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A coupled system for subharmonics of any order

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A set of two coupled differential equations with square law nonlinearities is shown to have dominant and stable subharmonic solutions. Mathematical expressions characterizing the subharmonic solutions and their regions of stability are obtained. It is further shown that under appropriate choice of system parameters a resulting system described through these coupled differential equations possesses an exact stable subharmonic of any real order. From this, a design theory is obtained for a system which yields an arbitrary dominant subharmonic. The theory is also directly applicable to the creation of arbitrary stable harmonics.

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INTRODUCTION

The occurrence in nonlinear systems of subharmonics at frequencies of m/n times the frequency of the driving source, where m and n are both integers, has been much analyzed.[1] Usually the nonlinear systems are described by a single differential equation of first and higher order and only approximate solutions are obtained. However, a variety of recent observations in acoustics,[2,3] and electrical systems[4,5] indicate that complex nonlinear systems may exactly produce subharmonics of a different kind. In this regard, Eller has analyzed a system of two coupled nonlinear oscillators.[4] His result predicts, in an approximate form, the existence of a one-half subharmonic or a pair of fractional harmonic components, the sum of whose frequencies is the frequency of the driving source.

The present work deals with the problems of fractional harmonics from a different point of view. Our attention is directed towards the design of a system which will generate an exact dominant subharmonic solution of any real order, possibly even nonrational. A dominant subharmonic maintains a large amplitude of its components at the subharmonic frequency compared to the amplitude of all other components in the Fourier expansion.

The analysis of the nonlinear systems considered here proceeds in several steps. First, we find a subharmonic solution that satisfies the equations of the system. Second, we study the conditions that determine a dominant subharmonic and thirdly, we test the stability of the dominant subharmonic. Finally, we present a design method for a system which generates an arbitrary dominant subharmonic.

Before proceeding though, we define precisely what we mean by a dominant subharmonic. If a system with input $u(t)$, periodic of period T, yields an output $y(t)$, periodic of period kT, $k>0$, then the output will be said to yield the $(1/k)$th harmonic of the input. In the case where $k>1$, then, on setting $n=k$, the output is said to be the nth *subharmonic* of the input. Considering the subharmonic case, on making Fourier series expansions for a single frequency (plus dc bias) input we have (with time normalized for zero phase in the input)

$$u(t)=U_0+U_1\cos\omega t\ , \tag{1a}$$

$$y(t)=\sum_{i=0}^{\infty}Y_i\cos[(i\omega/n)t+\varphi_i]. \tag{1b}$$

We will call the nth subharmonic *dominant* if

$$|Y_1|\gg|Y_i|,\quad i=0,2,3,\ldots\ . \tag{1c}$$

I. THE COUPLED EQUATIONS AND THEIR ANALYSIS

We take as the basic part of the description of our nonlinear system for the generation of subharmonic frequencies the following pair of coupled differential equations:

$$\ddot{x}_1+\Omega_1^2x_1+\alpha_1x_1^2+\beta_1x_1x_2+\gamma_1x_2^2=G_1+H_1\cos\omega t\ , \tag{2a}$$

$$\ddot{x}_2+\Omega_2^2x_2+\alpha_2x_2^2+\beta_2x_2x_1+\gamma_2x_1^2=G_2+H_2\cos\omega t\ . \tag{2b}$$

The equations yield the responses x_1 and x_2 of two coupled nonlinear oscillators driven by sources having a constant, dc, component and a sinusoidal component of angular frequency ω; Ω_1 and Ω_2 are the angular resonance frequencies of the corresponding linearized oscillators when they have no coupling; the six coefficients $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1$, and γ_2 account for the nonlinearity and coupling. Figure 1 is a block diagram of such a system where the single input $u(t)=U_0+U_1\cos\omega t$ has been decomposed through network N_0 into $u_1(t)=G_1+H_1\cos\omega t$ and $u_2(t)=G_2+H_2\cos\omega t$. Here the networks N_1 and N_2 represent oscillators to realize Eqs. (2).

We proceed by assuming subharmonic solutions of Eq. (2), x_1 and x_2 of the form

$$x_1=a_1+b_1\cos(\omega/n)t+c_1\cos\omega t \tag{3a}$$

$$x_2=a_2+b_2\cos(\omega/n)t+c_2\cos\omega t\ , \tag{3b}$$

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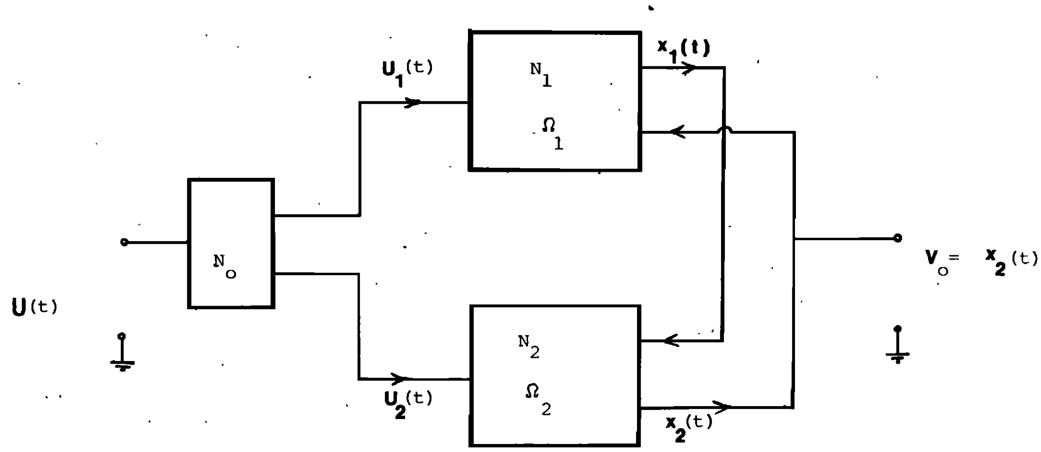

