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THE DRAZIN INVERSE AND SEMI - STATE EQUATIONS

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Abstract

The Drazin inverse is reviewed and it is shown that linear time - invariant semi-state equations can always be solved by its use whenever a solution exists (which is the case when proper initial semi - states are specified). Thus, a technique is given for setting up the semi - state equations for all finite linear time - invariant circuits. The concepts are illustrated by a simple circuit example.

I. INTRODUCTION

At this stage in the development of network and systems theory state - variable descriptions have become well recognized, playing a dominant role because of their convenient first order form. In the case of linear time - invariant systems, once the state - variable equations are obtained analysis proceeds simply through the use of the resulting exponential (of the A - matrix) impulse response. And perhaps more importantly convenient designs proceed through creation of the state - variable equations, as for example in the very practical design of active filters.

However, the state - variable equations of a circuit do not always result from a straight forward analysis of the circuit. Indeed because of possibly hidden dependencies between dynamic element variables, such as result from capacitor loops or inductor cut - sets, it may be troublesome to choose an independent set of state - variables or to find the transformation which takes a circuit description into state - variable form. To be sure a good portion of the early literature was devoted to the problem of "degrees of freedom" for which the theories eventually led to state - variable equations.

Here we show that some of these "problems" with state - variables can be avoided by considering a more basic set of equations, equations which we call semi - state equations. Basically semi - state equations are state - type equations with a possibly singular operator on the derivatives of the (semi -) state. As will be seen in Section III, every finite circuit has a semi - state description which is directly, and very simply, obtained through use of the circuit graph.

Now, what has led us to consider the semi - state equations is the existence of the Drazin inverse and its use in solving semi - state equations. In particular if the semi - state equations have a solution this solution can be found, and in an explicit and constructive manner, through the use of the Drazin inverse. In short, every finite time - invariant circuit can be solved through the use of the Drazin inverse on the semi - state equations, the latter being simply set up and the former being readily calculable.

In Section II we review the Drazin inverse while in Section III we introduce the semi - state equations. In Section IV we illustrate the concepts upon a simple example. We assume a standard knowledge of the graph theory of circuits and similarly of state variable theory, such as in [1] or [2].

II. THE DRAZIN INVERSE

At this point we define the Drazin inverse, this being the unique generalized inverse which satisfies equations (II-1a-c). Its nature is exhibited in terms of matrices by decomposition of a matrix into a nonsingular and a nilpotent part at equations (II-3a, b). Finally the section ends by giving at equation (II-10) the general solution of linear time - invariant differential equations in terms of Drazin inverses.

Given any associative ring R, as the ring of nxn matrices with real entries, Drazin [3, p. 507] calls an element a, $a\epsilon R$, "pseudo - invertible" if there exists another element a^D, $a^D \epsilon R$, now called the Drazin inverse, such that all three of the following equations hold

$$a^D a = a a^D \tag{II-1a}$$

$$a^m = a^{1+m} a^D \text{ for some positive integer } m \tag{II-1b}$$

$$a^D = (a^D)^2 a \tag{II-1c}$$

In the case where the ring R is the set of real nxn matrices the Drazin inverse exists, and is unique, for every matrix $a\epsilon R$ [3, p. 510][4, p. 411]. And the least integer I for which (II-1b) holds (for m = I) is called the index of a. We have

$$\text{rank}(a^I) = \text{rank}(a^{1+I}) \tag{II-2}$$

while if I = 0 then $a^D = a^{-1}$ is the standard inverse of a nonsingular matrix.

If a_J is the Jordan form for a, then

$$a = T^{-1} a_J T, \quad a_J = a_{NS} \dotplus a_{NP} \tag{II-3a}$$

where T is nonsingular, a_{NS} is nonsingular [of rank = rank (a^I)], $\dotplus$ denotes the matrix direct sum, and a_{NP} is nilpotent (here this being $a_{NP}^I = 0$). Then the Drazin inverse is given by [4, p. 412]

$$a^D = T^{-1}[a_{NS}^{-1} \dotplus 0]T \tag{II-3b}$$

That is, the Drazin inverse of a matrix a can be found by inverting the non-singular part of the Jordan form a_J transformed through the similarity transformation defined by T which brings the matrix a into this Jordan form.

