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A Nonlinear Impedance Converter

VACLAV DOLEZAL AND ROBERT W. NEWCOMB, FELLOW, IEEE

Abstract—**It is shown that under reasonable assumptions a common op amp circuit operating under nonlinear conditions possesses a nonlinear input admittance operator $Y(\cdot)$. By an appropriate choice of parameters a negative admittance may be obtained in the vicinity of the origin. Bounds on the behavior of $Y(\cdot)$ and its inverse are given.**

I. Introduction

Because of the commercial availability of operational amplifiers countless active circuits depend upon the op amp for their construction. Among these often appears the configuration of Fig. 1. For example, this configuration appears as a subcircuit in the integrator of Deboo [1], [2, p. 70], with more general two-port feedback in the inverse active networks of Rathore [3], and as the basic amplifier element, where the op amp internal pole is used for dynamics, in the inductor simulation of Anadamohan [4] and Sánchez [5], [6]. It also has been used for pulse generation in logarithm amplifiers [7] and a slight generalization, incorporating feedback amplification for insensitive active-R filters [8], as well as for high-order integrated circuits [9]. Beyond these linear uses the circuit generates the cubic-like negative resistance nonlinearity necessary for the design of the rings of coupled Van der Pol oscillators of Endo and Mori [10]. It is this latter use which has attracted our attention for general nonlinear analysis and design since the validity of the characterization of Endo and Mori [10, p. 15] had been verified analytically and experimentally by one of our students.

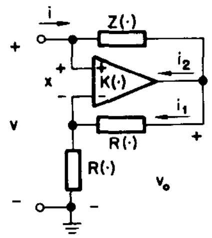

Fig. 1.

The static transfer characterization of practical op amps, for $v_0 = K(x)$ in Fig. 1, is known to be through a monotone (nondynamic) operator $K(\cdot)$, [1, especially p. 305]. Consequently, the ideas and techniques of monotone operator theory [12] are of interest. However, since feedback is used in Fig. 1 the resultant admittance operator $Y(\cdot)$ need not be monotone.

The circuit of Fig. 1 then serves as a practically important circuit which in its nonlinear regime also possesses properties of considerable theoretical interest and importance. Here we investigate some of these properties, showing in Theorem 1 that (under reasonable and practical hypotheses) the input admittance exists and for small inputs the circuit yields $v \approx -Zi$. Theorem 2 gives conditions on the range of input currents for which the input impedance operator is single valued.

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II. Technical Results

Let L be a real linear space.

Lemma 1. Let Z, R, K: $L \to L$ be (not necessarily linear) operators such that

(i) Z is invertible;

(ii) R is surjective;

(iii) $I+(1/2)K$ is invertible.

If Z, R, and K have a meaning of impedances and gain, respectively, then the one-port N in Fig. 1 has an admittance Y: $L \to L$, i.e., each voltage $v \in L$ applied to the terminals of N produces a unique current $i \in L$ given by

$$i = Yv \tag{1}$$

where

$$Y = Z^{-1}\left\{2\left(I+\tfrac{1}{2}K\right)^{-1} - I\right\}. \tag{2}$$

Proof: Choose $v \in L$ and define $i \in L$ by $i = Yv$ with Y being given by (2). Thus

$$Zi = 2\left(I+\tfrac{1}{2}K\right)^{-1}v - v. \tag{3}$$

Also, define $x \in L$ by

$$x = \left(I+\tfrac{1}{2}K\right)^{-1}v. \tag{4}$$

By assumption (ii), there exists $i_1 \in L$ such that

$$v - x = Ri_1. \tag{5}$$

Then, by (4),

$$x + \tfrac{1}{2}Kx = v. \tag{6}$$

Now, from (6) and (5) it follows that

$$Kx = 2Ri_1. \tag{7}$$

On the other hand, (3) can be written as

$$Zi = 2x - v. \tag{8}$$

Hence, by (5), $Zi = x + (x - v) = x - Ri_1$, i.e.,

$$x = Zi + Ri_1. \tag{9}$$

However, (5), (9), and (7) are Kirchhoff loop laws for N, i.e., i, i_1, i_2 are the currents forced by v.

Conversely, assuming that some $i, i_1, i_2 \in L$ satisfy Kirchhoff laws with the chosen $v \in L$, we conclude by reversing the argument that i is related to v by (1). Hence, the proof.