FIG. 1. A block diagram for a coupled system for subharmonic generation.

where n is any positive number satisfying the subharmonic condition $n>1$. Substituting Eqs. (3) into Eqs. (2), and equating the coefficients of the resulting $0, \omega, \omega/n$, $2\omega, 2\omega/n$, and $(1\pm 1/n)\omega$ frequency terms to zero to obtain the constraints such that Eqs. (3) could be an exact solution to Eqs. (2), yields a set of 12 equations with 18 parameters. Six of these are solution constants a_i, b_i, c_i $(i=1,2)$ and 12 are equation constants $\Omega_i, \alpha_i, \beta_i$, γ_i, G_i, and H_i. Through this set of equations we can obtain the necessary constraints on the solution parameters.

In this regard we note that the six equations involving the coefficients of the $(2\omega/n)$, 2ω, and $(1\pm 1/1/n)\omega$ frequency terms are particularly interesting as they are linear in α_i, β_i, and γ_i. Rewriting terms in matrix form $A\xi=0$, where $\xi^T=[\alpha_1,\beta_1,\gamma_1;\alpha_2,\beta_2,\gamma_2]$ the determinant of A is found to be the product of two determinants of the same form with the indices permuted. Setting this determinant tozero we obtain two equations replacing the original six from which they are derived. Thus the set of 12 equations which determine the constraints on the parameters reduces to eight, one each of the following four for each of the two index pairs $i,j=1,2$,

$$\Omega_i^2 a_i+\alpha_i a_i^2+\tfrac{1}{2}\alpha_i b_j^2+\tfrac{1}{2}\alpha_i c_i^2+\beta_i a_i a_j+\tfrac{1}{2}\beta_i c_i c_j +\gamma_i a_j^2+\tfrac{1}{2}\gamma_i b_j^2+\tfrac{1}{2}\gamma_i c_j^2+\tfrac{1}{2}\beta_i b_i b_j-G_i=0\,, \tag{4a}$$

$$-c_i\omega^2+\Omega_i^2 c_i+2\alpha_i a_i c_i+\beta_i a_i c_j+\beta_i a_j c_j+2\gamma_i a_j c_j-H_i=0\,, \tag{4b}$$

$$-b_i(\omega/n)^2+\Omega_i^2 b_i+2\alpha_i a_i b_i+\beta_i a_i b_j+\beta_i a_j b_i+2\gamma_i a_j b_j=0, \tag{4c}$$

$$\tfrac{1}{4}b_i c_i b_j c_j(\tfrac{3}{2}b_i c_j-\tfrac{3}{2}b_j c_i)+\tfrac{1}{8}(b_j^3 c_i^3-b_i^3 c_j^3)=0\,. \tag{4d}$$

One way of satisfying Eq. (4d) is to set, for $i=1, j=2$,

$$b_1 c_2=b_2 c_1 \quad\text{or}\quad b_1/b_2=c_1/c_2=k\,, \tag{5}$$

which, on substituting in the original six equations linear in α_i, β_i, and γ_i, yield for the index pairs $i,j=1,2$, $i\neq j$

$$\tfrac{1}{2}b_j^2[\alpha_i(c_i/c_j)^2+\gamma_i+\beta_i(c_i/c_j)]=0,$$
$$(2\omega/n\text{ frequency term})\,; \tag{6a}$$

$$\tfrac{1}{2}c_j^2[\alpha_i(b_i/b_j)^2+\gamma_i+\beta_i(b_i/b_j)]=0\,,$$
$$(2\omega\text{ frequency term})\,; \tag{6b}$$

$$\tfrac{1}{2}b_j c_j[\alpha_i(c_i/c_j)^2+\gamma_i+\beta_i(b_i/b_j)]=0\,,$$
$$[\omega(1\pm 1/n)\text{ frequency term}]\,. \tag{6c}$$

For b_j and c_j nonzero, these are

$$\alpha_i(b_i/b_j)^2+\beta_i(b_i/b_j)+\gamma_i=0\,, \tag{7}$$

or taking into account (5), both of the following must hold:

$$\alpha_1 k^2+\beta_1 k+\gamma_1=0\,, \tag{8a}$$

$$\gamma_2 k^2+\beta_2 k+\alpha_2=0\,. \tag{8b}$$

Since k is defined through two equations, some constraints are put on the six system coefficients α_1,α_2, β_1,β_2,γ_1, and γ_2. One has four degrees of freedom left and we can thus choose

$$\alpha_1=\gamma_2=\alpha,\quad \beta_1=\beta_2=\beta,\quad\text{and}\ \gamma_1=\alpha_2=\gamma\,. \tag{9}$$

k is then determined by solving one of the quadratic equations (8a) or (8b) giving in either case

$$k=-(\beta_1/2\alpha_1)\{1\pm[1-(4\alpha_1\gamma_1/\beta_1^2)]^{1/2}\}\,. \tag{10a}$$