Turning to the use of the Drazin inverse in solving differential equations we follow the notation of [5] in order to be able to quote results directly. Thus, consider the vector differential equation

$$A\dot{x} + Bx = f, \; x(0_-) = x_0, \; f(t) = 0 \text{ for } t < 0 \tag{II-4}$$

where A and B are constant nxn matrices (assumed real for physical systems' applications), x is an n - vector, and x_0 is its initially specified value. By definition, [5, p. 757], x_0 constitutes consistent initial conditions if a solution x(t) exists for (II-4) [for t > 0, say].

It is known [4, p. 418, Thm. 6] that $A\dot{x} + Bx = 0$ has a unique solution for consistent initial conditions x_0 if and only if there exists some constant c for which cA + B is nonsingular. Consequently, if $A\dot{x} + Bx = 0$ has unique solutions for consistent x_0 then there is some constant c for which the following exist

$$\tilde{A} = (cA+B)^{-1}A, \; \tilde{B} = (cA+B)^{-1}B, \; \tilde{f} = (cA+B)^{-1}f \tag{II-5}$$

To get a feeling for why the solution to (II-4) has the form it has at (II-10) we take Laplace transforms, denoted by the operator $\mathcal{L}[\cdot]$, to get (where s is the Laplace transform variable)

$$(sA+B)\mathcal{L}[x] = Ax_0 + \mathcal{L}[f] \tag{II-6a}$$

Now, since $(sA+B)^{-1}$ exists for s = c it also exists for all s with |s| large enough in a half plane; assuming such a region intersects the region of convergence of $\mathcal{L}[f]$ we can solve (II-6a) to get

$$\mathcal{L}[x(t)] = (sA+B)^{-1}(Ax_0 + \mathcal{L}[f]) \tag{II-6b}$$

This latter is evaluated through the Drazin inverse in terms of the expansion [5, p. 753]

$$(sA+B)^{-1} = \{\tilde{A}^D(s1_n + \tilde{A}^D\tilde{B})^{-1} + \tilde{B}^D(1_n - \tilde{A}\tilde{A}^D) \times \sum_{k=0}^{I}(-1)^k(\tilde{A}\tilde{B}^D)^k s^k\}(cA+B)^{-1} \tag{II-7}$$

where 1_n is the nxn matrix identity and, recalling from above, I is the index of A. Using the facts that $\tilde{A}$ and $\tilde{B}$ commute, as well as their Drazin inverses [by (II-1a)], application of (II-7) to the x_0 portion of (II-6b) yields

$$(sA+B)^{-1}Ax_0 = \tilde{A}\tilde{A}^D(s1_n + \tilde{A}^D\tilde{B})^{-1}x_0 + \left(\sum_{k=0}^{I}(-1)^k(\tilde{A}\tilde{B}^D)^k s^k\right)\tilde{A}\tilde{B}^D(1_n - \tilde{A}\tilde{A}^D)x_0 \tag{II-8}$$

which, as the left side vanishes at $s = \infty$, gives [5, p. 756, Eq. (28)]

$$(1_n - \tilde{A}\tilde{A}^D)x_0 = 0 \tag{II-9}$$

as the condition for x_0 to be consistent. Converting (II-6b) to the time - domain, using (II-5), (II-7), and (II-8), we finally get (with * denoting the convolution, and 1(·) = unit step function [used to guarantee the causal impulse response])

$$x(t) = \tilde{A}\tilde{A}^D e^{-\tilde{A}^D\tilde{B}t}x_0 + \tilde{B}^D(1_n - \tilde{A}\tilde{A}^D)\sum_{k=0}^{I}(-1)^k \times (\tilde{A}\tilde{B}^D)^k \tilde{f}^{(k)}(t) + e^{-\tilde{A}^D\tilde{B}t}1(t) * (\tilde{A}^D\tilde{f}(t)) \tag{II-10}$$

