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SYNTHESIS OF PASSIVE NETWORKS FOR NETWORKS ACTIVE AT p_o

by

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ABSTRACT

If a one or two-port network, N, has $q_+ (p_o) \leq 0$, synthesis methods are given for passive embedding networks, N_P, which yield a natural frequency at p_o. The synthesis is based on the Y matrix. However, the synthesis is general, since N's with Z are treated dually, and a procedure is given for converting those N without Y or Z to ones which have a Y or Z while leaving q_+ unchanged. A synthesis for some n-ports is also given which uses the one or two-port synthesis after shorting appropriate ports.

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I. INTRODUCTION

A. PROBLEM AND RESULTS

Until recently the design of active circuits has been a semi-haphazard process. One usually followed the designs of previous workers, incorporating small improvements which mainly resulted from trial and error. However in 1957, Thornton,[1*] initiated a study of the limitations of the natural frequencies of such devices. This was followed in early 1960 by the work of Desoer and Kuh,[2] Thornton's paper is concerned with determining the possible natural frequencies of an active resistive device with parasitic capacitance which is embedded in a transformer network. Although some special results are presented, a general treatment is only touched upon. In contrast Desoer and Kuh develop a criterion for an arbitrary active device, embedded in a passive network, to possess a natural frequency. Their cirtierion is that $q_+(p_o) \leq 0$ at the required frequency, p_o, (Ref. 2, p. 15). Thus one now knows a restriction on the switching speed of a flip-flop, say, whereas previously this was determined experimentally.

However, more is desired. We would like to be able to synthesize a passive network such that, when a given active device is embedded in it, a desired natural frequency results. This is the subject of this report. In particular, we wish to investigate the following question.

"Given an (active) network for which $q_+(p_o) \leq 0$ for p_o in Re $p \geq 0$ does there exist a finite passive embedding network such that the two networks combined support a natural frequency at p_o? If so, how is the passive network obtained?"

By supporting a natural frequency at p_o we will understand that a mode is supported at the frequency p_o in the sense of Desoer and Kuh[2], (p. 4). A difficulty with this concept will later be discussed,

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*Refers to the bibliography.

but using it, the following results are obtained. In Sections II, III, IV and V the above question is answered in the affirmative when the active network is a one or two port by actually synthesizing the desired passive network (in Section V several pathological cases are excluded which appear to have no physical significance). Section VI gives a partial solution for the n-port.

B. REVIEW OF q_+ AND CONVENTIONS

Consider an n-port N which at first is assumed to possess an admittance matrix Y(p). Now let N be excited by the voltage vector $v(t) = Ve^{pt}$ where V is a vector of complex constants and $p = \sigma + j\omega$. Let a superscript tilde, ~ , denote matrix transposition, a superscript asterisk, *, denote complex conjugation and $Y_H(p)$ denote the Hermitian part of Y.

We now define, for $\sigma \geq 0$,

$$Q_+(V, p) = \begin{cases} \tilde{V}^* Y_H(p)\, V + (\sigma/|p|)\,|\tilde{V} Y(p)\, V| & \text{if } \omega \neq 0 \\ \tilde{V}^* Y_H(p)\, V & \text{if } \omega = 0 \end{cases} \qquad \text{(I. 1)}$$

here $|\ |$ denotes the absolute value of a complex number. Physically, if $\sigma \neq 0$, $\sigma^{-1} e^{2\sigma t} Q_+$ represents the upper limit on energy into N for a given v(t) at a given instant. Desoer and Kuh work with, (Ref. 2, p. 15),

$$q_+(p) = \min_{||V|| = 1} Q_+(V, p) \qquad \text{(I. 2)}$$

where for $\tilde{V} = [V_1, \ldots, V_n]$ we have $||V||^2 = \Sigma |V_i|^2$. From the meaning of Q_+ we see that $\sigma^{-1} e^{2\sigma t} q_+$ represents the smallest of the upper limits on the energy into N at a given instant for all normalized non-zero V. From the meaning attached to q_+ it should be physically clear that q_+ should depend only upon the device and not the mode of description. In other words we should be able to define q_+ even though a Y (or Z) matrix doesn't exist. Such a quantity is clearly obtained from

$$\mathcal{Q}_+(V, I, p) = \begin{cases} (1/2)[\tilde{V}^*I + \tilde{I}^*V] + (\sigma/|p|)\,|\tilde{V}I| & \text{if } \omega \neq 0 \\ (1/2)[\tilde{V}^*I + \tilde{I}^*V] & \text{if } \omega = 0 \end{cases} \qquad (I.\,1')$$