For real solutions k has to be real, hence a final inequality constraint is obtained (for both $i=1$ and $i=2$);

$$\beta_i^2\geq 4\alpha_i\gamma_i\,. \tag{10b}$$

To obtain expressions for the subharmonic frequency (ω/n) Eq. (4c), after using Eq. (5), can be written in the form, for $i=1, j=2$,

$$a_2 k-a_1=\{[\Omega_1^2-(\omega/n)^2]/A_1\}k\,, \tag{11a}$$

where

$$A_1=\beta_1+2\alpha_1 k \tag{11b}$$

and, simultaneously for $i=2, j=1$,

$$a_1-a_2 k=\{[\Omega_2^2-(\omega/n)^2]/A_2\}k\,, \tag{11c}$$

where

$$A_2=2\alpha_2+\beta_2 k\,, \tag{11d}$$

whereby once k is fixed the relation between a_1 and a_2 is determined. On adding Eqs. (11a) and (11c), we obtain for $k\neq 0$,

$$(A_1+A_2)(\omega/n)^2=A_2\Omega_1^2+A_1\Omega_2^2\,. \tag{12}$$

Equation (12) gives the subharmonic frequency ω/n as a function of the two natural frequencies Ω_1 and Ω_2 and the system constants α_1, α_2, β_1, β_2, γ_1, and γ_2. Therefore, the order of the subharmonic is given by

$$n=[(A_1+A_2)/(A_2\Omega_1^2+A_1\Omega_2^2)]^{1/2}\omega\,. \tag{13}$$

Two cases are easily identifiable from Eq. (12). The first is when $A_1 \neq -A_2$. In this case subharmonic solutions exist and the order n of the subharmonic varies linearly with ω, the frequency of the driving source if A_1, A_2, Ω_1, and Ω_2 remain unchanged. The second case is when $A_1 = -A_2$. In this case $A_2\Omega_1^2 + A_1\Omega_2^2 = 0$ and the frequency of the subharmonic solution has to be determined by some other constraints. We shall not look into this in this work.

In general, for the subharmonic solution of Eq. (12) to hold exactly, we need to consider the constraints on the terms G_i and H_i of the driving source. To do this, we consider the zero and ω frequency coefficients equations. Substituting Eq. (7) into Eqs. (4a) and (4b), we obtain, again for $i,j = 1, 2,\ i \neq j$

$$\Omega_i^2 a_i + \alpha_i a_i^2 + \beta_i a_i a_j + \gamma_i a_j^2 = G_i \tag{14}$$

and

$$c_i[\Omega_i^2 - \omega^2 + 2\alpha_i a_i + \beta_i a_j + (1/k)\beta_i a_i + (2/k)\gamma_i a_j] = H_i\,. \tag{15}$$

We observe the following from these two equations: first, a consistent subharmonic, as defined by Eq. (12), requires that solutions for a_i, a_j in Eq. (14) must intersect those of the a_i, a_j in Eq. (11). This can be easily achieved since with k fixed and G_i free to be chosen we obtain two equations in the two unknown a_1 and a_2. Secondly, the H_i must be adjusted to give $c_i/c_j = k$ which is also certainly possible when the driving sources are free to be chosen and thus takes care of the needed constraints from Eq. (15). The c_i and the a_i, after being determined, are now fixed and by c_1/c_2, b_1/b_2 is known by (5). The initial conditions

$$x_i(0) = a_i + b_i + c_i \tag{16a}$$

$$\dot{x}_i(0) = 0 \tag{16b}$$

then determine the b_i and these initial conditions too must be adjusted to give the required ratio $b_1/b_2 = k$, which again is possible.

A. Relationship between n and ω

It has been shown through Eq. (13) that the order of the subharmonic n is a function of ω, the frequency of the driving source and Ω_1 and Ω_2, the two frequencies of natural oscillations. It is, however, often desired to generate subharmonic oscillations of the same order when ω varies within a certain range. In this case the natural frequency Ω_1 or Ω_2 of the system may be varied.

The relationship between the subharmonic frequency and the natural frequencies is depicted by Eq. (12) which is rewritten here as

$$\omega^2/n^2 = [A_2/(A_1 + A_2)]\Omega_1^2 + [A_1/(A_1 + A_2)]\Omega_2^2\,. \tag{17}$$

On dividing by $[A_1/(A_1 + A_2)]\Omega_2^2$, Eq. (17) can be put in the form

$$\frac{\omega^2}{(n^2\Omega_2^2 A_1)/(A_1 + A_2)} - \frac{\Omega_1^2}{(\Omega_2^2 A_1)/A_2} = 1\,, \tag{18}$$

which in the $\omega - \Omega_1$ plane represents a hyperbola

$$\omega^2/d^2 - \Omega_1^2/e^2 = 1\,, \tag{19a}$$

with

$$d = n\Omega_2[A_1/(A_1 + A_2)]^{1/2} \quad \text{and} \quad e = \Omega_2(A_1/A_2)^{1/2}\,. \tag{19b}$$

This hyperbola has two asymptotic lines represented by

$$\Omega_1 = \pm \frac{e}{d}\omega = \pm \frac{1}{n}\left(1 + \frac{A_1}{A_2}\right)^{1/2}\omega\,, \tag{20}$$

and foci, $F_{1,2}$, located at $\{[\pm n\Omega_2 A_1/(A_1 + A_2)] + A_1/A_2, 0\}$ in the $\omega - \Omega_1$ plane. Figure 2 is a sketch of the curve showing only the first quadrant.