It should be noted that if A is nonsingular, that is of index I = 0, then $1_n = \tilde{A}\tilde{A}^D$ and no ~'s are needed. In this nonsingular case the middle term on the right of (II-10) is absent, the solution (II-10) agrees with classical results, and any x_0 is consistent [by (II-9)]. Further, (II-10) shows that x exists for any distributional f for which the convolution exists, in particular this is the case if $\tilde{f}(t)$ has support bounded on the left (that is, is zero until a finite t) [6, p. 367]. It should be observed, as is classical, that if $f(t) \neq 0$ for $t < 0$ then in general $(x(0) \neq x_0$ but if $f(t) = 0$ for $t < 0$ then $x(0-) = x_0$.

III. THE SEMI - STATE EQUATIONS

In this section we use the general description for circuit elements [6, p. 48] in conjunction with a graph theoretical technique introduced for adjoint networks [7] to set up the semi - state equations.

Consider a finite linear time - invariant circuit, that is, a connection of a finite number of linear time - invariant capacitors, voltage controlled current sources (VCCS's), and elements which have equivalent circuits finitely represented by these (as resistors, inductors, transistors, gyrators, transformers, etc. [8]), as well as independent voltage and current sources exciting the linear circuit. If we denote graph branches as in Fig. 1 (where independent sources embed

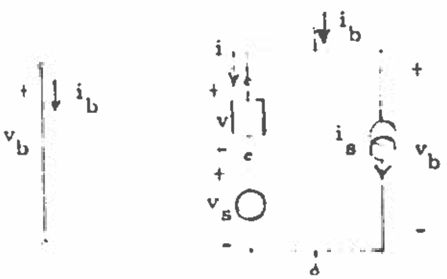


Fig. 1. General Circuit Graph Branch with Variables and Polarities

the terminals of linear circuit elements of variables v and i), let there be b individual branches in the graph of the circuit. Let v and i be the b - vectors of voltages across and currents through the linear circuit elements and v_b and i_b the total (graph) branch voltage and current b - vectors with v_s and i_s the corresponding independent source vectors. Proceeding in the time - domain with p the derivative operator and assuming bxb matrices M(p) and N(p) for the general description we write the <u>general description</u> of the linear circuit elements, branch by branch, as

$$M(p)v + N(p)i = 0 \qquad \text{(III-1)}$$

Next we choose a tree for the graph (the tree having t branches) and let v_t be the resulting t - vector of tree branch voltages, i_ℓ the resulting ℓ - vector of the ℓ link branch currents and C and $\mathcal{J}$ the corresponding cut - set and tie - set matrices. Then Kirchoff's laws for the branches and graph are

$$i_b = i + i_s = \mathcal{J}^T i_\ell \qquad \text{(III-2a)}$$

$$v_b = v + v_s = C^T v_t \qquad \text{(III-2b)}$$

where the superscript T denotes matrix transposition. Solving for v and i through the right sides of (III-2a, b) and substituting the results into (III-1) yields

$$[M(p)C^T, N(p)\mathcal{J}^T]\, x = M(p)v_s + N(p)i_s \qquad \text{(III-3a)}$$

where

$$x = \begin{bmatrix} v_t \\ i_\ell \end{bmatrix} \qquad \text{(III-3b)}$$

In terms of the unknown b - vector x (III-3a) is a set of b equations to be solved. We next put these into semi - state form.