By the nature of our problem we must base our work on q_+, but it is important to note that if we find some non-zero V for which $Q_+ \leq 0$ then $q_+ \leq 0$. Thus, if in a specific instance, we are only interested in the fact that q_+ is non-positive and not in its exact value, we may profit by using Q_+ and avoid the tedious job of finding a minimum.

Now consider an n-port N which is connected in parallel to a passive n-port N_P. The combined networks from a new n-port N_o whose terminal pairs are taken as the common terminal pairs of N and N_P. We say that N supports a mode $v(t) = Ve^{p_o t}$, $\sigma_o \geq 0$, if the voltage v(t) can appear across N_o when the terminal currents of N_o are zero. Such an N has been called active at p_o, (Ref. 2, p. 4).* If N and N_P have admittance matrices Y and Y_P then the following results have been established.

1. (Ref. 2, p. 15). If N is active at p_o then necessarily $q_+(p_o) \leq 0$.
2. (Ref. 2, p. 7). N is active at p_o if and only if there exists some N_P such that det $[Y(p_o) + Y_P(p_o)] = 0$.

The second of these results is the key to the synthesis methods, since it gives an analytical way of determining if we have solved our problem. It should be pointed out that the definition of natural frequency used here differs from the usual one which rests upon initial conditions in energy storage elements, (Ref. 3, p. 7), since a network for which the determinant is zero for all p, such as an open circuit, can support any mode. This introduces a subtlety which is expanded upon in Example E-1 of Section II.

*In Ref. 2, $\sigma_o = 0$ was omitted from the definition of active to exclude L-C resonant circuits, however the results of Ref. 2 remain valid for $\sigma_o = 0$. Certainly we desire a synthesis of N_P for $\sigma_o = 0$.

For transforming one network into another we will have use for ideal transformer networks and gyrators. Consider two networks N_c and N whose admittance matrices are related by

$$Y_c = \tilde{T} Y T \tag{I. 3}$$

where T is a real matrix. Then N_c is obtained by connecting a transformer network to N, (Ref. 4, p. 233 and Ref. 5, p. 301 or Ref. 6, p. 85).

The notation for the gyrator must be clarified. Let

$$E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{I. 4}$$

Then the polarities for the gyrator are made clear by Fig. 1 for which

$$\begin{aligned} Z &= \gamma E \qquad \gamma = \text{gyration resistance} \\ Y &= -\gamma^{-1} E \end{aligned} \tag{I. 5}$$

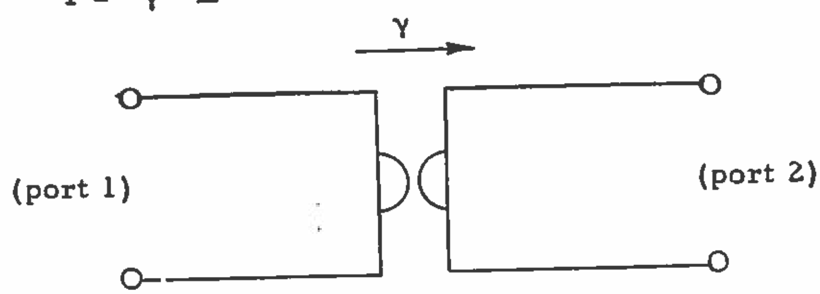

FIG. 1. --Gyrator notation.

We will also adhere to the following notation. 1_n will denote the unit matrix of order n, 0_n the corresponding zero matrix and $\dot{+}$ will mean the direct sum of two matrices. Further we will assume, as in Ref. 2, that Y(p) has rational elements with real coefficients.

C. METHOD

We will assume that a network N is given which possesses an admittance matrix Y at p_o and for which $q_+(p_o) \leq 0$. Clearly a dual situation holds if only an impedance matrix Z exists. In Section V, for $n = 2$, we will show how to obtain a Z or Y if neither exists.

