STABILITY PROPERTIES OF A CLASS OF LARGE SCALE NEURAL NETWORKS

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ABSTRACT

Asymptotic stability in the large for a class of large scale neural networks having the topology of the cerebellum is established. The method used in establishing these results obtains bounds on the solutions of two sub-networks in the form of functions of t which tend exponentially to 0 as t tends to infinity.

1. INTRODUCTION

In [1] a generalized way of describing large scale neural networks was derived and the existence and uniqueness of solutions was proven for a class of such networks. In this work we will concentrate on the stability properties of a subclass of these neural networks, those which are topologically similar to the cerebellum.

2. DESCRIPTION OF THE NETWORK

The neural network under question is shown in Fig. 2.1 where π_i has n_i neurons, i=1,2,3,4 and which corresponds in the case of the cerebellum to Purkinje cells, basket cells, Golgi cells and granule cells, respectively. As discussed in [1], this neural network is described by the system of differential equations

$$T\dot{X} + X = W\cdot\Phi[X]\cdot X + V\cdot e \qquad 2.1a$$

with $X = [X_1, X_2, X_3, X_4]^T = [x_1, \ldots, x_N]^T$,

$N = n_1 + n_2 + n_3 + n_4$

$$W = \begin{bmatrix} 0 & -W_{12} & 0 & W_{14} \\ 0 & 0 & 0 & W_{24} \\ 0 & 0 & 0 & W_{34} \\ 0 & 0 & -W_{43} & 0 \end{bmatrix} \qquad V = \begin{bmatrix} V_{11} & 0 \\ V_{21} & 0 \\ V_{31} & V_{32} \\ 0 & V_{42} \end{bmatrix} \qquad 2.1b$$

and where $\Phi[X] = \operatorname{diag}_{1\le j\le N} [\phi[x_j]]$

and $\phi[x_j] = \begin{cases} 0 & \text{if } x_j \le 0 \\ 1 & \text{if } x_j > 0 \end{cases}$

W_{ij} and V_{ij} are matrices with non-negative entries, and e is the external input to the system, to be taken zero in the following. That is, we will examine the behavior of the autonomous system 2.2 below, which – through the partitioning of 2.1b – is broken down into two interconnected sub-systems given by 2.3 and 2.4.

$$T\dot{X} + X = W\cdot\Phi[X]\cdot X \qquad 2.2$$

$$T_1\dot{X}_1 + X_1 = -W_{12}\Phi[X_2]\cdot X_2 + W_{14}\Phi[X_4]\cdot X_4 \qquad 2.3a$$

$$T_2\dot{X}_2 + X_2 = W_{24}\Phi[X_4]\cdot X_4 \qquad 2.3b$$

$$T_3\dot{X}_3 + X_3 = W_{34}\Phi[X_4]\cdot X_4 \qquad 2.4a$$

$$T_4\dot{X}_4 + X_4 = -W_{43}\Phi[X_3]\cdot X_3 \qquad 2.4b$$

In the following we will examine the stability properties of generalizations of the systems 2.3 and 2.4 and afterwards the stability properties of the interconnected system 2.2.

3. APPROACH

Our approach to the stability question is to find bounds to the solutions of the differential equations in 2.3 and 2.4 and examine the behavior of these bounds with time.

As it turns out, the solutions of both the subsystems 2.3 and 2.4 are bounded by functions of the form $e^{-At}\cdot P(t)$ where P(t) is a vector polynomial in t and A is a positive diagonal matrix, hence asymptotic stability in the large will be established.

4. ANALYTIC APPROACH TO THE STABILITY QUESTION

<u>Definition 1</u>. For every $(X,Y) \in \mathbb{R}^N \times \mathbb{R}^N$ $X \le Y$ is defined by $x_i \le y_i$ $\forall$ i=1,2,...N. This allows the concept of monotonicity of vectors for use in defining the neuro-mime functions.