Hence, for a given subharmonic order n, there is a minimum driving source frequency, $\omega_{min} = d$, below which no subharmonic can be generated. However, at frequencies well above ω_{min}, when Ω_2 is fixed, Ω_1 varies almost linearly with ω. For a large value of n, ω_{min} be-

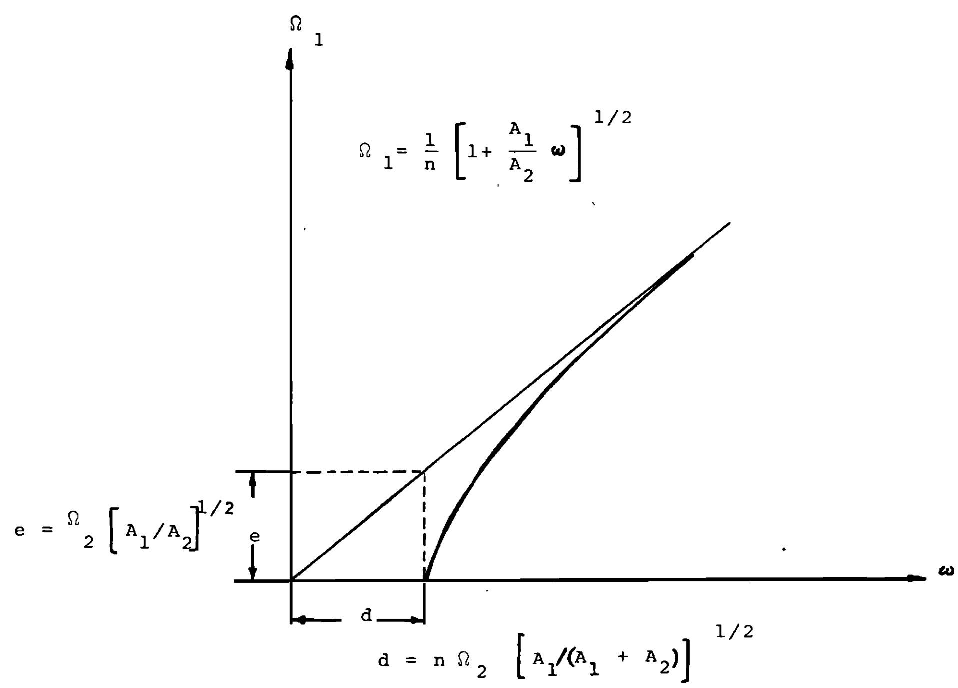

FIG. 2. Relationship between Ω_1 and ω for fixed n and Ω_2.

comes larger and the change in Ω_1 required to keep n fixed when ω varies is less than the corresponding change in ω.

B. Dominant subharmonics

The subharmonic solutions, Eq. (3), describe dominant subharmonics if for $i=1,2$

$$|a_i| \ll |b_i| , \tag{21a}$$

$$|c_i| \ll |b_i| . \tag{21b}$$

It may be argued that since a_1 and a_2 represent only the dc levels in the output, why should condition (21a) be necessary? Since for design purposes, it is convenient to make the amplitude, b_1 and b_2, of the output subharmonic components determinable through the initial conditions, Eq. (16a) makes condition (21a) necessary.

We now apply conditions (21) to the subharmonic solutions previously obtained. From Eq. (11a) and Eq. (14)

$$a_1 = \frac{k}{1-k(\Omega_1^2/\Omega_2^2)}\left(\frac{(\omega/n)^2-\Omega_1^2}{A_1} - \frac{G_1-G_2}{\Omega_2^2}\right) , \tag{22}$$

where k is as given by Eq. (10a). The absolute value of k, which could be a design parameter, can be chosen less than unity leading, from (10a), to

$$k = -(\beta_1/2\alpha_1)\{1-[1-(4\alpha_1\gamma_1/\beta_1^2)]^{1/2}\} , \tag{23a}$$

where

$$4\alpha_1\gamma_1 \leq \beta_1^2 . \tag{23b}$$

Thus, with $|k|<1$, the value of a_1 is essentially determined by (i) the difference between the squares of the subharmonic frequency and the natural frequency Ω_1 and (ii) the difference between the dc components G_1 and G_2 of the driving sources. Therefore, for (ω/n) close to Ω_1, a_1 can be kept small by proper adjustment of G_1 and G_2. By symmetry, when (ω/n) is close to Ω_2 and $|k|$ is small, the value of a_2 can be kept small through G_1 and G_2. Consequently, the linear portions of the two coupled oscillators should be tuned near the derived subharmonic frequency.

The amplitudes c_1 and c_2 can be made small by exciting the system at small sinusoidal excitation levels of H_i, as Eq. (15) and (2) indicates. The amplitudes b_1 and b_2 at the subharmonic frequency are governed by the constraint $b_1/b_2 = c_1/c_2 = k$ and the initial condition constraint as shown by Eq. (5) and (16a) and b_2 is arbitrary. However, stability analysis of the solution, as given in the next section, imposes some limits on the amplitude b_2 and consequently $b_1 = kb_2$. The analysis shows that the conditions on b_2 that lead to a stable dominant subharmonic solution depend upon the relationship between the subharmonic frequency ω/n and the natural frequencies Ω_1 and Ω_2.