Consequently, for $n = 2$, the assumption on Y is no restriction. At p_o, Y is a matrix of complex numbers and we write

$$Y = Y_{RS} + Y_{RSS} + jY_{IS} + jY_{ISS} \tag{I. 6}$$

where the subscripts R and I refer to real and imaginary parts and S and SS refer to symmetric and skew-symmetric matrices.

The synthesis of the passive network N_P will begin by assuming that $Y_{ISS} = 0$. Then for $Y_{ISS} \neq 0$ we will transform N such that $Y_{ISS} = 0$, when possible. By the use of transformers and gyrators we will transform N into a canonical network N_c. A passive network N_{Pc} will be obtained for N_c; the passive network for N will then consist of N_{Pc} and the transforming network as illustrated in Fig. 2.

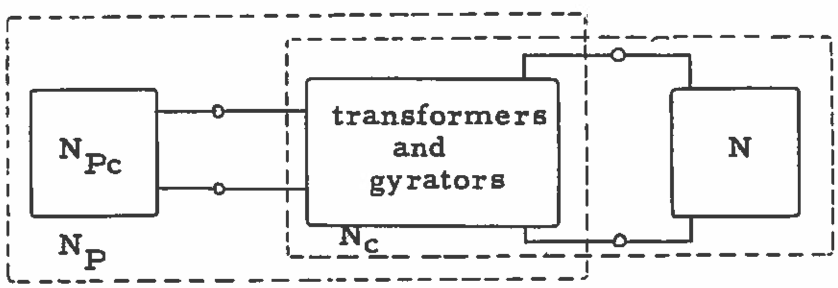

FIG. 2. --Construction of N_c and N_P.

From the physical interpretation of q_+ it should be clear that if N can be obtained from N_c then N and N_c have the same q_+. This will be justified analytically for the actual transformations that we will use.

As a consequence of the canonical forms actually used, we will have many cases to consider. For some of the cases several synthesis methods are available for N_{Pc}. In the body of the report we will present only the simplest synthesis methods, reserving some of the alternates for the appendix. At the very beginning we can assume $Y_{RSS} = 0$, since it can be absorbed in N_c. We will usually do this, but in cases where fewer gyrators will be used, Y_{RSS} terms will appear in the canonical forms.

By glancing at Eq. (I.1), it can readily be appreciated that we must consider two regions in ω_o. If $\omega_o \neq 0$, there is no loss in generality in assuming $\omega_o > 0$ since Y(p) has real coefficients.

II. SYNTHESIS OF N_P; n = 1

Here Y is a scalar which at p_o can be written as

$$Y(p_o) = g + jb \tag{II. 1}$$

Region 1: $\omega_o = 0$

As Y is real at p_o, b = 0. The condition $q_+ (p_o) \leq 0$ gives $g \leq 0$. We then let $Y_P(p) = -g$.

Region 2: $\omega_o > 0$

The condition $q_+(p_o) \leq 0$ now requires

$$g \leq 0 \tag{II. 2}$$

$$(\sigma_o b)^2 \leq (\omega_o g)^2$$

If $\sigma_o = 0$ we cancel b by an inducatance or a capacitance and g by a positive resistance. If $\sigma_o > 0$ we form

$$Y_P(p) = (1/2)[(-g/\sigma_o) - (b/\omega_o)]\ p + (1/2)[(-g/\sigma_o) + (b/\omega_o)](\sigma_o^2 + \omega_o^2)/p \tag{II. 3}$$

Y_P is positive real as a result of the constraints of Eq. (II.1). It should be noticed that the second of Eq. (II.2) is equivalent to the angle constraints for positive real functions, (Ref. 7, p. 114). Alternative Y_P are easily found, but they may not hold for all allowed g and b as this one does.

We can now appreciate a difficulty which can occur. It may happen that two different active networks have the same admittance matrix at p_o. When N_P is connected to these, the resulting determinant may be identically zero for all p for one while merely falling to zero at p_o for the other. The latter situation is the one actually desired, but, since we can only assume $Y(p_o)$ known, we can not tell which situation occurs in general. Of course if the properties of a device are known for all p we can actually check to see what happens. This is illustrated by the following example.