In formulating the general description (III-1) let us assume for convenience that equivalences have been used such that only capacitors and VCCS's are present in which case the only elements which yield actual p dependence of M and N are the capacitors (which may have come from inductor equivalences). Consequently, we insure that the entries of M(p) are polynomial of at most degree one in p by using an admittance description which similarly guarantees that N(p) is to be the bxb identity. The left - hand bxb coefficient matrix in (III-3a) can thus be written

$$\mathcal{C}p + \mathcal{B} = [M(p)C^T, N(p)\mathcal{J}^T] \qquad \text{(III-4)}$$

where $\mathcal{C}$ and $\mathcal{B}$ are bxb constant matrices. The describing equations (III-3a) are then

$$\mathcal{C}\dot{x} + \mathcal{B}x = f, \quad f = M(p)v_s + N(p)i_s \qquad \text{(III-5)}$$

Note that in actual fact we did not need to use the capacitor and source replacements made in order to achieve these equations; the replacements were merely a convenience (more theoretical than practical) and (III-5) would have resulted even had we not used this convenience.

Usually the actual input u is only a subset of the possible independent source variables v_s and i_s, in which case there is a projection matrix $\mathcal{P}$ such that simultaneously

$$u = \mathcal{P}\begin{bmatrix} v_s \\ i_s \end{bmatrix} \qquad \text{(III-6a)}$$

and

$$\begin{bmatrix} v_s \\ i_s \end{bmatrix}_{\text{actual}} = \mathcal{P}^T u \qquad \text{(III-6b)}$$

This gives

$$f = [M(p), N(p)]\mathcal{P}^T u = (\mathcal{E} + \mathcal{F}p)u \qquad \text{(III-7)}$$

Likewise, there are matrices $\mathcal{J}$ and $\mathcal{K}$ which give the output y as a linear combination of the semi -

state variables x and the input u since any voltage and current in the circuit can be expressed as a linear combination of the tree branch voltages, the link currents and the source inputs (by (III-2a, b)); $y = \mathcal{J}x + \mathcal{S}u$

Summarizing:

$$\mathcal{G}\dot{x} + \mathcal{B}x = \mathcal{F}u + \mathcal{E}\dot{u} \qquad \text{(III-8a)}$$

$$y = \mathcal{J}x + \mathcal{S}u \qquad \text{(III-8b)}$$

where $\mathcal{G}$, $\mathcal{B}$, $\mathcal{F}$, $\mathcal{E}$, $\mathcal{J}$, $\mathcal{S}$ are real constant matrices. These equations we will call a set of <u>semi state equations</u>, with x the semi - state variables, for the circuit with input u and output y. Equation (II-10) gives a complete solution in terms of the Drazin inverse, with $A = \mathcal{G}$, $B = \mathcal{B}$, $f = \mathcal{F}u + \mathcal{E}\dot{u}$, for any consistent initial conditions. Because there are many possible trees it is clear that the semi - state equations are not unique for a given circuit.

Note that if there are no capacitor loops, the last ℓ rows of $\mathcal{G}$ are zero in which case i_ℓ can be expressed, through the (2, 1) entry of $\mathcal{B}$ (and source entries) in terms of v_t (N(p) being the identity). Further, if the output is dependent only upon v_t then the dimension of the semi - state can be reduced to the number of tree branches, t, with x being replaced by v_t with final semi - state equations still being of the form of (III-8). This reduction will clearly show up in the following example. Note further that by replacing voltage sources by gyrator loaded current sources $\mathcal{E} = 0$ can be guaranteed. Also, by using separate branches for the excitation sources, $\mathcal{S} = 0$ can be assured. Finally if u is actually any nonsingular linear transformation of actual voltage or currents of sources, as for example for scattering variables, then (III-6b) will hold (though $\mathcal{P}$ can no longer be considered as solely a projection matrix). Thus we have our <u>main</u> result:

> If the input u is a linear combination of independent source voltages and currents and the output y is a linear combination of any voltages and currents in the circuit, then for any linear, finite, time-invariant circuit there exists a semi - state vector x and constant matrices $\mathcal{G}$, $\mathcal{B}$, $\mathcal{P}$, $\mathcal{J}$ such that

$$\mathcal{G}\dot{x} + \mathcal{B}x = \mathcal{P}u \qquad \text{(III-9a)}$$

x/ $$y = \mathcal{J}x \qquad \text{(III-9b)}$$

Subject to

$$x(0-) = x_0 \qquad \text{(III-9c)}$$

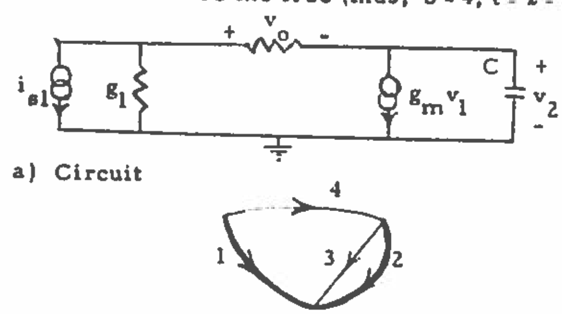
$$u(t) = 0 \quad \text{for } t < 0 \qquad \text{(III-9d)}$$

then for any consistent initial conditions x_0 there exists for any distributional input u a unique semi-state x, this being found via the Drazin inverse via (II-10).

IV. EXAMPLE

Most of the important points from the previous sections can be illustrated by the following simple example.

Consider the circuit of Fig. 2a) where the input is chosen as $i_{s1} = u$, the output is $y = v_o$ and the circuit graph is taken as in part b) of the figure where branches 1 and 2 are the tree (thus, $b = 4$, $t = \ell = 2$).



a) Circuit

b) Graph

Fig. 2. Example Circuit

The equations $i_b = \mathcal{J}^T i_\ell$ and $v_b = \mathcal{C}^T v_t$ are written as, with the subscript b on a vector denoting its entries are from v_b or i_b (as opposed to v or i),

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}_b = \begin{bmatrix} 0 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_3 \\ i_4 \end{bmatrix}_b \qquad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_b \qquad \text{(IV-1)}$$

and the general description $M(p)v + N(p)i = -Y(p)v + 1_4 i = 0$ is

$$\begin{bmatrix} -g_1 & 0 & 0 & 0 \\ 0 & -pC & 0 & 0 \\ -g_m & 0 & 0 & 0 \\ 0 & 0 & 0 & -g_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} +$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(IV-2)}$$

By simple multiplication, using (IV-1) & (IV-2), $[M\mathcal{C}^T, N\mathcal{J}^T]x = Mv_s + Ni_s$, (III-3a), becomes

$$\begin{bmatrix} -g_1 & 0 & 0 & -1 \\ 0 & -pC & -1 & 1 \\ -g_m & 0 & 1 & 0 \\ -g_4 & g_4 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_3 \\ i_4 \end{bmatrix}_b = \begin{bmatrix} i_{s1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(IV-3a)}$$

while

$$u = i_{s_1} = P\begin{bmatrix} v_s \\ i_s \end{bmatrix} = [0\ 0\ 0\ 0\ 1\ 0\ 0\ 0]\begin{bmatrix} v_s \\ i_s \end{bmatrix} \qquad \text{(IV-3b)}$$

On writing the left coefficient matrix as $\mathcal{G}p + \mathcal{B}$ and the right source term as $P^T u$ we have on multiplying by -1

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & pC & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_3 \\ i_4 \end{bmatrix}_b + \begin{bmatrix} g_1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ g_m & 0 & -1 & 0 \\ g_4 & -g_4 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_3 \\ i_4 \end{bmatrix}_b = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u;$$

$$x = \begin{bmatrix} v_1 \\ v_2 \\ i_3 \\ i_4 \end{bmatrix}_b, \quad u = i_{s1} \qquad \text{(IV-4a)}$$

which is (III-8a). Also we see

$$y = v_o = [0\ 0\ 0\ g_4^{-1}] \begin{bmatrix} v_1 \\ v_2 \\ i_3 \\ i_4 \end{bmatrix}_b = [1\ -1\ 0\ 0] \begin{bmatrix} v_1 \\ v_2 \\ i_3 \\ i_4 \end{bmatrix}_b \qquad \text{(IV-4b)}$$