C. Stability analysis

The coupled differential equations of Eq. (2) can be written, after applying the constraint of Eq. (9a), $\alpha_1 = \gamma_2 = \alpha$, $\alpha_2 = \gamma_1 = \gamma$, $\beta_1 = \beta_2 = \beta$, as

$$\ddot{x}_1 + \Omega_1^2 x_1 + \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2 = G_1 + H_1 \cos\omega t , \tag{24a}$$

$$\ddot{x}_2 + \Omega_2^2 x_2 + \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2 = G_2 + H_2 \cos\omega t . \tag{24b}$$

For dominant subharmonics, solutions to Eqs. (24) can be written as

$$x_1 = b_1 \cos(\omega/n)t + \text{(terms of smaller amplitudes)}, \tag{25a}$$

$$x_2 = b_2 \cos(\omega/n)t + \text{(terms of smaller amplitudes)}. \tag{25b}$$

The stability of a solution x_1, x_2 for Eq. (24) is tested by adding a small perturbation to the solution and then seeing whether the perturbation grows or decays in time. Therefore, replacing x_1 and x_2 by $x_1 + y_1$ and $x_2 + y_2$ in Eq. (24), where y_1 and y_2 are initially small perturbations, and dropping second-order terms in y_1 and y_2, the following equations are obtained:

$$\ddot{y}_1 + \Omega_1^2 y_1 + (2\alpha x_1 + \beta x_2)y_1 + (\beta x_1 + 2\gamma x_2)y_2 = 0 , \tag{26a}$$

$$\ddot{y}_2 + \Omega_2^2 y_2 + (2\gamma x_2 + \beta x_1)y_2 + (\beta x_2 + 2\alpha x_1)y_1 = 0 . \tag{26b}$$

If we now substitute for x_1 and x_2 from Eq. (25) into Eq. (26) and let $b_1 = b_2 k$, we obtain the following equation:

$$\ddot{y}_1 + \Omega_1^2 y_1 + A_1 b_2[\cos(\omega/n)t]y_1 + A_2 b_2[\cos(\omega/n)t]y_2 = 0 , \tag{27a}$$

$$\ddot{y}_2 + \Omega_2^2 y_2 + A_2 b_2[\cos(\omega/n)t]y_2 + A_1 b_2[\cos(\omega/n)t]y_1 = 0 , \tag{27b}$$

where, as defined by Eq. (11b) and (11d),

$$A_1 = 2\alpha k + \beta , \tag{27c}$$

$$A_2 = \beta k + 2\gamma . \tag{27d}$$

Letting $z = (\omega/2n)t$, then Eq. (27) can be written in the form of a pair of coupled Mathieu equations[6]:

$$y_1'' + [L_1 - 2q_1\cos(2z)]y_1 - [2p_1\cos(2z)]y_2 = 0 , \tag{28a}$$

$$y_2'' + [L_2 - 2q_2\cos(2z)]y_2 - [2p_2\cos(2z)]y_1 = 0 , \tag{28b}$$

where the prime indicates differentiation with respect to z and where

$$L_1 = 4\Omega_1^2/(\omega/n)^2 , \tag{29a}$$

$$L_2 = 4\Omega_2^2/(\omega/n)^2 , \tag{29b}$$

$$q_1 = p_2 = -2(\omega/n)^{-2}A_1 b_2 , \tag{29c}$$

$$q_2 = p_1 = -2(\omega/n)^{-2}A_2 b_2 . \tag{29d}$$

Thus the stability analysis of the dominant subharmonics reduces to a study of coupled Mathieu equations, the well-known properties of which can be used. Therefore, following a method used by Eller[4] and McLachlan,[7] solutions to the pair of equations, (28a) and (28b), can be assumed to be of the form

$$y_1(z) = \exp(\mu z)[C_1 \sin(M_1 z - \sigma_1) + \Phi_1] , \tag{30a}$$

$$y_2(z) = \exp(\mu z)[C_2 \sin(M_2 z - \sigma_2) + \Phi_2] , \tag{30b}$$

where C_1 and C_2 are constants, M_1 and M_2 are non-negative frequencies, σ_1 and σ_2 are small functions of the normalized time z and μ is in general a real damping number. We also write for M_1 and M_2

$$L_1 = 4\Omega_1^2/(\omega/n)^2 = M_1^2 + \Sigma_1 , \tag{31a}$$

$$L_2 = 4\Omega_2^2/(\omega/n)^2 = M_2^2 + \Sigma_2 . \tag{31b}$$

The quantities μ, Φ, Σ, p, q with either subscript are assumed to be small.

