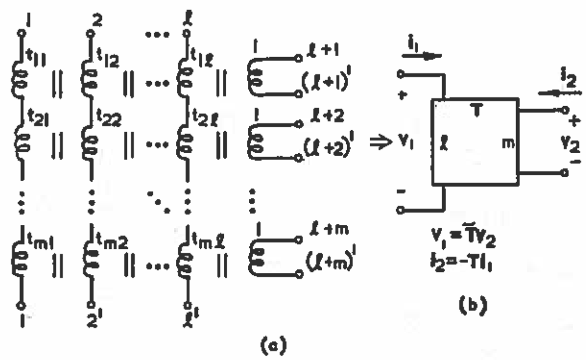

Fig. 5.—(l + m)-Port Transformer Representation.

As can be seen by observing these examples this definition of a network is somewhat cumbersome to work with but it is extremely general, allowing almost anything which can be described through voltages and currents to be a network. The advantage of such a description lies in the fact that the most general properties of any conceivable network can be investigated on a theoretical basis. This allows the conditions for validity of various descriptions to be determined at first glance.

3.—Defined Properties of Networks.

The most important property of a network appears to be that of linearity. This is rigorously defined by Definition 1.

Definition 1:

$\mathbf{N}$ is *linear* if for all $[\mathbf{v}_1, \mathbf{i}_1]$ and $[\mathbf{v}_2, \mathbf{i}_2]\epsilon\mathbf{N}$ and for all real constants α

$$\alpha[\mathbf{v}_1, \mathbf{i}_1] = [\alpha\mathbf{v}_1, \alpha\mathbf{i}_1]\epsilon\mathbf{N} \quad \text{........................(4a)}$$
$$[\mathbf{v}_1, \mathbf{i}_1] + [\mathbf{v}_2, \mathbf{i}_2] = [\mathbf{v}_1 + \mathbf{v}_2, \mathbf{i}_1 + \mathbf{i}_2]\epsilon\mathbf{N} \quad \text{............(4b)}$$

Physically linearity corresponds to the notion of superposition. As examples the transformer given above and the norator and nullator are linear networks.

An important class of linear networks are those described by an $n \times n$ impedance matrix $\mathbf{z}(t, \tau)$ through

$$\mathbf{v}(t) = \int_{-\infty}^{\infty} \mathbf{z}(t, \tau)\mathbf{i}(\tau)d\tau$$

where $\mathbf{z}$ must be such that $\mathbf{i}\epsilon\mathcal{D}_+$ implies $\mathbf{v}\epsilon\mathcal{D}_+$.

Linearity appears to be an idealization but only recently have any kind of general results been obtained without assuming linearity (Ref. 19).

The second notion of importance is related to that of continuity and is here called by the name of solvability.

Definition 2:

$\mathbf{N}$ is *solvable* if for all $\mathbf{e}\epsilon\mathcal{D}_+$

$$\mathbf{e} = \mathbf{v} + \mathbf{i} \quad \text{...(5)}$$

is satisfied by a unique $[\mathbf{v}, \mathbf{i}]\epsilon\mathbf{N}$.

Here the voltage **e** can be considered as that applied to the augmented network $\mathbf{N}_a$ as shown in Fig. 6, where $\mathbf{1}_n$ denotes the unit matrix of order n and represents n uncoupled unit resistors.

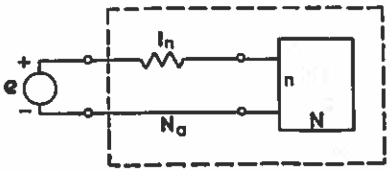

Fig. 6.—The Augmented Network $\mathbf{N}_a$.

Physically the concept of solvability corresponds to the notion that a given **e** applied to the augmented network gives rise to a unique current **i** flowing into the network. As examples the transformer and the gyrator are solvable while the nullator and norator are not. Further, the negative resistor of resistance -1 is not solvable, nor is the ideal diode. However, almost all physical networks are solvable. Another concept of interest is that of time invariance which is precisely defined in the following manner.

Definition 3:

N is *time-invariant* if for all $[\mathbf{v}, \mathbf{i}]\epsilon\mathbf{N}$ there exists a $[\mathbf{v}_0, \mathbf{i}_0]\epsilon\mathbf{N}$ such that for all real, finite λ

$$[\mathbf{v}(t), \mathbf{i}(t)] = [\mathbf{v}_0(t+\lambda), \mathbf{i}_0(t+\lambda)] \quad \text{.................(6)}$$

Physically this states that a shift in the time scale for the currents results in a similar shift for the time scale of the voltages. As an example the transformer is time-invariant if and only if the turns ratio **T** is independent of time t. Likewise $\mathbf{z}(t, \tau)$, below Definition 1, defines a time-invariant **N** if and only if the argument is $t - \tau$, that is $\mathbf{z}(t, \tau) = \mathbf{Z}(t - \tau)$.