E-1: Let N have $Y(p) = -1$ and consider $p_o = (1/2) + j(\sqrt{3}/2)$. If we choose N_P to have $Y_1(p) = +1$ then $Y + Y_1 = 0$ for all p. Then N supports $e^{p_o t}$ even though no energy storage elements need to be considered. In contrast let N_P have $Y_2(p) = p + 1/p$ then $Y(p_o) + Y_2(p_o) = 0$ but this is not true for all p. Now consider another active network N' described by $Y' = -Y_2(p)$. Then at p_o N' and N are indistinguishable. However, $Y' + Y_1$ has only an isolated zero at p_o.

III. SYNTHESIS OF N_P; $n = 2$, $Y_{ISS} = 0$

We recall that we will generally assume $Y_{RSS} = 0$ as Y_{RSS} can be lumped in N_c. Clearly q_+ remains the same before and after Y_{RSS} is deleted, since Q_+ is independent of Y_{RSS}. With this assumption, we will generally transform $Y(p_o)$ to a canonical form $Y_c(p_o)$ through the use of Eq. (I.3), with T non-singular. This operation also leaves q_+ invariant since V in Q_+ is replaced by TV which assumes all values with V. In two cases the canonical form will require a cascade connection of gyrators in addition to the transformers. For these situations the invariance of q_+ is proven in Appendix 1.

Region 1: $\omega_o = 0$

Here $Y(p_o) = Y_{RS}$. We then diagonalize this to obtain $Y_c(p_o) = g_1 \dotplus g_2$. The condition $q_+ \leq 0$ requires that at least one of g_1 or g_2 be ≤ 0; through our diagonalization process we can assume it to be g_1. We then form, for all p, $Y_{Pc} = (-g_1) \dotplus 0$.

Region 2: $\omega_o > 0$

If $\sigma_o = 0$ we diagonalize Y_{IS} to get $Y_c(p_o) = (Y_{RS})_c + j[b_1 \dotplus b_2]$. b_1 and b_2 are then cancelled by inductances and capacitances. $(Y_{RS})_c$ is then diagonalized to $g_1 \dotplus g_2$ where we can assume $g_1 \leq 0$ by $q_+ \leq 0$. g_1 is then cancelled and the construction of the passive network in the form of Fig. 2 should be clear.

If $\sigma_o > 0$ we are apparently forced to consider the following mutually exclusive cases. Unfortunately there are many subcases, each leading to a different canonical form. We first note that Y_{RS} cannot be positive definite since $q_+ \leq 0$.

Case 1: Y_{RS} positive semi-definite (rank 0 or 1)

Case 2: Y_{RS} negative definite (rank 2)

Case 3: Y_{RS} negative semi-definite (rank 1)

Case 4: Y_{RS} indefinite (rank 2)

Case 1: Y_{RS} positive semi-definite (rank 0 or 1)

Clearly $q_+ = 0$. If Y_{RS} has rank zero we can diagonalize Y_{IS} to get $Y_c(p_o) = j[b_1 \dotplus b_2]$. $q_+ = 0$ then requires that at least one of b_1, b_2 be zero, and Y_c then has a zero determinant.

If Y_{RS} has rank one, we can first diagonalize Y_{RS} to $1 \dotplus 0$. The requirement $q_+ = 0$ then requires that the (2,2) term of the transformed Y_{IS} is zero. If the (1,2) term of the new Y_{IS} is also zero we have $Y_c(p_o) = [(1 + jb) \dotplus 0]$ which has a zero determinant. If the (1,2) term is not zero, we further transform by adding the second row and column to the first to have the (1,1) term zero in the new Y_{IS}. Thus we have arrived at

$$Y'(p_o) = \begin{bmatrix} 1 & jb \\ jb & 0 \end{bmatrix}$$

We now connect a gyrator as shown in Fig. 3 to obtain (see Appendix 2)

$$Y_c(p_o) = \begin{bmatrix} 1 & jb \\ -jb & b^2 \end{bmatrix}$$

which has a zero determinant.

