Thus

$$\sum_s b_s b_{-s} \epsilon^{-is} = \begin{cases} N & (s \equiv 0 \pmod N) \\ 0 & (s \not\equiv 0 \pmod N). \end{cases}$$

It follows that

$$(3) \qquad b_s b_{-s} = 1 \qquad \text{(for all } s).$$

Since a_r is real, (2) implies $b_{-r} = \sum_s a_s \epsilon^{rs} = \bar{b}_r$, so that (3) becomes $|b_s| = 1$ (for all s).

Therefore all real sequences satisfying the several hypotheses are of the form (1), where the coefficients b_s are complex numbers of absolute value 1 such that $b_{-s} = \bar{b}_s$.

Alternately we may put the general solution in the following form. If N is odd then

$$a_r = \frac{u_0}{N} + \frac{2}{N} \sum_{s=1}^{\frac{1}{2}(n-1)} \left(u_s \cos \frac{2rs\pi}{N} + v_s \sin \frac{2rs\pi}{N} \right),$$

where u_s, v_s are real and $u_s^2 + v_s^2 = 1$. If N is even there is an additional term

$$\frac{2}{N} u_n \cos r\pi \qquad (u_n = \pm 1, \ N = 2n).$$

Also solved by the proposers.

Term by Term Differentiation without Uniform Convergence

4994 [1961, 934]. *Proposed by Peter Ungar, New York University*

For $a < x < b$, let

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \lim_{n\to\infty} f_n'(x) = \phi(x).$$

Prove. If all the functions named above and also $f'(x)$ are continuous, then $f'(x) = \phi(x)$.

Solution by R. D. McWilliams, Florida State University. Let $[c, d] \subset (a, b)$, and for each positive integer n let H_n be the set of all $t \in [c, d]$ such that $|f_m'(t) - \phi(t)| < 1$ for all $m \geqq n$. Since $[c, d]$ is a set of the second category in itself, some H_n must be dense in a subinterval $[p, q]$ of $[c, d]$, and hence $\{f_n'\}$ is uniformly bounded on $[p, q]$. By the Lebesgue bounded convergence theorem, for each $x \in (p, q)$,

$$\int_p^x \phi(t) dt = \lim_{n\to\infty} \int_p^x f_n'(t) dt = f(x) - f(p),$$

and hence $f'(x) = \phi(x)$. Since $[c, d]$ is arbitrary, it follows that $f'(x) = \phi(x)$ for all x in a dense subset of (a, b) and hence, by continuity, for all $x \in (a, b)$.

Also solved by Robert Breusch, J. Czipszer, N. G. de Bruijn, David Greenstein, J. B. Linder, Norman Meyers, R. W. Newcomb, D. J. Newman, G. B. Parrish, Barbara L. Osofsky, W. C. Waterhouse, and the proposer.

Editorial Note. The theorem as usually given carries the additional hypothesis that f' be uniformly convergent. See, *e.g.*, A. Taylor, *Advanced Calculus*, Theorem 5, p. 601.

The proof given above is an illustration of Osgood's theorem, see Riesz and Nagy, *Functional Analysis*, New York 1955, p. 63.

Newcomb remarks that there is a similar theorem regarding distributions, given as Theorem IV, p. 20 of Mikusiński and Sikorski, *The Elementary Theory of Distributions* (I), Rozprawy Matematyczne XII, Warsaw, 1957.

Euler's Function

4995 [1961, 1010]. *Proposed by Oystein Ore, Yale University*

When $\phi(x)$ denotes Euler's function it is readily verified that $n=14$ is the smallest even number such that the equation $\phi(x)=n$ has no solution. Prove that for each exponent α there is a smallest odd integer k_α such that the equation $\phi(x)=2^\alpha k_\alpha$ has no solution. Determine k_2, k_3, k_4. Try to find bounds for k_α.

I. *Solution by J. L. Selfridge, University of Washington and University of California at Los Angeles.* If $k\cdot 2^n+1$ is prime then $\phi(x)=2^\alpha k$ has solutions for any $\alpha\geqq n$. If 2^s+1 is prime then $\phi(x)=2^\alpha(2^s+1)$ has solutions for any $\alpha\geqq s$. If p is prime and $p\cdot 2^n+1$ is composite for every $n\leqq\alpha$, and $p\neq 2^s+1$ for any $s\leqq\alpha$, then $\phi(x)=2^\alpha p$ has no solution. These statements are easy to verify. (See Solution II.)

It is known (Sierpiński, *Elem. Math.* 15 (1960) p. 73) that if k belongs to certain arithmetic progressions then any term of the sequence $k+1, 2k+1, \cdots, 2^nk+1, \cdots$ is divisible by one of a set of 6 or 7 fixed primes. For example, if p is prime, $p\equiv 1 \pmod{(2^{32}-1)\cdot 641}$, and $p\equiv -1 \pmod{(2^{32}+1)/641}$, then $p\cdot 2^n+1$ is composite for every n, so then $\phi(x)=2^\alpha p$ has no solution for any α, and $k_\alpha\leqq p$ for all α. For the prime 271129, the sequence $271129\cdot 2^n+1$ has the covering set 3, 5, 7, 13, 17, 241. Thus $k_\alpha\leqq 271129$ for all α.

Armed with the statements above, a glance at Robinson's table (*Proc. Amer. Math. Soc.* 9(1958), p. 674) shows that $k_0=3$, $k_1=7$, $k_2=17$, $k_3=k_4=k_5=19$, $k_6=k_7=31$, and $k_\alpha=47$ for $8\leqq\alpha<512$. Computations on the SWAC show that $47\cdot 2^n+1$ is composite for $n<583$, that $47\cdot 2^{583}+1$ is prime, that for each odd $k\neq 257$ such that $47<k<383$ there is a prime $k\cdot 2^n+1$ with $n\leqq 66$, and that $383\cdot 2^n+1$ is composite for every $n<2313$. Hence $k_\alpha=47$ for $8\leqq\alpha<583$ and $k_\alpha=383$ for $583\leqq\alpha<2313$.

II. *Solution by P. T. Bateman, University of Illinois and University of Pennsylvania.* We begin with several lemmas.

LEMMA 1. *Suppose α is a given positive integer. Then there are infinitely many primes p such that $2p+1, 2^2p+1, \cdots, 2^\alpha p+1$ are all composite.*