The next definition has raised many controversies (Ref. 5). We state what seems to us the most physically satisfying by taking a passive network to be one for which the total energy input is non-negative for all time.

Definition 4:

N is *passive* if for all $[\mathbf{v}, \mathbf{i}]\epsilon\mathbf{N}$ and for all finite t

$$\mathcal{E}(t) = \int_{-\infty}^{t} \tilde{\mathbf{v}}(\beta)\mathbf{i}(\beta)d\beta \geq 0 \quad \text{.......................(7)}$$

As examples the transformer is passive even if the turns ratio actually varies with time. The norator is not passive but the nullator and gyrator are.

Closely related to the concept of passivity is that of losslessness. Physically we can consider the augmented network, seeing that if $\mathbf{e}(t)$ tends to **0** at $t = \infty$ for a lossless **N** then all the energy flowing into **N** will eventually be dissipated in the unit resistors at the input. Consequently, at infinity the energy into **N** will also be zero. This is the content of Definition 5, where for convenience we require solvability as well as passivity.

Definition 5:

N is *lossless* if

(*P*) **N** is passive and
(*S*) **N** is solvable and
(*L*) For every $\mathbf{e}\epsilon\mathcal{D}$, $\mathbf{e} = \mathbf{v} + \mathbf{i}$,

$$\mathcal{E}(\infty) = \int_{-\infty}^{\infty} \tilde{\mathbf{v}}(\beta)\mathbf{i}(\beta)d\beta = 0 \quad \text{.......................(8)}$$

As examples, the transformer for any time varying turns ratio is lossless as is a linear time-invariant capacitor whose capacitance is non-negative. The nullator is passive but not solvable, hence not lossless.

Physically reciprocity means that responses to excitations are invariant to an interchange of response and excitation points. In terms of the pairs which we are using, this is phrased in the following way.

Definition 6:

N is *reciprocal* if for all $[\mathbf{v}_1, \mathbf{i}_1]$, $[\mathbf{v}_2, \mathbf{i}_2]\epsilon\mathbf{N}$

$$\tilde{\mathbf{v}}_1 * \mathbf{i}_2 = \tilde{\mathbf{v}}_2 * \mathbf{i}_1 \quad \text{............................(9)}$$

Here * denotes convolution. As examples, the transformer is reciprocal while the gyrator is not. The norator is non-reciprocal as is the ideal diode which is also non-linear.

The next definition has philosophical content but, as we will see, for linear passive networks the property is a consequence of solvability. Antecedance is closely related to the concept of causality, the latter of which means that no responses can occur before an excitation. However, no mathematics is capable of stating what is a response or what is an excitation, at least in our opinion, and consequently we follow the ideas of Bunge (Ref. 12) and refer to the concept as antecedance.

Definition 7:

N is *antecedal* if for all fixed t_0 there exists $[\mathbf{0}, \mathbf{i}(t)]$, $[\mathbf{v}(t), \mathbf{0}]\epsilon\mathbf{N}$ for $-\infty < t < t_0$ and for all such pairs

$$[\mathbf{0}, \mathbf{i}(t)]\epsilon\mathbf{N} \text{ implies } \mathbf{i}(t) = \mathbf{0} \text{ for all } t < t_0 \quad \text{.........(10a)}$$

and

$$[\mathbf{v}(t), \mathbf{0}]\epsilon\mathbf{N} \text{ implies } \mathbf{v}(t) = \mathbf{0} \text{ for all } t < t_0 \quad \text{.........(10b)}$$

4.—Derived Properties of Networks.

Having defined and used the allowed pair formulation to formulate precisely the properties of networks, we can specialize to certain classes of networks. In the linear and solvable case we can obtain descriptions which are much simpler to work with. These will be somewhat familiar, but more general than customarily met.

The first result is that every linear and solvable network **N** defines, through the augmented network, a linear continuous mapping $\mathcal{Y}_a[\ ]$ of **e** into **i**. That is, when **N** is linear and solvable, there exists a transformation $\mathcal{Y}_a[\ ]$ having the following three properties, which hold for all real constants α and β and all **e**, $\mathbf{e}_1$, $\mathbf{e}_2\epsilon\mathcal{D}_+$ and convergent sequences $\{\mathbf{e}_k\}$, $\mathbf{e}_k\epsilon\mathcal{D}_+$

$$\mathbf{i} = \mathcal{Y}_a[\mathbf{e}] \quad \text{...............................(11a)}$$

$$\alpha\mathcal{Y}_a[\mathbf{e}_1] + \beta\mathcal{Y}_a[\mathbf{e}_2] = \mathcal{Y}_a[\alpha\mathbf{e}_1 + \beta\mathbf{e}_2] \quad \text{..............(11b)}$$

$$\lim_{k\to\infty} \mathcal{Y}_a[\mathbf{e}_k] = \mathcal{Y}_a[\lim_{k\to\infty} \mathbf{e}_k] \quad \text{....................(11c)}$$