It is known[6] that, for the single Mathieu equation $y'' + (L - 2q\cos 2z)y = 0$, the domain of stability located near the resonance frequency 4 is determined by

$$L \leq 4 - q^2/12, \quad \text{when } L < 4, \tag{32a}$$

and

$$L \leq 4 + 5q^2/12, \quad \text{when } L > 4. \tag{32b}$$

For the case of the pair of coupled Mathieu equations of Eqs. (28), when the subharmonic frequency ω/n is in the vicinity of both the nautral frequencies Ω_1 and Ω_2, the domains of stability can be located in a similar manner to those for a single Mathieu equation.[4] Hence, for Eqs. (28), the domains of stability near the resonance frequency 4 can be determined by the following for $i = 1, 2$:

$$L_i \leq 4 - q_i^2/12, \quad \text{when } L_i < 4, \tag{33a}$$

and

$$L_i \leq 4 + 5q_i^2/12, \quad \text{when } L_i > 4. \tag{33b}$$

From Eqs. (29) we note that if $L_i < 4$, $\Omega_i < \omega/n$ and if $L_i > 4$, $\Omega_i > \omega/n$. On substituting for L_i from (29), Eqs. (33 a,b) are equivalent to

$$A_i^2 < 12\left(\frac{(\omega/n)^2 - \Omega_i^2}{b_2^2(\omega/n)^{-2}}\right), \quad \text{for } \Omega_i < \omega/n, \tag{34a}$$

$$A_i^2 < \tfrac{12}{5}\left(\frac{\Omega_i^2 - (\omega/n)^2}{b_2^2(\omega/n)^{-2}}\right), \quad \text{for } \Omega_i > \omega/n. \tag{34b}$$

Therefore, if the system of coupled differential equations is to possess a stable subharmonic solution, the parameters of the system A_1 and A_2, must satisfy the inequalities (34) for the situation on hand. There are the following possible cases: $\Omega_1, \Omega_2 < \omega/n$; $\Omega_1, \Omega_2 > \omega/n$; $\Omega_1 < \omega/n$ while $\Omega_2 > \omega/n$; and $\Omega_1 > \omega/n$ while $\Omega_2 < \omega/n$. It is important to note here that for any of the four cases, the above equations place bounds on the value b_2 can take for given A_1 and A_2.

For example, for the value of $b_2 = 5$ and $n = 3$, Fig. 3 illustrates the choice for A_1 (below the Ω_1 curves) and A_2 (below the Ω_2 curves) which guarantee a stable subharmonic system. For a system to be designed with $\Omega_1 < \omega/n$ and $\Omega_2 > \omega/n$, as shown in Fig. 3, and to be driven at a source frequency of ω, the maximum values of A_1 and A_2 can be read off from the appropriate curves for the desired values of $n = 3$ and $b_2 = 5$; alternatively these can be calculated from (34 a,b). Then values of $A_1 = 2\alpha k + \beta$ and $A_2 = \beta k + 2\gamma$ are chosen less than the calculated maximum values.

It is worth noting that the effects of the case discussed in Sec. IA, where n is desired to be fixed even when ω varies within a certain range, can be seen clearly in Fig. 3. If Ω_2 if fixed, to keep n and A_1 and A_2 of the system constant the curves for Ω_1 have to be shifted right or left by varying Ω_1 accordingly.

Once b_2 is set, b_1 is fixed to satisfy $b_1 = kb_2$ where k is as defined by Eq. (23a). Since, by Eqs. (21a,b), b_i is large compared to a_i or c_i for $i = 1, 2$, a choice of $a_i \leq b_i/10$ and $c_i \leq b_i/10$ is considered reasonable. Note that if $c_i = 0$ then the system has $H_i = 0$ and there would be no excitation at the fundamental frequency; hence $c_i \neq 0$. Now the initial conditions for the system can be determined from Eq. (16a).

II. SUBHARMONIC SYSTEM

Given an input $u(t) = U_0 + U_1 \cos\omega t$, (1a), the coupled equations so far developed can be used to generate an output $y(t)$ with a dominant nth subharmonic by choosing an output point, for example, $y(t) = x_2(t)$.

The block diagram for the system is shown in Fig. 1 where it is seen there are three basic blocks. In Fig. 1 the input $u(t)$ is processed by the block N_0 to yield the forcing functions for the differential equations (2), $u_1(t) = G_1 + H_1\cos\omega t$, $u_2(t) = G_2 + H_2 \cos\omega t$. Thus, N_0 acts to scale the input to give the appropriate forcing functions to the coupled system. The two separate nonlinear differential equations of Eqs. (2a) and (2b) are realized by the blocks N_1 and N_2 of Fig. 1, respectively. Each of these blocks is essentially a forced second-order oscillator with square-law nonlinearities and cross coupling between the states x_1 and x_2. One of these states x_2 is taken as the output.

III. DESIGN TECHNIQUE

A procedure can now be set up based upon our analysis for the generation of stable dominant nth subharmonics. We describe this in possible steps and then carry out an example.

Step 1:

Determine the input $u(t) = U_0 + U_1\cos\omega t$ available and the output $y(t) = Y\cos(\omega/n)t$ desired.

Step 2:

Choose an actual output acceptable as $y = x_2$

$$x_2(t) = a_2 + b_2\cos(\omega/n)t + c_2\cos\omega t$$

$$b_2 = Y, \quad a_2, c_2 \ll b_2, \quad c_2 \neq 0.$$

Step 3:

Choose Ω_1 and Ω_2 near ω/n with

$$\Omega_1 < \omega/n, \ \Omega_2 > \omega/n.$$

Step 4:

Form, according to (34),

$$A_{1\max}^2 = 12\left(\frac{(\omega/n)^2 - \Omega_1^2}{b_2^2/(\omega/n)^2}\right),$$

$$A_{2\max}^2 = \tfrac{12}{5}\left(\frac{\Omega_2^2 - (\omega/n)^2}{b_2^2/(\omega/n)^2}\right).$$

Choose (for stability)

$$0 < A_i < A_{i\max}, \quad i = 1, 2.$$

Step 5:

Choose k; by (10) choose $k < 0$, $|k| < 1$.

