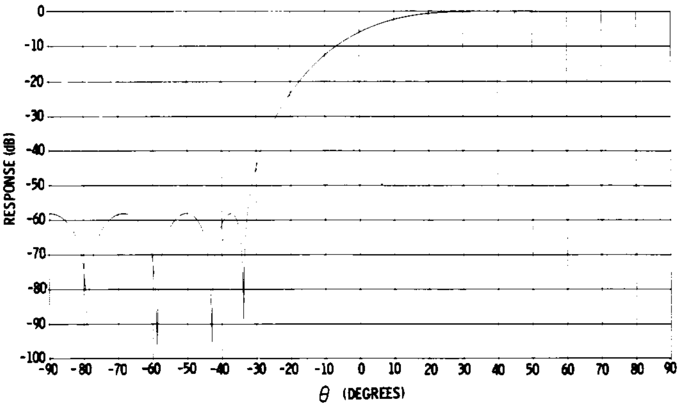

Fig. 2. Line array pattern, 5 active elements, (γ = 0.3, T = 0.325 λ, scan angle = ±57°).

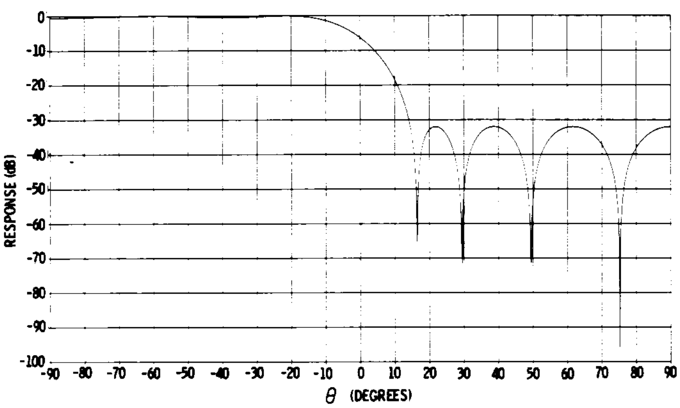

Fig. 3. Line array pattern, 5 active elements, (γ = 0.6, T = 0.40 λ, scan angle = ±75°).

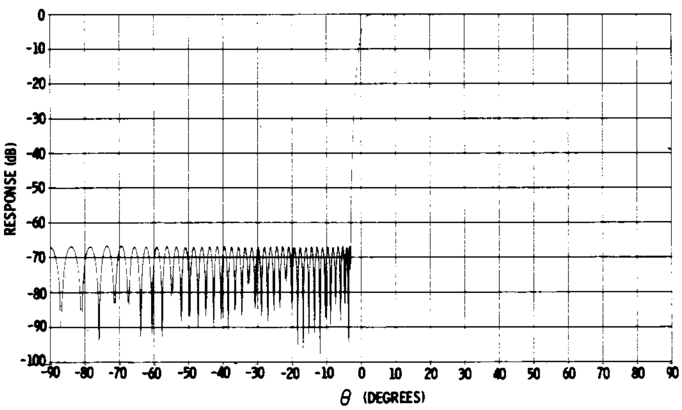

Fig. 4. Line array pattern, 39 active elements, (γ = 0.9, T = 0.475 λ, scan angle = ±87°).

Fig. 2 shows the pattern plot for an element designed with a γ of 0.3 and a T-value of 0.325 λ. There are 5-active elements with currents of −0.0740035, −0.572755, ±j, 0.572755, 0.0740035. The transition angle is 33° which allows a scan angle of ±57° for the planar array. Fig. 3 also uses 5 active elements but γ = 0.6 here so that the transition angle is reduced to 15° permitting a ±75° scan angle. Comparing Fig. 2 and 3 shows a design trade of scan angle against sidelobe level, the active element count being held constant. Fig. 4 shows a particular design for γ = 0.9 which reduces the transition angle to 3° and raises the scan angle to ±87°. The active element count rises to 39 for this rather extreme example.

Real-world realization of any array concept faces a host of problems, not the least of which is mutual coupling between elements [3]–[5]. It should be noted that the predominant element spacing in Fig. 1(a) is twice the T value of the design. For the three examples given, spacings in the range of 0.65 λ to 0.95 λ are encountered. These values are not at all unusual. The spacing at the center element only is T rather than $2T$. The role played by the quadrature relationship of the center element current to all other currents relative to mutual coupling effects is not clear. It is rather clear however that the fixed amplitude and phase characteristic of this procedure can be exploited in the design of feed networks.

If the scan angle is constrained to the order of ±50° or ±60° line element count remains quite reasonable, even at the lower sidelobe specification levels. The loss of projected aperture will, in most cases, limit the maximum angle of scan. It would appear that a vertical, orthogonal pair of such planar arrays could be used for full azimuthal coverage.