The first of these results from the fact that **i** is uniquely determined by **e**. Thus there is some mapping, called $\mathcal{Y}_a[\ ]$ which transforms a given $\mathbf{e}\epsilon\mathcal{D}_+$ into **i**, the latter of which is in $\mathcal{D}_+$ by our network assumptions. Eq. (11*b*) merely states that the augmented network is linear with **N**. Eq. (11*c*) follows from the fact that $[\mathbf{0}, \mathbf{0}]\epsilon\mathbf{N}$, if the network is linear. Thus if we choose a sequence such that $\lim \mathbf{e}_k = \mathbf{0}$, then since $\mathbf{0}\epsilon\mathcal{D}_+$, $\lim \mathbf{v}_k = -\lim \mathbf{i}_k = \mathbf{0}$ by the uniqueness incorporated in the solvability constraint, here $\mathbf{e}_k = \mathbf{v}_k + \mathbf{i}_k$. Then, if $\lim \mathbf{e}_k = \mathbf{e} \neq \mathbf{0}$ we have $\mathcal{Y}_a[\lim(\mathbf{e}_k - \mathbf{e})] = = \mathbf{0} = \lim \mathcal{Y}_a[\mathbf{e}_k - \mathbf{e}] = \lim \mathcal{Y}_a[\mathbf{e}_k] - \mathcal{Y}_a[\mathbf{e}]$ which is Eq. (11*c*).

The augmented network then defines a linear continuous map of $\mathbf{e}\epsilon\mathcal{D}C\mathcal{D}_+$ into $\mathbf{i}\epsilon\mathcal{D}_+C\mathcal{D}'$ in which case a result of L. Schwartz (Ref. 13) shows that there exists an $n \times n$ matrix $\mathbf{y}_a(t, \tau)$ such that:

$$\mathbf{i}(t) = \int_{-\infty}^{\infty} \mathbf{y}_a(t, \tau)\mathbf{e}(\tau)d\tau \quad \text{.......................(12a)}$$

for all $\mathbf{e}\epsilon\mathcal{D}$. Here $\mathbf{y}_a(t, \tau)$ is a matrix of distributions in two variables called the time-varying augmented admittance matrix and the integral is a familiar short-hand notation for an unfamiliar distributional mapping precisely defined by Schwartz (Ref. 14, p. 221). Since $\mathcal{D}$ is dense in $\mathcal{D}'$, for any $\mathbf{e}\epsilon\mathcal{D}'$ there exists a sequence $\{\mathbf{e}_k\}$ with $\mathbf{e} = \lim \mathbf{e}_k$. By taking:

$$\mathbf{i}(t) = \lim_{k\to\infty} \int_{-\infty}^{\infty} \mathbf{y}_a(t, \tau)\mathbf{e}_k(\tau)d\tau = \int_{-\infty}^{\infty} \mathbf{y}_a(t, \tau)\mathbf{e}(\tau)d\tau \quad \text{......(12}b\text{)}$$

we can extend $\mathbf{N}$ to those $\mathbf{e}\epsilon\mathcal{D}'$ for which this limit exists in the distributional sense. In particular this will always be possible when $\mathbf{e}$ consists of impulses, in which case we see that the i, j component of $\mathbf{y}_a(t, \tau)$ is physically the current entering port i of $\mathbf{N}_a$ at time t when an impulse of voltage is applied at port j at time τ, all other ports of $\mathbf{N}_a$ being unexcited, that is short circuited. Since $\mathbf{v} = \mathbf{e} - \mathbf{i}$, Eq. (12*a*) shows that:

$$\mathbf{v}(t) = \int_{-\infty}^{\infty} [\delta(t - \tau)\mathbf{1}_n - \mathbf{y}_a(t, \tau)]\mathbf{e}(\tau)d\tau \quad \text{..............(12}c\text{)}$$

where δ is the unit impulse.