We note that the last two rows of (IV-4a) can be solved for i_ℓ in terms of v_t, thus,

$$i_\ell = \begin{bmatrix} i_3 \\ i_4 \end{bmatrix}_b = \begin{bmatrix} g_m & 0 \\ g_4 & -g_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_b \qquad \text{(IV-5)}$$

Substituting into the first two rows of (IV-4a) gives

$$\begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix}_b + \begin{bmatrix} g_1 + g_4 & -g_4 \\ g_m - g_4 & g_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_b = \begin{bmatrix} -i_{s1} \\ 0 \end{bmatrix} \qquad \text{(IV-6a)}$$

with

$$v_o = [1\ \ -1] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_b \qquad \text{(IV-6b)}$$

This is another set of semi-state equations, with $x^T = [v_1, v_2]_b = v_t^T$, obtainable directly as the node-datum equations of the circuit.

To illustrate the use of the Drazin inverse we apply it to this last set of semi-state equations. We note that $A = 0 \dotplus C$ has index $I = 1$ and that $A^D = 0 \dotplus (1/C)$. We have

$$(cA + B)^{-1} = \begin{bmatrix} g_1 + g_4 & -g_4 \\ g_m - g_4 & cC + g_4 \end{bmatrix}^{-1} = [cC(g_1 + g_4) + g_4(g_1 + g_m)]^{-1} \begin{bmatrix} cC + g_4 & g_4 \\ g_4 - g_m & g_1 + g_4 \end{bmatrix} \qquad \text{(IV-7)}$$

which exists for $c = 0$ for example; choosing $c = 0$ and assuming B nonsingular

$$\tilde{A} = (cA + B)^{-1} A = C(g_1 + g_m)^{-1} \begin{bmatrix} 0 & 1 \\ 0 & (g_1 + g_4)g_4^{-1} \end{bmatrix} \qquad \text{(IV-8a)}$$

$$\tilde{B} = (cA + B)^{-1} B = 1_2 \qquad \text{(IV-8b)}$$

$$\tilde{f} = (cA + B)^{-1} f = i_{s1}(g_1 + g_m)^{-1} \begin{bmatrix} 1 \\ (g_4 - g_m)g_4^{-1} \end{bmatrix} \qquad \text{(IV-8c)}$$

By writing $\tilde{A} = b \begin{bmatrix} 0 & 1 \\ 0 & a \end{bmatrix}$ with $a = (g_1 + g_4)g_4^{-1}$, $b = C(g_1 + g_m)^{-1}$, we find $T = \begin{bmatrix} 0 & t_{12} \\ -at_{22} & t_{22} \end{bmatrix}$ for $\tilde{A} = T^{-1}[ab \dotplus 0]T$ in which case $\tilde{A}^D = \begin{bmatrix} 0 & (a^2 b)^{-1} \\ 0 & (ab)^{-1} \end{bmatrix}$ for any nonzero t_{12}, t_{22}. Thus

$$\tilde{A}^D = g_4(g_1 + g_m)C^{-1}(g_1 + g_4)^{-1} \times \begin{bmatrix} 0 & g_4(g_1 + g_4)^{-1} \\ 0 & 1 \end{bmatrix} \qquad \text{(IV-9)}$$