Step 6:

Choose α, β, γ. By (11b,d) and (9)

$$A_1 = \beta + 2\alpha k, \quad A_2 = 2\gamma + \beta k.$$

Thus

$$\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \frac{1}{2k^2}\begin{bmatrix}k & -1 \\ 0 & 2k\end{bmatrix}\begin{bmatrix}A_1 \\ A_2 - 2\gamma\end{bmatrix}.$$

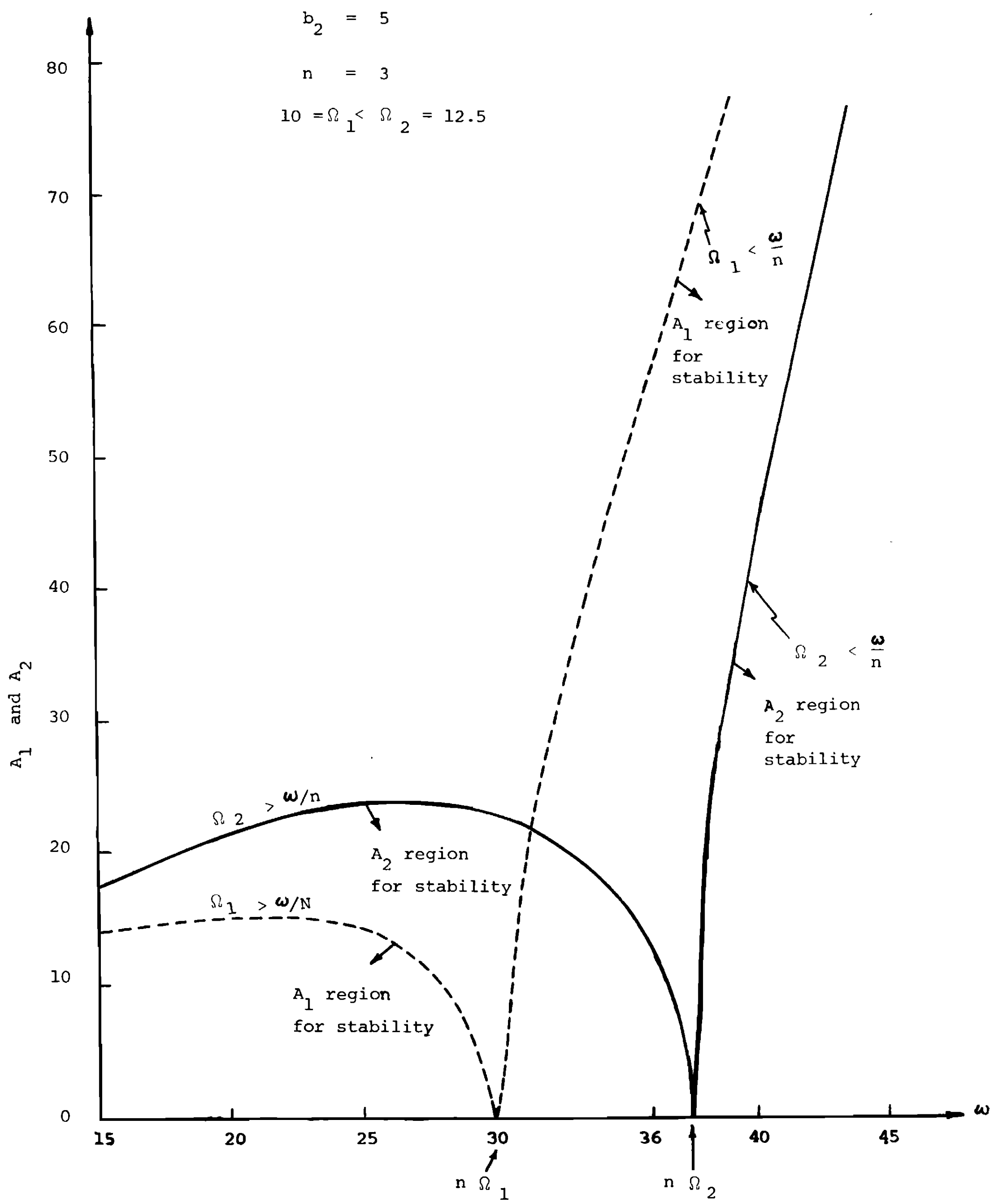

FIG. 3. Bounds on system parameters A_1 and A_2 showing maximum values of A_1 and A_2 for stable operation of the system for the case $10=\Omega_1<\Omega_2=12.5$ and $b_2=5$, $n=3$.

By (10b)

$$\beta^2 \geq 4\alpha\gamma ,$$

which on using the last result gives, assuming $kA_1+A_2>0$,

$$\gamma \leq \gamma_{\max}=\frac{1}{2[kA_1+A_2]} A_2^2 .$$

Therefore, choose $\gamma<\gamma_{\max}$, and then determine α and β as per the above matrix equation. Note that all of α, β, γ can be determined to be positive if so desired.