Before proceeding it is convenient to introduce some more notation. We define $\mathbf{a} = \mathbf{A}\circ\mathbf{b}$ and $\mathbf{C} = \mathbf{A}\circ\mathbf{B}$ for $n \times n$ matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and n-vectors $\mathbf{a}$ and $\mathbf{b}$ by

$$\mathbf{a}(t) = \int_{-\infty}^{\infty} \mathbf{A}(t, \tau)\mathbf{b}(\tau)d\tau = \mathbf{A}\circ\mathbf{b} \quad \text{..................(13}a\text{)}$$

$$\mathbf{C}(t, \tau) = \int_{-\infty}^{\infty} \mathbf{A}(t, \lambda)\mathbf{B}(\lambda, \tau)d\lambda = \mathbf{A}\circ\mathbf{B} \quad \text{...........(13}b\text{)}$$

The second of these is often referred to as Volterra composition, the first being commonly referred to as a superposition integral. The operation of forming either of these we shall call composing. The $\circ$ in $\mathbf{A}\circ\mathbf{B}$ can be treated as a multiplication with $\delta\mathbf{1}_n$ acting as a unit in the resulting algebra. Thus we call $\mathbf{A}^{-1}$ an inverse of $\mathbf{A}$ under composition if:

$$\mathbf{A}^{-1}\circ\mathbf{A} = \mathbf{A}\circ\mathbf{A}^{-1} = \delta(t - \tau)\mathbf{1}_n \quad \text{...............(13}c\text{)}$$

In this notation:

$$\mathbf{i} = \mathbf{y}_a\circ\mathbf{e} \quad \text{.............................(14}a\text{)}$$

$$\mathbf{v} = (\delta\mathbf{1}_n - \mathbf{y}_a)\circ\mathbf{e} \quad \text{..........................(14}b\text{)}$$

Since any network possessing such a $\mathbf{y}_a$ must be linear and solvable, we comment that a necessary and sufficient condition for the existence of $\mathbf{y}_a(t, \tau)$ is that $\mathbf{N}$ be linear and solvable.

The augmented admittance, although containing the properties of $\mathbf{N}$, is more characteristic of $\mathbf{N}_a$ than $\mathbf{N}$. We can thus pre-compose Eq. (14*a*) by $(\delta\mathbf{1}_n - \mathbf{y}_a)$ and Eq. (14*b*) by $\mathbf{y}_a$. After noting that these are equal we see that:

$$\mathbf{y}_a\circ\mathbf{v} = (\delta\mathbf{1}_n - \mathbf{y}_a)\circ\mathbf{i} \quad \text{.......................(15}a\text{)}$$

which is a special case of the *general description*:

$$\mathbf{a}\circ\mathbf{v} = \mathbf{b}\circ\mathbf{i} \quad \text{.............................(15}b\text{)}$$

Here $\mathbf{a}$ and $\mathbf{b}$ are now square matrices.

Consequently any linear and solvable network possesses a general description; however, other networks such as the -1 resistor do too. This latter has $v = -\delta\circ i = -i$. It is convenient to allow non-square matrices $\mathbf{a}$ and $\mathbf{b}$ in Eq. (15*b*), in which case the nullator can be described by $\begin{bmatrix}0\\ \delta\end{bmatrix}\circ v = \begin{bmatrix}\delta\\ 0\end{bmatrix}\circ i$. The general description is extremely convenient for setting up network equations, for instance the transformer has:

$$\delta\begin{bmatrix}-\tilde{\mathbf{T}} & \mathbf{1}_m\\ 0 & \mathbf{0}_m\end{bmatrix}\circ\begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{bmatrix} = \delta\begin{bmatrix}\mathbf{0}_l & 0\\ \mathbf{1}_l & \mathbf{T}\end{bmatrix}\circ\begin{bmatrix}\mathbf{i}_1\\ \mathbf{i}_2\end{bmatrix} \quad \text{...........(16)}$$

where $\mathbf{0}_m$ is the mth order zero matrix.

From the general description we can derive all the standard descriptions. For instance we can pre-compose by a non-singular, under $\circ$, matrix $\mathbf{c}$ to write

$$(\mathbf{c}\circ\mathbf{a})\circ\mathbf{v} = (\mathbf{c}\circ\mathbf{b})\circ\mathbf{i} \quad \text{.......................(17}a\text{)}$$

which in fact shows that $\mathbf{a}$ and $\mathbf{b}$ of Eq. (15*b*) need not be unique, since $\mathbf{c}\circ\mathbf{a}$ serves as a new $\mathbf{a}$. Choosing $\mathbf{c} = \mathbf{a}^{-1}$ yields the time-variable impedance matrix $\mathbf{z}(t, \tau)$

$$\mathbf{v} = \mathbf{z}\circ\mathbf{i} \quad \text{.............................(17}b\text{)}$$

$$\mathbf{z} = \mathbf{a}^{-1}\circ\mathbf{b} \quad \text{.............................(17}c\text{)}$$

