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An Accelerated Convergence Method
(for DC Circuit Analysis) (?)

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internal notes

Abstract: an accelerated iteration is given for efficient fixed point solution of circuit equations

I. Introduction

In the analysis of nonlinear circuits and systems the solution of fixed point equations

$$x = F(x) \qquad (1)$$

is important. In particular, when $F(\cdot)$ is an algebraic function on the real numbers the bias condition of an electronic circuit can often be determined while if $F(\cdot)$ is a functional maps many dynamical systems can be considered. For example, on integrating the state variable equations

$$\dot{x} = f(t, x, u) \quad , \quad u = \text{input} \qquad (2)$$

we have

$$x = x(0) + \int_0^t f(\tau, x(\tau), u(\tau))\, d\tau = F(x) \qquad (3)$$

in which case $F(\cdot)$ is seen to be given through an integral operator on the function space of the state $x(\cdot)$.

If $F(\cdot)$ is a contraction mapping, for example when $f(t, \cdot, u)$ of (2) satisfies a Lipschitz condition [1, p. 82], then it is known that Picard's iteration can be used to find the fixed point, Picard's iteration being

$$x^{(k+1)} = F(x^{(k)}) \quad , \quad k = 0, 1, 2, \ldots \qquad (4)$$

where $x^{(k)}$ is the kth iterate with given $x^{(0)}$, the initial iterate [1, p. 73].

Although (4) will always yield a fixed point and the rate of convergence is known in terms of the contraction mapping constant, practically it is known that this rate of convergence is slower than necessary. For this reason various alternatives have been proposed [2]. Here we call attention to a very simple and efficient accelerated convergence algorithm.

II. The Method of Accelerated Convergence

We return to (4) and note that we desire

$$x^{(k+1)} = F(x^{(k+1)}) \tag{5a}$$

Consequently, expand $F(\cdot)$ in a Taylor series about $x^{(k)}$ and evaluate at $x^{(k+1)}$, using the appropriate type of derivative to suit the underlying space:

$$F(x^{(k+1)}) = F(x^{(k)}) + J(x^{(k)})(x^{(k+1)} - x^{(k)}) + \cdots \tag{5b}$$

where $J(\cdot)$ is the appropriate Jacobian (matrix).

Dropping the higher (unwritten) terms in (5b) and equating to (5a) we solve for $x^{(k+1)}$ to get

$$x^{(k+1)} = \left[I - J(x^{(k)})\right]^{-1} \left\{ F(x^{(k)}) - J(x^{(k)}) \cdot x^{(k)} \right\} \qquad (6)$$

where I is the identity (matrix). Equation (6) yields our <u>m</u>ethod of accelerated convergence with the $(k+1)$st iterate, $x^{(k+1)}$, given through (6) in terms of $x^{(k)}$. Examples show that the use of (6) can cut the iteration time to a third that of the Picard's method.

III. An Example

II. Discussion

In [3, pp. 120–122] an accelerated convergence technique is given through

$$x^{(k+1)} = \Theta(x^{(k)})\, F(x^{(k)}) - [I - \Theta(x^{(k)})]\cdot x^{(k)} \qquad (7)$$

which agrees with ours of (6) if

$$\Theta(x) = [I - J(x)]^{-1} \qquad (8a)$$

is chosen. However, [3] gives little insight as to why (7) is chosen; the reasoning of (5) & (6) gives clear motivation. Further, and importantly, rather than (8a), [3, p. 122] recommends in the n-dimensional case the choice

$$\Theta(x) = \text{diagonal}[\theta_1(x), \ldots, \theta_n(x)] \qquad (8b_1)$$

$$\theta_i(x) = 1/[1 - (\partial F_i(x)/\partial x_i)] \qquad (8b_2)$$

The choice of (8b) has some advantage in that only n derivatives are needed instead of the n^2 derivatives of (8a) in forming the Jacobian; further the simple scalar inversions of ($8b_2$) avoids the $n \times n$ matrix inversion of (8a).

In short the derivation of (6) gives motivation for (7) with the choice of (8b) yielding relatively simple manipulations for a method of accelerated convergence which may be somewhat speeded up by (8a) if the matrix inversion is convenient.

References

[1]. M. Vidyasagar, "Nonlinear Systems Analysis," Prentice-Hall, Inc., Englewood Cliffs, N.J., 1978.

[2]. A. J. Jimenez and S. W. Director, "New Families of Algorithms for Solving Nonlinear Circuit Equations," IEEE Transactions on Circuits and Systems, Vol. CAS-25, No. 1, January 1978, pp. 1-7.

[3] E. Isaacson and H. B. Keller, "Analysis of Numerical Methods," John Wiley & Sons, N.Y., 1966.