<u>Definition 2</u>. The class N (the neuro-mime functions) is defined as follows.

$$N = \{f \mid f: \mathbb{R}^N \to \mathbb{R}^M_+,\ f \text{ continuous},\ f \text{ monotonically non-decreasing},\ f(0) = 0\}$$

For example the functions $W_{ij}\Phi[X_j]\cdot X_j$ in 2.3 and 2.4 are all neuro-mime functions. Note that neuro-mime functions are zero for negative arguments.

<u>Definition 3</u>. Given a vector polynomial

$P(t) = \sum_{i=1}^{N} \underline{a}_i t^{i-1}$ we define

$$|P(t)| = \sum_{i=1}^{N} |\underline{a}_i| t^{i-1} \text{ where}$$

$$|X| = \begin{bmatrix} |x_1| \\ |x_2| \\ \vdots \\ |x_N| \end{bmatrix}$$

<u>Lemma 1</u>. For every function $f \in N$, every vector polynomial P(t), and all positive diagonal matrices A and B, there exists a vector polynomial $\hat{P}(t)$ and a positive diagonal matrix $\hat{B}$ so that the following is true.

$$0 \le e^{-At}\int_0^t e^{As} f[e^{-Bs}\cdot P(s)]\, ds \le e^{-Bt}\,\hat{P}(t) \quad \forall\, t \ge 0$$

Proof: For every connected bounded subset $\mathcal{L}$ of E_+^N that includes the origin and every function $f \in N$ there exists a positive number H such that

$$0 \le f(X) \le \begin{cases} 0 & \text{if} \quad X \in \mathcal{L}_-^N \\ H\cdot X & \text{if} \quad X \in \mathcal{L} \\ f(X) & \text{if} \quad X \in \mathcal{L}_+^N \cap \mathcal{L} \end{cases} \qquad \text{£1.1}$$

Then designating

$$I(t) = e^{-At}\int_0^t e^{As} f(e^{-Bs}P(s))\,ds \qquad \text{£1.2}$$

we have for $t \ge 0$

$$0 \le I(t) \le e^{-At}\int_0^t e^{As} f(e^{-Bs}|P(s)|)\,ds \qquad \text{£1.3}$$

By choosing $T_0 \ge 0$ so that $e^{-Bt}|P(t)| \in \mathcal{L}\ \forall\, t \ge T_0$, the integral in £1.3 is broken down as follows

$$I(t) \le e^{-At}\Big[\int_0^{T_0} e^{As} f(e^{-Bs}|P(s)|)\,ds + \int_{T_0}^t e^{As} f(e^{-Bs}|P(s)|)\,ds\Big] \qquad \text{£1.4}$$

The first integral inside the brackets is clearly bounded by a positive vector P_0 and taking into account £1.1 we can write

$$0 \le I(t) \le e^{-At}[P_0 + \int_{T_0}^t H\cdot e^{As}\cdot e^{-Bs}\cdot |P(s)|\,ds] \quad \text{or}$$

$$0 \le I(t) \le e^{-At}[P_0 + e^{(A-B)s}\,\bar{P}(s) \Big|_{T_0}^t] \qquad \text{£1.5}$$

where $\bar{P}(s)$ is a vector polynomial. We can subsequently write for £1.5

$$0 \le I(t) \le e^{-At}[P_0 - e^{(A-B)T_0}\,\bar{P}(T_0)] + e^{-Bt}\bar{P}(t) \quad \text{or}$$

$$0 \le I(t) \le e^{-At}|P_0 - e^{(A-B)T_0}\,\bar{P}(T_0)| + e^{-Bt}|\bar{P}(t)| \qquad \text{£1.6}$$

Choosing now $\hat{B} = \min[A,B]$ with the meaning $\hat{B}_{ij} = \min[A_{ij}, B_{ij}]$ we have

$$0 \le I(t) \le e^{-\hat{B}t}\{|P_0 - e^{(A-B)T_0}\,\bar{P}(T_0)| + |\bar{P}(t)|\}$$

or

$$0 \le I(t) \le e^{-\hat{B}t}\,\hat{P}(t)$$

Q.E.D.