Similarly, the time-variable admittance matrix $\mathbf{y}(t, \tau)$ results from $\mathbf{c} = \mathbf{b}^{-1}$

$$\mathbf{i} = \mathbf{y}\circ\mathbf{v} \quad \text{.............................(17}d\text{)}$$

$$\mathbf{y} = \mathbf{b}^{-1}\circ\mathbf{a} \quad \text{.............................(17}e\text{)}$$

Because of its apparent importance for synthesis, we wish to draw attention to the time-varying scattering matrix $\mathbf{s}(t, \tau)$. For this we define the incident and reflected voltages for $\mathbf{N}$ by:

$$2\mathbf{v}^i = \mathbf{v} + \mathbf{i} = \mathbf{e} \quad \text{.........................(18}a\text{)}$$

$$2\mathbf{v}^r = \mathbf{v} - \mathbf{i} \quad \text{.............................(18}b\text{)}$$

or

$$\mathbf{v} = \mathbf{v}^i + \mathbf{v}^r \quad \text{.............................(18}c\text{)}$$

$$\mathbf{i} = \mathbf{v}^i - \mathbf{v}^r \quad \text{.............................(18}d\text{)}$$

Inserting this last pair of equations into Eq. (15*b*) gives $\mathbf{a}\circ(\mathbf{v}^i + \mathbf{v}^r) = \mathbf{b}\circ(\mathbf{v}^i - \mathbf{v}^r)$ or

$$(\mathbf{b} + \mathbf{a})\circ\mathbf{v}^r = (\mathbf{b} - \mathbf{a})\circ\mathbf{v}^i \quad \text{....................(19}a\text{)}$$

If by definition $\mathbf{s}(t, \tau)$ maps incident voltages into reflected ones, that is if:

$$\mathbf{v}^r = \mathbf{s}\circ\mathbf{v}^i \quad \text{.............................(19}b\text{)}$$

then clearly:

$$\mathbf{s} = (\mathbf{b} + \mathbf{a})^{-1}\circ(\mathbf{b} - \mathbf{a}) \quad \text{....................(19}c\text{)}$$

If we further make the choice of $\mathbf{a}$ and $\mathbf{b}$ given in Eq. (15*a*), direct substitution into Eq. (19*c*) yields:

$$\mathbf{s}(t, \tau) = \delta(t - \tau)\mathbf{1}_n - 2\mathbf{y}_a(t, \tau) \quad \text{.................(19}d\text{)}$$

From this we can conclude that $\mathbf{N}$ *has a time variable scattering matrix if and only if* $\mathbf{N}$ *is linear and solvable.*

Eq. (19*d*) usually gives an easier method of calculating $\mathbf{s}$ than Eq. (19*c*). As an important example consider the transformer $(l + m)$-port, augmented as shown in Fig. 7.

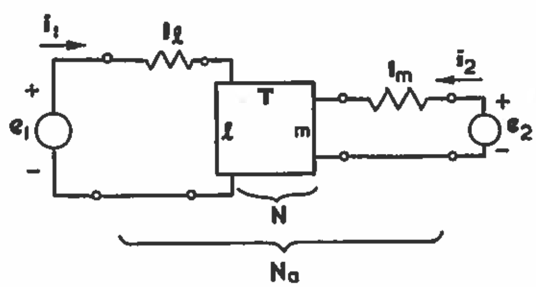

Fig. 7.—The Augmented Transformer.

This is described by $\mathbf{e}_1 = \mathbf{i}_1 + \tilde{\mathbf{T}}\mathbf{v}_2$, $\mathbf{e}_2 = -\mathbf{T}\mathbf{i}_1 + \mathbf{v}_2$. If the second of these is solved for $\mathbf{v}_2$ and inserted into the first we get $\mathbf{e}_1 = \mathbf{i}_1 + \tilde{\mathbf{T}}\mathbf{T}\mathbf{i}_1 + \tilde{\mathbf{T}}\mathbf{e}_2$. Inverting this yields $\mathbf{i}_1$ as a function of $\mathbf{e}_1$ and $\mathbf{e}_2$, and $\mathbf{i}_2$ is found from $\mathbf{i}_2 = -\mathbf{T}\mathbf{i}_1$. Thus

$$\mathbf{i}_1 = (\mathbf{1}_l + \tilde{\mathbf{T}}\mathbf{T})^{-1}\mathbf{e}_1 - (\mathbf{1}_l + \tilde{\mathbf{T}}\mathbf{T})^{-1}\tilde{\mathbf{T}}\mathbf{e}_2 \quad \text{....................(20}a\text{)}$$

$$\mathbf{i}_2 = -\mathbf{T}(\mathbf{1}_l + \tilde{\mathbf{T}}\mathbf{T})^{-1}\mathbf{e}_1 + \mathbf{T}(\mathbf{1}_l + \tilde{\mathbf{T}}\mathbf{T})^{-1}\tilde{\mathbf{T}}\mathbf{e}_2 \quad \text{...........(20b)}$$