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On the State Description of Nonlinear Networks—Further Results

ROBERT W. NEWCOMB AND IZZET CEM GÖKNAR

Abstract—**Starting from specified types of element constitutive relations, necessary and sufficient conditions are given, which lead to validity tests, for the existence of state-variable equations in terms of capacitor charges and inductor fluxes or in terms of capacitor voltages and inductor currents as state variables. The relation between the equilibrium states and operating points is also discussed. These lead to resolution of problems previously posed.**

I. Introduction

Previously it has been shown in [1], through an example, that state-variable-like equations for a circuit in terms of capacitor voltages and inductor currents may yield solutions which do not satisfy the circuit's constraints and which are different than the solutions of the state-variable-like equations for the same circuit in terms of capacitor voltages and inductor flux linkages. Since all steps in deriving the two sets of equations are mathematically justifiable, this has raised the question, unanswered at that point, of how is an analyst to know whether equations on hand will yield proper results.

Here this question is answered. Specifically, necessary and sufficient conditions are given on the functions characterizing a network in order for resulting state-like equations to actually be state-variable equations for the network. The relation between the equilibrium states of the given circuit and the operating points of the algebraic subnetwork is also presented. These results, when applied to the example of [1], resolve the rather paradoxical situation.

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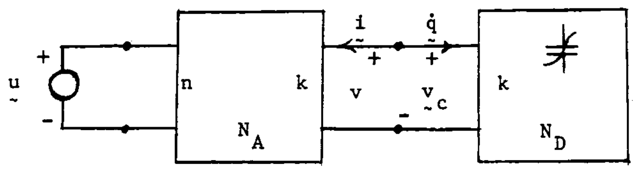

Fig. 1. Nonlinear network: interconnection of dynamical k-port with algebraic $(n+k)$-port.

II. Existence Conditions

Consider the nonlinear circuit of Fig. 1 consisting of the interconnection of a dynamic k-port N_D with an algebraic $(n+k)$-port N_A. It is assumed that N_A and N_D are respectively characterized by

$$i = f(v, u, t), \quad y = g(v, u, t) \tag{1a}$$

$$q = c(v_c, t) \tag{1b}$$

where f maps $\mathbf{R}^k \times \mathbf{R}^n \times \mathbf{R}_+$ into $\mathbf{R}^k$ and c maps $\mathbf{R}^k \times \mathbf{R}_+$ into $\mathbf{R}^k$, $\mathbf{R}^k$ being the real k-dimensional Euclidean space and $\mathbf{R}_+ \triangleq \{t \in \mathbf{R}: t \geq 0\}$. In (1), generally (u, y) and (v, i) are hybrid pairs of port voltages and currents, q the vector of charges and flux linkages and, v_c the vector of inductor currents and capacitor voltages. Without loss of generality, using transformations of current sources into voltage sources and of inductors into capacitors through the use of linear, time-invariant gyrators, it can be assumed that N_D consists only of capacitors and that N_A is driven by voltage sources [2]; hence u, v, u_c are assumed to be voltage vectors, i a current vector, q a charge vector and (1) result from the constitutive relations of the elements in the network. Using (1) with Kirchhoff's laws and mathematical relations the following chain of equalities is obtained:

$$\dot{q} \underset{\text{KCL}}{=} -i \underset{\text{(1a)}}{=} -f(v, u, t) \underset{\text{KVL}}{=} -f(v_c, u, t) \underset{\text{(1b)}}{=} -f(c^{-1}(q, t), u, t)$$

$$\underset{\substack{\text{chain}\\\text{rule}}}{=} \frac{\partial c(v_c, t)}{\partial v_c} \cdot \dot{v}_c + \frac{\partial c(v_c, t)}{\partial t} \tag{2}$$

where $\partial c/\partial v_c$ is the $k \times k$ Jacobian matrix, $c^{-1}(\cdot, t)$ is the functional inverse of $c(\cdot, t)$ for each t and $\dot{}$ is d/dt.

Separating the pertinent terms the following theorems can be stated.

Theorem 1:

$$\dot{q} = -f(c^{-1}(q, t), u, t) \tag{3}$$

is a state-variable equation for all initial $q_0 \in \mathbf{R}^k$ if and only if $c^{-1}(\cdot, \cdot)$ exists as a unique map of $\mathbf{R}^k \times \mathbf{R}_+$ into $\mathbf{R}^k$.

Theorem 2:

$$\dot{v}_c = -\left[\frac{\partial c(v_c, t)}{\partial v_c}\right]^{-1} \left\{ f(v_c, u, t) + \frac{\partial c(v_c, t)}{\partial t} \right\} \tag{4}$$

is a state-variable equation for all initial $v_{c_0} \in \mathbf{R}^k$ if and only if $(\partial c(\cdot, \cdot))/\partial v_c$ exists as a unique and everywhere nonsingular map of $\mathbf{R}^k \times \mathbf{R}_+$ into $\mathbf{R}^{k \times k}$ and $(\partial c(\cdot, \cdot))/\partial t$ as a unique map of $\mathbf{R}^k \times \mathbf{R}_+$ into $\mathbf{R}^k$.