From now on, let $X \neq \varnothing$, and let L be a fixed linear subspace of the space of all *bounded* functions x: $X \to R^1$. Also, let

$$\|x\| = \sup\{|x(t)| : t \in X\} \tag{10}$$

for each $x \in L$.

In particular, for X we can take the interval $[a, b]$, $[0, \infty)$, or R^1. Correspondingly, L can be the space of all continuous functions on $[a, b]$, or the space of all bounded measurable functions on $[0, \infty)$, etc.

Also note that if we define as a linear operator R: $L \to L$ by

$$(Rx)(t) = rx(t) \tag{11}$$

with $r \neq 0$, then R is invertible and describes a linear resistor having resistance r.

We will need the following two lemmas whose proof is quite straightforward.

Lemma 2. Let χ: $R^1 \to R^1$ be a continuous, nondecreasing function. Let the operator K: $L \to L$ be defined by

$$(Kx)(t) = \chi(x(t)). \tag{12}$$

Then the operator $I + \frac{1}{2}K$: $L \to L$ is invertible, and

$$\left[\left(I+\tfrac{1}{2}K\right)^{-1}z\right](t) = \varphi^{-1}(z(t)) \tag{13}$$

where φ: $R^1 \to R^1$ is defined by

$$\varphi(\xi) = \xi + \tfrac{1}{2}\chi(\xi). \tag{14}$$

Lemma 3. Let χ: $R^1 \to R^1$ by a continuous, nondecreasing function. Assume that there exist constants $a > 0$ and $\kappa > -2$ such that

$$\kappa\xi^2 \leqslant \xi\chi(\xi) \tag{15}$$

for every $\xi \in [-a, a]$. If φ: $R^1 \to R^1$ is defined by (14), then

$$\eta\varphi^{-1}(\eta) \leqslant \frac{2}{2+\kappa}\eta^2 \tag{16}$$

for every $\eta \in [-b, b]$, where $b = (1+\kappa/2)a$.

Now we can prove a theorem on simulating the negative of an impedance.

Theorem 1. Assume that the following conditions are satisfied:

(i) The operator Z: $L \to L$ is invertible and

$$\|Z^{-1}x_1 - Z^{-1}x_2\| \leqslant \mu\|x_1 - x_2\| \tag{17}$$

for all $x_1, x_2 \in L$ and some fixed $\mu > 0$;

(ii) The operator R: $L \to L$ is surjective;

(iii) The function χ: $R^1 \to R^1$ is continuous, nondecreasing, satisfies the inequality (15) for all $\xi \in [-a, a]$ with some fixed $a > 0$, $\kappa > -2$, and the operator K: $L \to L$ is defined by (12).

Then the one-port N in Fig. 1 possesses an admittance Y, and we have

$$\|Yv - Z^{-1}(-v)\| \leqslant \frac{4\mu}{2+\kappa}\|v\| \tag{18}$$

for each $v \in L$ with $\|v\| \leqslant (1+\kappa/2)a$.

Proof: By Lemma 2, $I + \frac{1}{2}K$ is invertible, and consequently, the admittance Y of N exists by Lemma 1.

Next, choose $v \in L$ with $\|v\| \leqslant b = (1+\kappa/2)a$. Then we have by (2), (17),

$$\begin{aligned}\lambda &= \|Yv - Z^{-1}(-v)\| \\ &= \left\|Z^{-1}\left\{2\left(I+\tfrac{1}{2}K\right)^{-1} - I\right\}v - Z^{-1}(-v)\right\| \\ &\leqslant 2\mu\left\|\left(I+\tfrac{1}{2}K\right)^{-1}v\right\|.\end{aligned} \tag{19}$$

Choosing $t \in X$, we have $|v(t)| \leqslant \|v\| \leqslant b$, and by (13) and (16) in Lemma 3,

$$\begin{aligned}\left|\left[\left(I+\tfrac{1}{2}K\right)^{-1}v\right](t)\right| &= |\varphi^{-1}(v(t))| \\ &\leqslant \frac{2}{2+\kappa}|v(t)| \leqslant \frac{2}{2+\kappa}\|v\|.\end{aligned}$$

Hence,

$$\|(I+\tfrac{1}{2}K)^{-1}v\| \leqslant \frac{2}{2+\kappa}\|v\|. \tag{20}$$

Introducing (20) into (19), (18) follows.