When the coefficient matrix in this, with $\mathbf{T} = \mathbf{T}(t)$, is multiplied by $\delta(t - \tau)$ it is $\mathbf{y}_a(t, \tau)$. Eq. (19*d*) then gives for the scattering matrix of the transformer, using a superscript -1 here for the normal matrix inverse:

$$\mathbf{s}(t, \tau) = \delta(t - \tau) \begin{bmatrix} (\mathbf{1}_l + \tilde{\mathbf{T}}\mathbf{T})^{-1}(\tilde{\mathbf{T}}\mathbf{T} - \mathbf{1}_l) & 2(\mathbf{1}_l + \tilde{\mathbf{T}}\mathbf{T})^{-1}\tilde{\mathbf{T}} \\ 2\mathbf{T}(\mathbf{1}_l + \tilde{\mathbf{T}}\mathbf{T})^{-1} & (\mathbf{1}_m + \mathbf{T}\tilde{\mathbf{T}})^{-1}(\mathbf{1}_m - \mathbf{T}\tilde{\mathbf{T}}) \end{bmatrix} \quad (21a)$$

where we have used $\mathbf{T}(\mathbf{1}_l + \tilde{\mathbf{T}}\mathbf{T})^{-1} = (\mathbf{1}_m + \mathbf{T}\tilde{\mathbf{T}})^{-1}\mathbf{T}$ which can be checked by cross-multiplication. A special case of interest is when $\mathbf{T}$ is orthogonal, that is $m \times m$ and satisfying $\tilde{\mathbf{T}}(t)\mathbf{T}(t) = \mathbf{1}_m$. In this case:

$$\mathbf{s}(t, \tau) = \delta(t - \tau) \begin{bmatrix} \mathbf{0}_m & \tilde{\mathbf{T}}(t) \\ \mathbf{T}(t) & \mathbf{0}_m \end{bmatrix} \quad \text{....................(21b)}$$

The importance of these lies in their use for synthesis, as can be seen through Fig. 8 where an $(n + m)$-port of scattering matrix:

$$\Sigma(t, \tau) = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \quad \text{....................(22)}$$

partitioned as the ports, that is $\Sigma_{11} = n \times n$, $\Sigma_{12} = n \times m$, $\Sigma_{21} = m \times n$ and $\Sigma_{22} = m \times m$ is loaded by an m-port $\mathbf{N}_l$ of scattering matrix $\mathbf{s}_l$.

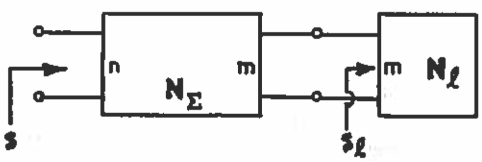

Fig. 8.—Terminated Network.

Since the reflected and incident voltages are interchanged for the final m ports of $\mathbf{N}_\Sigma$ and $\mathbf{N}_l$, because voltages are equal but currents are negatives, solving equations of the form of Eq. (19*b*) gives:

$$\mathbf{s} = \Sigma_{11} + \Sigma_{12} \circ \mathbf{s}_l \circ (\delta \mathbf{1}_m - \Sigma_{22} \circ \mathbf{s}_l)^{-1} \circ \Sigma_{21} \quad \text{...........(23)}$$

Some special cases of interest are when $\Sigma_{11} = \mathbf{0}_n$ with $\Sigma_{22} = \mathbf{0}_m$, when $\mathbf{N}_\Sigma$ is an orthogonal transformer, and when $\mathbf{N}_l$ is a set of m uncoupled $+1$ resistors:

$$\mathbf{s} = \Sigma_{12} \circ \mathbf{s}_l \circ \Sigma_{21} \quad \text{if} \quad \Sigma_{11} = \mathbf{0}_n,\ \Sigma_{22} = \mathbf{0}_m \quad \text{........(24a)}$$

$$\mathbf{s}(t, \tau) = \tilde{\mathbf{T}}(t)\mathbf{s}_l(t, \tau)\mathbf{T}(\tau) \quad \text{if } \mathbf{N}_\Sigma = \text{orthogonal transformer} \quad \text{...(24b)}$$

$$\mathbf{s} = \Sigma_{11} \quad \text{if} \quad \mathbf{s}_l = \mathbf{0}_m \quad \text{....................(24c)}$$