Lemma 2. The system

$$T_1\dot{X}_1 + X_1 = f_1(X_2)$$
$$T_2\dot{X}_2 + X_2 = -f_2(X_1) \qquad \text{£2.1}$$

where T_1, T_2 are diagonal positive matrices and f_1, $f_2 \in N$ is asymptotically stable in the large. Furthermore, the solutions $X_1(t;X_{10})$ and $X_2(t;X_{20})$ are upper and lower bounded by functions of the form $e^{-At}P(t)$, A being a positive diagonal matrix and P(t) a vector polynomial in t.

Proof. From £2.1 we have [2,p.343]

$$X_1 = e^{-T_1^{-1}t}\,X_{10} + e^{-T_1^{-1}t}\int_0^t e^{T_1^{-1}s}\,T_1^{-1} f_1[X_2(s)]\,ds \qquad \text{£2.2a}$$

$$X_2 = e^{-T_2^{-1}t}\,X_{20} - e^{-T_2^{-1}t}\int_0^t e^{T_2^{-1}}\,T_2^{-1}\, f_2[X_1(s)]\,ds \qquad \text{£2.2b}$$

from which, due to the nature of the functions f_1 and f_2, we have

$$X_1 \ge e^{-T_1^{-1}t}\,X_{10} \qquad \text{£2.3a}$$

$$X_2 \le e^{-T_2^{-1}t}\,X_{20} \qquad \text{£2.3b}$$

Using £2.2a and £2.3b and having in mind that f_1 is a non-decreasing function we have

$$X_1 \le e^{-T_1^{-1}t}\,X_{10} + e^{-T_1^{-1}t}\int_0^t e^{T_1^{-1}s}\,T_1^{-1} f_1[e^{-T_2^{-1}s}X_{20}]\,ds \qquad \text{£2.4}$$

Applying lemma 1 in £2.4 we have

$$X_1 \le e^{-T_1^{-1}t}\,X_{10} + e^{-\tilde{T}t}\,\tilde{P}_1(t) \qquad \text{£2.5}$$

where

$$\tilde{T} = \min\,[T_1^{-1}, T_2^{-1}] \qquad \text{£2.6}$$

or

$$X_1 \le e^{-Tt}\{|X_{10}| + |P(t)|\} = e^{-Tt}\,P_1(t) \qquad \text{£2.7}$$

with $T = \min\,[\tilde{T}, T_1^{-1}]$

Using £2.7 and £2.2b we have

$$X_2 \ge e^{-T_2^{-1}}X_{20} - e^{-T_2^{-1}t}\int_0^t e^{T_2^{-1}}\, f_2[e^{-Ts}P_1(s)] \qquad \text{£2.8}$$

Applying lemma 1 in £2.8 we have

$$X_2 \ge e^{-T_2^{-1}t}\,X_{20} - e^{-\tilde{T}t}\,\tilde{P}_2(t) \quad \text{or}$$

$$X_2 \ge -e^{-Tt}\{|X_{20}| + |\tilde{P}_2(t)|\} = e^{-Tt}\,P_2(t) \qquad \text{£2.9}$$

The bounds on X_2, for example, are these in £2.9 and £2.3b

Q.E.D.

Lemma 3. The system

$$T_1\dot{X}_1 + X_1 = f_1(X_3) \qquad \text{£3.1a}$$
$$T_2\dot{X}_2 + X_2 = -f_2(X_2) + f_3(X_3) \qquad \text{£3.1b}$$

where f_1, f_2, $f_3 \in N$ and

$$e^{-Tt}P(t) \le X_3 \le e^{-Tt}\dot{P}(t) \qquad £3.1c$$

is asymptotically stable in the large

Proof. From £3.1a we have

$$X_1 = e^{-T_1^{-1}t}X_{10} + e^{-T_1^{-1}t}\int_0^t e^{T_1^{-1}s}\,T_1^{-1}f_1(X_3(s))ds \qquad £3.2$$