By virtue of (18), our one-port N simulates the impedance $-Z$ for all voltages whose norm does not exceed $(1+\kappa/2)a$. Indeed, if κ is large, then Yv approximates $Z^{-1}(-v)$. However, the operator $Y_0=Z^{-1}(-I)$ has a meaning of an admittance, and it is invertible. Hence, the corresponding impedance is $Z_0=Y_0^{-1}=(-I)Z=-Z$.

Under the assumptions of Theorem 1, our one-port N is voltage-controlled. However, if the assumptions on χ are slightly strengthened, then N becomes also current-controlled in a certain neighborhood of the origin. To show this, we will need the following result:

Lemma 4. Let χ: $R^1 \to R^1$ be a continuous, nondecreasing function with $\chi(0)=0$. Assume that there exist constants $a>0$ and $\kappa>2$ such that

$$\kappa(\xi_2-\xi_1)^2 \leqslant (\xi_2-\xi_1)(\chi(\xi_2)-\chi(\xi_1)) \tag{21}$$

for all $\xi_1, \xi_2 \in [-a,a]$. If φ: $R^1 \to R^1$ is defined by (14), then

$$0 \leqslant (\eta_2-\eta_1)(\varphi^{-1}(\eta_2)-\varphi^{-1}(\eta_1)) \leqslant \frac{2}{2+\kappa}(\eta_2-\eta_1)^2 \tag{22}$$

for all $\eta_1, \eta_2 \in [-b,b]$ with $b=(1+\kappa/2)a$.

Moreover, we have

(a) Let ψ: $R^1 \to R^1$ be defined by

$$\psi(\eta)=2\varphi^{-1}(\eta)-\eta. \tag{23}$$

Then

$$-(\eta_2-\eta_1)^2 \leqslant (\eta_2-\eta_1)(\psi(\eta_2)-\psi(\eta_1)) \leqslant \frac{2-\kappa}{2+\kappa}(\eta_2-\eta_1)^2 \tag{24}$$

for all $\eta_1, \eta_2 \in [-b,b]$, and ψ is 1-to-1 on $[-b,b]$.

(b) Let $c=(\kappa/2-1)a$, and let σ: $[-c,c] \to [-b,b]$ be defined by: If $\xi \in [-c,c]$, let $\sigma(\xi)=\eta$, where $\psi(\eta)=\xi$. Then $\psi(\sigma(\xi))=\xi$ for each $\xi \in [-c,c]$,

$$\frac{2+\kappa}{2-\kappa}(\xi_2-\xi_1)^2 \leqslant (\xi_2-\xi_1)(\sigma(\xi_2)-\sigma(\xi_1)) \leqslant -(\xi_1-\xi_2)^2 \tag{25}$$

for all $\xi_1, \xi_2 \in [-c,c]$, and

$$|\sigma(\xi)+\xi| \leqslant \frac{4}{\kappa-2}|\xi| \tag{26}$$

for each $\xi \in [-c,c]$.

(The proof is a simple exercise with inequalities and is omitted.)

To state the theorem, we introduce the following notation: If $d>0$, let the ball of radius d be defined as

$$\mathfrak{B}_d=\{x: x\in L, \|x\| \leqslant d\}. \tag{27}$$

Theorem 2. Assume that the following conditions are satisfied:

(i) The operator Z: $L \to L$ is invertible and

$$\|Zx\| \leqslant \nu\|x\| \tag{28}$$

for all $x \in L$ with some fixed $\nu>0$;

(ii) The operator R: $L \to L$ is surjective;

(iii) The function χ: $R^1 \to R^1$ is continuous, nondecreasing with $\chi(0)=0$ and satisfies the inequality (21) for all $\xi_1, \xi_2 \in [-a,a]$ with some fixed $a>0$, $\kappa>2$, and the operator K: $L \to L$ is defined by (12). Then the one-port N in Fig. 1 possesses an impedance W: $\mathfrak{B}_r \to \mathfrak{B}_b$ with $r=a(\kappa-2)/2\nu$ and $b=(1+\kappa/2)a$, i.e., each current $i \in \mathfrak{B}_r$ forced to the terminals of N produces a unique voltage v in $\mathfrak{B}_b$ and $v=Wi$. Moreover,

$$\|Wi+Zi\| \leqslant \frac{4\nu}{\kappa-2}\|i\| \tag{29}$$

for every $i \in \mathfrak{B}_r$.