The first of these shows how we can "multiply" scattering matrices, either on the left, if $\Sigma_{21} = \mathbf{1}_n$, or on the right, if $\Sigma_{12} = \mathbf{1}_n$. Such multiplication has been used to advantage by Belevitch (Ref. 15) in the synthesis of time-invariant networks. Eq. (24*b*) shows how "congruency" transformations on $\mathbf{s}$ can be realized in the time domain. The last of these, Eq. (24*c*), shows that if we wish to synthesize a given $\mathbf{s}$, we could border it such that Σ of Eq. (22) corresponds to a lossless network. Terminating in unit resistors, for $\mathbf{N}_l$ of Fig. 8, shows that $\mathbf{s}$ is realized at the input. This is another method that has been used advantageously by Belevitch (Ref. 16), as well as by Oono and Yasuura (Ref. 17), in the time-invariant case. Although some general time-varying results obtain from these ideas, as will be reported elsewhere, research is still in progress. However, the ideas can be illustrated in the analysis of the network of Fig. 9.

If we assume $\mathbf{T}(t) = \begin{bmatrix} t_{11}(t) & t_{12}(t) \\ t_{21}(t) & t_{22}(t) \end{bmatrix}$ orthogonal, that is $t_{22} = t_{11}$, $t_{21} = -t_{12}$, $t_{11}^2 + t_{12}^2 = 1$, then the capacitor loaded 2-port is described by:

$$\Sigma(t, \tau) = \begin{bmatrix} t_{11}(t) & -t_{12}(t) \\ t_{12}(t) & t_{11}(t) \end{bmatrix} \begin{bmatrix} \mathbf{s}_c(t, \tau) & 0 \\ 0 & \mathbf{s}_c(t, \tau) \end{bmatrix} \begin{bmatrix} t_{11}(\tau) & t_{12}(\tau) \\ -t_{12}(\tau) & t_{11}(\tau) \end{bmatrix}$$

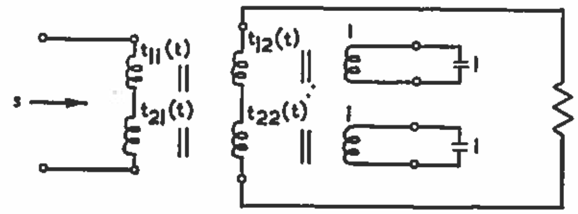

Fig. 9.—Example Network.

Here, by the use of the Laplace transform for which $S_c(p) = (1 + Y(p))^{-1}(1 - Y(p)) = (1 - p)/(1 + p)$, we obtain the scattering coefficient for the capacitor as:

$$s_c(t, \tau) = -\delta(t - \tau) + 2e^{-(t-\tau)}u(t - \tau)$$

where u is the unit step function. When the unit resistor is connected, Eq. (24*c*) applies and we have:

$$\begin{aligned} s(t, \tau) &= t_{11}(t)s_c(t, \tau)t_{11}(\tau) + t_{12}(t)s_c(t, \tau)t_{12}(\tau) \\ &= [t_{11}(t) + t_{12}(t)][-\delta(t - \tau) + 2e^{-(t-\tau)}u(t - \tau)][t_{11}(\tau) + t_{12}(\tau)] \end{aligned}$$

Finally for a linear solvable and passive network $\mathbf{N}$ we outline the proof that $\mathbf{N}$ is antecedal. The idea follows that of Youla (Ref. 5, p. 111). We have:

$$\int_{-\infty}^{t} \tilde{\mathbf{e}}(\tau)\mathbf{e}(\tau)d\tau = \int_{-\infty}^{t} \tilde{\mathbf{v}}(\tau)\mathbf{v}(\tau)d\tau + \int_{-\infty}^{t} \tilde{\mathbf{i}}(\tau)\mathbf{i}(\tau)d\tau + 2\int_{-\infty}^{t} \tilde{\mathbf{v}}(\tau)\mathbf{i}(\tau)d\tau \quad \text{...(25)}$$

When $\mathbf{e}, \mathbf{v}, \mathbf{i} \in \mathcal{D}_+$ all integrals exist for finite t and when $\mathbf{N}$ is passive the last term is non-negative. Since the other three terms are always non-negative, if the left side is zero all terms are zero. Thus, if we let $\mathbf{e}(t) = \mathbf{0}$ for $t < t_0$, t_0 an arbitrary but fixed constant, then letting $t < t_0$ in Eq. (25) shows that $\mathbf{i}(t)$, as well as $\mathbf{v}(t)$, is zero for $t < t_0$. Intuitively then Eq. (12*a*) shows that $\mathbf{y}_a(t, \tau) = \mathbf{0}_n$ for $t < \tau$, a fact which can be proven rigorously through the use of "support" and scalar product arguments from the theory of distributions (which we however omit for conciseness). Consequently, by Eq. (19*d*):

$$\mathbf{s}(t, \tau) = \mathbf{0}_n \quad \text{for } t < \tau \text{ when } \mathbf{N} \text{ is passive} \quad \text{...........(26)}$$

This shows that $\mathbf{v}^r(t) = \mathbf{0}$ for $t < t_0$ when $\mathbf{v}^i(t) = 2\mathbf{e}(t) = \mathbf{0}$ for $t < t_0$ in which case Eq. (18*c*) shows that $\mathbf{v}(t) = \mathbf{0}$ for $t < t_0$ when $\mathbf{i}(t) = \mathbf{0}$ for $t < t_0$ and vice versa. Definition 7 then shows that $\mathbf{N}$ is antecedal, if it is linear, solvable and passive.