For the time-autonomous case (4) becomes

$$\dot{v}_c = -\left[\frac{\partial c(v_c)}{\partial v_c}\right]^{-1} \{f(v_c, u)\} \tag{5}$$

and $(v_0, u_0) \in \mathbf{R}^k \times \mathbf{R}^n$ is called the *operating-point* of N_A if and only if $f(v_0, u_0) = 0$. The following result[1] discussed in the literature [3], can now be stated as a theorem.

Theorem 3:

Let $(\partial c(\cdot))/\partial v_c$ exist as a unique and everywhere nonsingular map of $\mathbf{R}^k$ into $\mathbf{R}^{k \times k}$. Then every $(v_0, u_0) \in \mathbf{R}^k \times \mathbf{R}^n$ is an equilibrium state of (5) if and only if it is an operating point of N_A.

[1]This result can be paraphrased from the literature as [3, p. 844]: "Let an algebraic network N_R be obtained by open circuiting all capacitors and short circuiting all inductors from a dynamic network N. Then every equilibrium state of N is an operating point of N_R."

Several comments are now in order.

1) Should the existence and properties of the above quantities hold at a point and in a neighborhood of that point then Theorems 1, 2, and 3 will hold at that point.

2) If the capacitor k-port is defined with the constitutive relation $v_c = \boldsymbol{\gamma}(q, t)$, state equations in terms of charge variables may still exist without $\boldsymbol{\gamma}(\cdot, \cdot)$ being a nonsingular map, i.e., without having $c(\cdot, \cdot)$ functionally defined.

3) Equations like (4) (or (5)) may still exist without having $(\partial c(\cdot, \cdot))/\partial v_c$ everywhere nonsingular; they are probably best not called state equations. In fact it was shown in [1] that

$$\dot{v}_c = -\boldsymbol{\Gamma}(v_c) f(v_c, u) \tag{6}$$

where

$$\boldsymbol{\Gamma}(v_c) \triangleq \left.\frac{\partial \boldsymbol{\gamma}(q)}{\partial q}\right|_{q = \boldsymbol{\gamma}^{-1}(v_c)}$$

was singular and that not every solution of (6) is a solution of the network. For example, at $(v_0, u_0) \in \mathbf{R}^k \times \mathbf{R}^n$ it may happen that $f(v_0, u_0) \neq 0$ is orthogonal to $\boldsymbol{\Gamma}(v_0)$, giving an equilibrium state of (6) and not an operating point of N_A, thus violating KCL as demonstrated in [1]. The nonsingularity of $(\partial c(\cdot, \cdot))/\partial v_c$ prevents this ambiguity and forces KCL to be satisfied through the chain of equalities (2).

III. Discussion

The necessary and sufficient conditions given above show that different criteria hold when using charge q on capacitors than when using voltage v_c on the same capacitors for the state, at least when N_D is voltage controlled. Consequently, it appears that in general one can not quote a preference (unless the dynamic k-port has a charge controlled constitutive relation) and each case must be checked on its own.

For example, the circuit in [1] has $i = v$, $q = c(v_c, t) = q_1 + (v_c - E)^{1/3}$ for which $v_c = c^{-1}(q, t) = (q - q_1)^3 + E$ has $c^{-1}(\cdot, \cdot)$ existing on all of $\mathbf{R} \times \mathbf{R}_+$ where $E \neq 0$. But $(\partial q)/\partial v_c = \frac{1}{3}(v_c - E)^{-2/3}$ does not exist at $v_c = E$, as a map into $\mathbf{R}^{1 \times 1}$, in which case the necessary and sufficient conditions of Theorem 2 are violated (but only at $v = v_c = E$).

On the other hand, the example $i = v$, $q = c(v_c, t) = (v_c - E)^2$ does not have $c^{-1}(\cdot, \cdot)$ uniquely existing on any point in $\mathbf{R} \times \mathbf{R}_+$ in which case (3) cannot be written, though (4) can be used everywhere except at $v_c = E$ where the matrix inverse of $(\partial c)/\partial v_c = 2(v_c - E)$ does not exist.

It should be pointed out that there are some quite interesting situations which this treatment brings out. For example, $i = v - E$ and $q = \frac{1}{2}(v_c - E)^2$ gives $(v_c - E)\dot{v}_c = -(v_c - E)$. If $v_{c_0} \geq E > 0$ this differential equation has multiple solutions, $v_c = E$ for all t and $v_c = -t + v_{c_0}$ which bifurcate one to the other at $t = v_{c_0} - E$. Hence $\dot{v}_c = -1$ is not an adequate state-variable description; in this case state equations in q or v_c do not exist.

In summary necessary and sufficient conditions on element constitutive relations have been given such that the relevant state-variable equations can "readily" be tested for validity of description of the network.

Acknowledgment

The authors wish to thank the reviewer for pointing out that a similar conclusion on the example circuit of [1] has recently appeared in [4, pp. 1060-1061] and that [4, p. 1069 eqs. (3.9), (3.8)] are contained in our Theorems 1 and 2, the latter also applying to time-variable capacitors while giving necessary and sufficient conditions.

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