Step 7:

Form, from (11a) and (5)

$$a_1=(a_2-\{[\Omega_1^2-(\omega/n)^2]/A_1\})k ,$$

$$b_1=b_2k, \quad c_1=c_2k .$$

Step 8:

Form, from (14)

$$G_1=\Omega_1^2a_1+\alpha a_1^2+\beta a_1a_2+\gamma a_2^2$$

$$G_2=\Omega_2^2a_2+\gamma a_2^2+\beta a_1a_2+\alpha a_1^2$$

and from (15)

$$H_1=c_1[\Omega_1^2-\omega^2+2\alpha a_1+\beta a_2+k^{-1}(\beta G_1+2\gamma a_2)]$$

$$H_2=c_2[\Omega_2^2-\omega^2+2\gamma a_2+\beta a_1+k^{-1}(\beta G_2+2\alpha a_1)] .$$

Step 9:

Form the required initital conditions, from (16) for $i=1,2$

$$x_i(0)=a_i+b_i+c_i$$

$$\dot{x}_i(0)=0 .$$

As an example, let it be desired to design the system for a dominant subharmonic solution of order 3.

Specifically, let an input signal $\cos 36t$ be fed into a system such as that shown in Fig. 1 and let the system be designed so that the output is

$$y = v_0 = x_2 = 0.01 + 5\cos 12t + 0.02\cos 36t$$
$$= a_2 + b_2 \cos(\omega/n)t + c_2 \cos\omega t \,. \qquad (35)$$

Our design technique proceeds as follows:

Since Ω_1 and Ω_2 should be in the vicinity of ω/n, we choose $\Omega_1 = 10$ and $\Omega_2 = 12.5$. Next, using the stability equations (34) or the curves of Fig. 3, we determine at $\omega = 36$ the design parameters A_1 and A_2 for b_2 set equal to 5 and n equal to 3. In this case, $A_1 < 55.15$ and $A_2 < 13.01$. We, therefore, choose $A_1 = 37.5$ and $A_2 = 5$, numbers which are convenient for the choices of α, β, γ. Now from Eqs. (27), $A_1 = 2\alpha k + \beta$ and $A_2 = \beta k + 2\gamma$. As k is a design parameter whose amplitude is much less than unity, we choose $k = -0.12$ and then determine α, β, and γ. A set of values that will satisfy the values for A_1 and A_2 are $\beta = 40$, $\alpha = 10.416$ and $\gamma = 4.9$.

Next we determine a_1 from Eq. (11a) or (11c) or from [using (12)]

$$a_1 = [a_2 - (\Omega_1^2 - \Omega_2^2)/(A_1 + A_2)]k \,, \qquad (36)$$

which gives $a_1 = -0.16$, and from Eq. (14) we calculate G_1 and G_2 to be -15.748 and 1.814, respectively. Knowing k and c_2, c_1 is calculated to be -0.0024 and from Eq. (15) we determine H_i and H_2 to be 2.75 and -22.43, respectively. Hence, the designed system is represented by the coupled equations:

$$\ddot{x}_1 + 100x_1 + 10.416x_1^2 + 40x_1x_2 + 4.9x_2^2$$
$$= -15.748 + 2.75\cos\omega t \,, \qquad (37a)$$
$$\ddot{x}_2 + 156.25x_2 + 10.416x_1^2 + 40x_1x_2 + 4.9x_2^2$$
$$= 1.814 - 22.43\cos\omega t \,. \qquad (37b)$$

These equations should have the desired x_2 of Eq. (35) and

$$x_1 = -0.16 - 0.6\cos 12t - 0.0024\cos 36t \,.$$

Finally, to test the accuracy of the design procedure, the desired dominant subharmonic solution, x_1 and x_2, is substituted into the designed equations. Within limits of the calculation's rounding errors, the equations hold quite well.

IV. DISCUSSION

It has been shown that a system described by two coupled nonlinear differential equations excited by an input containing a constant and sinusoidal term can be designed to generate a dominant subharmonic frequency of any real order. A dominant subharmonic occurs when the frequency of the driving force is close to an n multiple of the natural frequencies of the uncoupled linearized system. For other values of the input frequency, the response of the system still contains subharmonic components but possibly not in a dominant form. The order n of the dominant subharmonic is not restricted to be of rational form, but can be any real number. For a fixed system, however, it does vary continuously as the input frequency ω is changed. A design procedure based on the results of the analysis has been presented which will achieve any subharmonic order for a given input.

It should be observed that the development holds equally well for superharmonics in which case the results can be used to generate from a given frequency ω any harmonically related frequency ω/n; that is, n can be any positive real number. In the situation where $n = 2$ or 3, there are other sets of equations which can also be developed by virtue of the fact that $1 \pm (1/n) = (1/n)$ or $(2/n)$.

As seen at Eq. (13), n is a linear function of frequency. Consequently, the system is convenient for automatically carrying out a frequency compression. Such could prove of value for coding of signals as, for example, in the field of hearing improvement where it is thought that such a compression may bring signals, otherwise unhearable, into the range of a partially deaf person suffering high-frequency hearing loss. Preliminary results indicate that if properly done, this could prove a valuable technique for hearing improvement.[8,9]

Some comments on practical constructions are in order. Presently, one can purchase squaring devices which can be used to realize the nonlinearities and cross couplings needed for Eqs. (2). These squaring devices do suffer from imbalances which, however, can be tuned out via tuning of other terms in (2). The linear portions of (2) can be realized using active operational amplifier (op-amp) circuits which have the added advantage that any dissipative losses can be tuned out through the activity of the op-amps. Should, though, dissipation actually be present, a much more extensive analysis shows that through the presence of proper added linear coupling terms, subharmonics can also be made to result. The techniques and results follow those of this paper but, being quite detailed, may be worth a separate study.

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