Eq. (25) also shows that if $\mathbf{e}$ is square integrable, that is:

$$\int_{-\infty}^{\infty} \tilde{\mathbf{e}}(\tau)\mathbf{e}(\tau)d\tau < \infty$$

then $\mathbf{v}$ and $\mathbf{i}$ are also square integrable if $\mathbf{N}$ is passive, and thus also $\mathbf{v}^i$ and $\mathbf{v}^r$ by Eq. (18). Now if $\mathbf{N}$ is time-invariant, then $\mathbf{s}(t, \tau)$ is a function only of the difference argument $t - \tau$ (Ref. 13). In this case we can take Laplace transforms of Eq. (19*b*), which is now a convolution, giving:

$$\mathbf{V}^r(p) = \mathbf{S}(p)\mathbf{V}^i(p) \quad \text{..........................(27)}$$

where p is the complex frequency variable and $\mathbf{S}(p)$ is the Laplace transform of $\mathbf{s}(t, 0)$. In the passive case then, choosing $2\mathbf{e} = \mathbf{v}^i$ square integrable, we see that every entry $s_{ij}(p)$ in $\mathbf{S}(p)$ is the ratio of Laplace transforms of square integrable functions. By a theorem of Laplace transform theory, such transforms are analytic everywhere in the right-half plane, Re $p > 0$, and almost everywhere on the imaginary axis, Re $p = 0$, Re denoting the real part (Ref. 18, p. 80). Consequently every linear, solvable, time-invariant and passive network has a transform scattering matrix $\mathbf{S}(p)$ which is analytic in Re $p > 0$. A simple consequence of this is the fact that the linear, solvable, time-invariant network defined by:

$$v^r(t) = \int_{-\infty}^{\infty} e^{(t-\tau)^2} u(t-\tau) v^i(\tau) d\tau$$

can not be passive, since $e^{t^2}u(t)$ has no Laplace transform!

5.—Conclusions and Discussion.

This paper outlines a rigorous theory of networks, set up with the synthesis of time-varying networks in mind. Although the definitions of network properties given hold when considering any type of network, they should not be looked upon as the only possible ones or even the most appropriate. For instance it may be desirable to generalize the notion of solvability such that it is more meaningful for nonlinear networks. In any event, using the notions given, very general results concerning time-variable networks can be obtained. Here we merely introduce the scattering matrix, outlining its apparent importance for synthesis. However, it is possible to give more of the properties of $\mathbf{s}(t, \tau)$ for passive networks, this being reserved for a more detailed treatment elsewhere.

The definition of a network and those of its properties are open for many philosophical discussions, which space however does not permit. Nevertheless we reiterate that we feel the emphasis at the beginning of a theory of networks must be placed upon pairs of variables $[\mathbf{v}, \mathbf{i}]$, in place of upon input-output concepts. Thus the input to a resistor in one usage might be the current with the voltage as a response, while in another context the voltage might best be considered an input. The symbolism $[\mathbf{v}, \mathbf{i}]$ is intended to place equal weight upon $\mathbf{v}$ and $\mathbf{i}$, and all the definitions reflect this. Consequently the principal of duality can be immediately used in deriving results. We also point out the necessity of placing a restriction on the domain of definition of the variables, our $\mathcal{D}_+$ constraint. If we had allowed arbitrary distributions from the very beginning, then many of the familiar properties would no longer be valid. For instance the linear time-invariant inductor would not be reciprocal since $i_1 = u$ gives $v_1 = \delta$ while $i_2 = I$ = constant gives $v_2 = 0$ and $l\delta * I = lI \neq 0 = 0 * u$. Further, impulses can not be applied to the non-linear resistor $r = i$, nor step functions physically to ideal differentiators.

Some other important concepts can also be introduced, such as circuits as inter-connections of networks following Kirchhoff's laws, and equivalent circuits, that is circuits which yield the same network. Since emphasis is placed upon terminal behavior, a network can then be looked upon as an equivalence class of circuits.

Epilogue.

But build on the past:
Discard the bad,
Make use of good
In which it's clad.

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