Proof: By Lemma 2 and 1, N possesses an admittance Y: $L \to L$ given by (2). Choose $i \in \mathfrak{B}_r$. Using the notation of Lemma 4, we have $\|i\| \leqslant r=a(\kappa-2)/2\nu=c/\nu$, so that, by (28), $\|Zi\| \leqslant c$.

Next, define $v \in L$ by

$$v(t)=\sigma[(Zi)(t)]. \tag{30}$$

Invoking (25), it follows that

$$|v(t)| \leqslant \frac{\kappa+2}{\kappa-2}|(Zi)(t)| \leqslant \frac{\kappa+2}{\kappa-2}\|Zi\| \leqslant \frac{\kappa+2}{\kappa-2}c=\frac{\kappa+2}{2}a=b.$$

Thus $v \in \mathfrak{B}_b$.

On the other hand, if voltage v is applied to N, it forces a current $i' \in L$, and, by virtue of (1), (2), $i'=Z^{-1}\{2(I+\tfrac{1}{2}K)^{-1}-I\}v$. However, by Lemma 4,

$$\left[\left\{2(I+\tfrac{1}{2}K)^{-1}-I\right\}v\right](t)=\psi(v(t)) =\psi(\sigma[(Zi)(t)])=(Zi)(t)$$

for each $t \in X$. Hence, $i'=i$, i.e., the operator W: $\mathfrak{B}_r \to \mathfrak{B}_b$ defined by

$$(Wi)(t)=\sigma[(Zi)(t)] \tag{31}$$

is the impedance of N.

Moreover, invoking (26) in Lemma 4, we have for any $t \in X$,

$$|(Wi+Zi)(t)|=|\sigma[(Zi)(t)]+(Zi)(t)| \leqslant \frac{4}{\kappa-2}|(Zi)(t)| \leqslant \frac{4}{\kappa-2}\|Zi\| \leqslant \frac{4\nu}{\kappa-2}\|i\|.$$

Thus (29) follows and the proof is complete.

The inequality (29) clearly shows that, for κ large, the impedance W simulates $-Z$.

III. Discussion

For reasonable assumptions on the components in the circuit we have shown various properties of the op amp circuit of Fig. 1. This circuit acts as a nonlinear impedance converter, giving an input impedance approximating

$-Z(\cdot)$ where $Z(\cdot)$ is the "positive feedback" impedance operator in Fig. 1. Bounds on the closeness of this approximation under the assumptions of Theorem 1 (which requires a large enough gain near the origin by (15)) are given in (18) in terms of the input admittance and the applied voltage. The dependence upon the input voltage is such that larger deviations may occur for larger voltages, as one might physically expect when the op amp saturates. In contrast, and under more general assumptions, (2) gives the input impedance operator exactly as $Z_{\text{in}}(\cdot)=(2[I+\frac{1}{2}K]^{-1}-I)(Z(\cdot))$ which shows the correction factor, $2[I+\frac{1}{2}K]^{-1}(Z(\cdot))$, to $-Z(\cdot)$ due to finite and nonlinear gains $K(\cdot)$. In the case that the feedback impedance operator is bounded linearly via (28), and the gain again increases fast enough near the origin, via (21), then a linear current bound, (29), can be given near the origin on $Z_{\text{in}}=W$ as it approximates $-Z$.

Finally, let us make the following comments:

(A) The assumption on invertibility of Z made in Theorem 2 can be dropped. To see this, we merely use a lemma similar to Lemma 1 which concerns the impedance rather than the admittance of our circuit.

(B) Referring to (30), it is apparent that for a given i the corresponding voltage v can be established graphically provided the op amp characteristic χ is obtained by a measurement. The same applies to finding i for a given v. Since the construction is quite straightforward, we omit the details.

(C) In estimates (18) and (29) we have used the uniform norm, because it is most appropriate in our setting. However, it is easy to see that similar estimates like (18), (29) hold if the L_2-norm is used. Then, of course, assumption $\|v\| \leqslant (1+\kappa/2)a$ and $\|i\| \leqslant a(\kappa-2)/2\nu$ has to be replaced by the requirement that $|v(t)| \leqslant (1+\kappa/2)a$ and $|i(t)| \leqslant a(\kappa-2)/2\nu$ for all $t \in X$, respectively.

(D) In Theorem 1 assumption (ii), R is surjective, may be dropped and the voltage divider action of the two $R(\cdot)$'s used. In this case, however, the current i, becomes a meaningless quantity.

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