Note that $\tilde{A}^D \neq A^D[(cA + B)^{-1}]^D = A^D B =$

$$C^{-1}\begin{bmatrix}0 & 0\\ g_m-g_4 & g_4\end{bmatrix}$$ counter to what one might expect. Then, since $\tilde{A}\tilde{A}^D = \begin{bmatrix}0 & 1/a\\ 0 & 1\end{bmatrix}$

$$\tilde{A}\tilde{A}^D = \begin{bmatrix}0 & g_4(g_1+g_4)^{-1}\\ 0 & 1\end{bmatrix} \quad \text{(IV-10a)}$$

$$1_2 - \tilde{A}\tilde{A}^D = \begin{bmatrix}1 & -g_4(g_1+g_4)^{-1}\\ 0 & 0\end{bmatrix} \quad \text{(IV-10b)}$$

$$e^{-\tilde{A}^D\tilde{B}t} = \begin{bmatrix}1 & a^{-1}[e^{-t/ab}-1]\\ 0 & e^{-t/ab}\end{bmatrix} =$$

$$\begin{bmatrix}1 & g_4(g_1+g_4)^{-1}[e^{-\{g_4(g_1+g_m)t/C(g_1+g_4)\}}-1]\\ 0 & e^{-\{g_4(g_1+g_m)t/C(g_1+g_4)\}}\end{bmatrix} \quad \text{(IV-10c)}$$

$$\tilde{A}\tilde{A}^D e^{-\tilde{A}^D\tilde{B}t} =$$

$$\begin{bmatrix}0 & g_4(g_1+g_4)^{-1}e^{-\{g_4(g_1+g_m)t/C(g_1+g_4)\}}\\ 0 & e^{-\{g_4(g_1+g_m)t/C(g_1+g_4)\}}\end{bmatrix} \quad \text{(IV-10d)}$$

$$\tilde{A}^D\tilde{f} = g_4C^{-1}(g_1+g_4)^{-2}i_{sl}\begin{bmatrix}g_4-g_m\\ g_4\end{bmatrix} \quad \text{(IV-10e)}$$

from which x(t) can be evaluated using (II-10). For example if $i_{sl} = 0$ then

$$x(t) = \begin{bmatrix}v_1\\ v_2\end{bmatrix} = \tilde{A}\tilde{A}^D e^{-\tilde{A}^D\tilde{B}t} x_0 =$$

$$v_2(0)\, e^{-\{g_4(g_1+g_m)t/C(g_1+g_4)\}}\begin{bmatrix}g_4(g_1+g_4)^{-1}\\ 1\end{bmatrix} \quad \text{(IV-11a)}$$

which is readily checked directly from the (single time - constant) circuit in this case. Note that

$$x_0 = \begin{bmatrix}g_4(g_1+g_4)^{-1}\\ 1\end{bmatrix} v_2(0) \quad \text{(IV-11b)}$$

is the consistent initial condition vector in this case.

V. DISCUSSION

In Section III we have shown that every finite linear time - invariant circuit has a semi - state description, equations (III-8), which according to Section II can be uniquely solved for consistent initial conditions through the use of the Drazin inverse, as per equation (II-10).

The semi - state equations have the distinct advantage of resulting directly from the circuit element laws through the application of Kirchhoff's laws to the topological configuration. Thus, the manipulations to reduce the equations to state variable form are avoided. Since these eliminations to obtain state variables essentially reduce the coefficient matrix G of the semi - state derivative, $\dot{x}$, to be nonsingular, our theory takes them into account through the Drazin inverse which in essence really just inverts the nonsingular part of a matrix, according to equation (II-3b). To be sure there are a number of other generalized inverses which one may wish to consider [9][10], but as illustrated here the Drazin inverse is particularly effective, and, by virtue of its uniqueness, most convenient.

By way of generalization, it is clear that the semi-state equations (III-8 or 9) hold for finite time - varying linear circuits since the general description M(p, t)v+ N(p, t)i = 0 holds with M and N polynomial in the derivative operator p of degree one. However, a generalization of the solution (II-10) needs to be developed. Similarly, by going to an infinite dimensional semi - state, (III-9) will hold for distributed circuits. A similar quantity to the semi - state, called the pseudostate, has been previously introduced in [11] for proving the stability of finite passive time - variable circuits.

In short we believe the semi - state description gives a powerful tool of circuit analysis which when coupled with the Drazin inverse opens up new possibilities, not the least of which is improved computational techniques.

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