or

$$X_1 \ge e^{-T_1^{-1}t} \qquad £3.3$$

Using £3.1c, £3.2, lemma 1, and proceeding as in lemma 2, we have

$$X_1 \le e^{-Tt}P(t) \qquad £3.4$$

where $T = \min[\dot{T}, T_1^{-1}]$ £3.5

From £3.1b we have

$$X_2 = e^{-T_2^{-1}t}X_{20} - e^{-T_2^{-1}t}\int_0^t e^{T_2^{-1}s}T_2^{-1}f_2(X_1(s))ds$$

$$+ e^{-T_2^{-1}t}\int_0^t e^{T_2^{-1}}\,T_2^{-1}\,f(X_3(s))ds \qquad £3.6$$

which - keeping in mind that f_2, f_3 are non-negative - yields

$$e^{-T_2^{-1}t}X_{20} - e^{-T_2^{-1}t}\int_0^t e^{T_2^{-1}}f_2(X_1(s))ds \le X_2 \le$$

$$e^{-T_2^{-1}t}X_{20} + e^{-T_2^{-1}}\int_0^t e^{T_2^{-1}s}T_2^{-1}f_3(X_3(s))ds \qquad £3.7$$

from which, using £3.4, £3.1c, and lemma 1, and proceeding as in lemma 2, we have

$$e^{-T_2^{-1}}X_{20} - e^{-T_*t}P_*(t) \le X_2 \le e^{-T_2^{-1}t}X_{20} + e^{-T^*t}P^*(t) \qquad £3.8$$

where $T_* = \min[T_2^{-1}, T]$

and $T^* = \min[T_2^{-1}, \dot{T}]$

Theorem: System 2.2 is asymptotically stable in the large.

Proof: The functions $W_{ij}\phi[X_j]\cdot X_j$ belong to class N.

Moreover, system 2.2 can be broken down to systems 2.3 and 2.4 which in turn are asymptotically stable in the large by lemmas 2 and 3.

5. DISCUSSION

Neural networks are of interest and importance because of their ability to describe biological type systems. Practically they are becoming significant because of the recent simple electronic realizations of neuristors suitable for VLSI technology [3]. In many such neural networks the neurons can be simply modeled individually by a first order, single time-constant, differential equation with step function nonlinearity, $T_i\dot{x}_i + x_i - \sum_j w_{ij}\phi[x_j]\cdot x_j = \sum_k v_{ik}e_k$, the Morshita neuron. Here we have shown that interconnections of these neurons with weightings in the scheme of Fig. 2.1 leads to an asymptotically stable, in the large, network.

Beyond that we have shown that any system that has the topology of the cerebellum, as described in Fig. 2.1, is asymptotically stable in the large provided that the individual elements in the network are described by the following differential equation $T_i\dot{x}_i + x_i + f_i(x_j)$ where $f_i \in N$.

The result is achieved through the means by which the sub-system of 2.3 is fed by the "self-interacting" system of 2.4 with the connection, shown in Fig. 2.1, described by the W matrix where the weight matrices W_{ij} have only nonnegative entries, and only positive signals get so weighted because of the unit step functions $\phi[\cdot]$. Physically there are two types of neurons, those whose outputs are excitatory or inhibitory. These we take into account by the sign on the W_{ij}; thus in Fig. 2.1 η_4 is a class of excitatory neurons while η_2 and η_3 are classes of inhibitory ones. Here we have taken the time constant matrices as diagonal, meaning that each neuron is represented by a first order time-invariant system with their interactions completely determined by the topology of the structure, through W. It does appear that non-diagonal T are possible but for this the lemmas need strengthening. In any event for present models of the cerebellum diagonal T suffice and we see that then any signal present in the cerebellar neural network will eventually decay to zero in the absence of restoring inputs.

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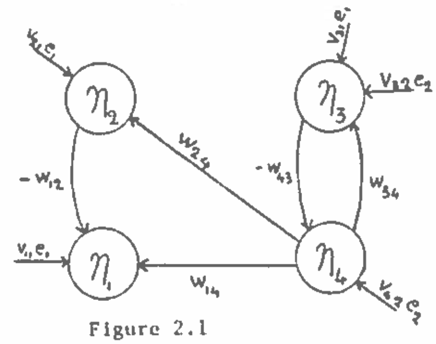

Cerebellum Topology

Figure 2.1

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