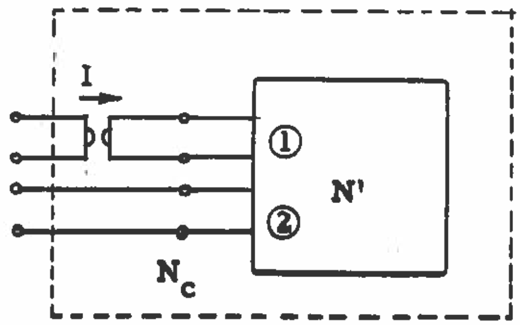

FIG. 3.--Gyrator connection to obtain Y_c.

Case 2: Y_{RS} negative definite (rank 2)

In this case we will save some gyrators by considering Y_{RSS} to be present. We simultaneously diagonalize Y_{RS} and Y_{IS} to get, (Ref. 8, p. 107),

$$Y_c(p_o) = -1_2 + gE + j[b_1 \dotplus b_2] \qquad \text{(III. 1)}$$

where E is as defined in Eq. (I.4). Such an admittance always has $q_+ < 0$ since we can find a non-zero V such that $|\tilde{V}Y_cV| = 0$, Appendix 3. Consequently there are no constraints on b_1 and b_2. We have three subcases to consider.

Case 2_a: $b_1b_2 = 0$

There is no loss in generality in assuming $b_1 = 0$. Then we let, for all p,

$$Y_{Pc} = [1 \dotplus 0] - gE$$

to obtain a zero determinant.

Case 2_b: $b_1b_2 > 0$

We let, for all p,

$$Y_{Pc} = \{[1 + (b_1/b_2)] \dotplus 0\} + \{[(b_1/b_2)(b_2^2 + 1)]^{1/2} - g\}E$$

which yields a zero determinant.

Case 2_c: $b_1 b_2 < 0$

We derive a new canonical form from Eq. (III.1) by normalizing the imaginary (2,2) term to $-b_1$.

$$Y_c'(p_o) = [-1 \dotplus (b_1/b_2)] + g'E + j[b_1 \dotplus (-b_1)]$$

where $g' = (-b_1/b_2)^{1/2} g$. We first add, for all p,

$$Y_{Pc}' = 1 \dotplus (-b_1/b_2)$$

We then add rows and columns of the resulting matrix to obtain a zero input admittance. This corresponds to connecting port one to port two and is illustrated in Fig. 4.

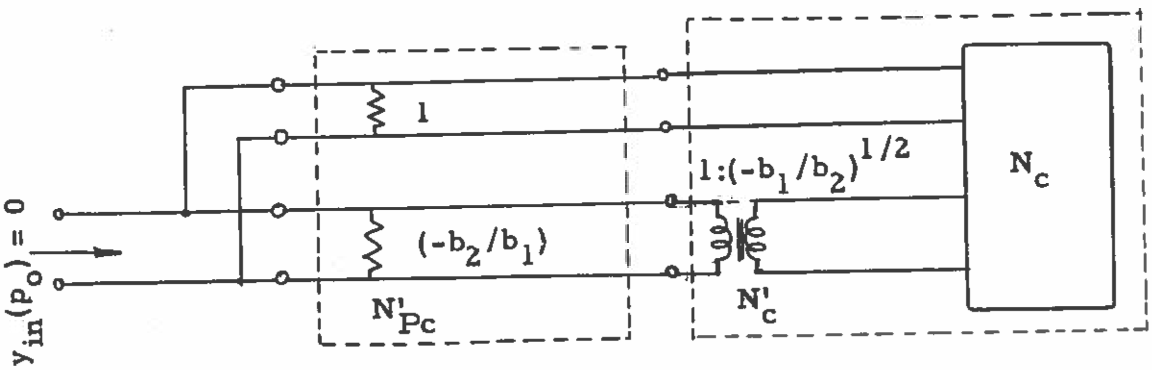

FIG. 4.--Illustration of Case 2_c synthesis.

It should be noted that if, in Eq. (III.1), $g = 0$ and $0 < |b_i| \leq (\omega_o/\sigma_o)$ for i = 1 or 2 then we can add a passive network to port i to get a zero determinant. This would then avoid the gyrator in Case 2_b.

The following will exhibit a simple Case 2 synthesis while clarifying the general procedure to be used.

E-2: Let N be the network so denoted in Fig. 5, for which

$$Y(p) = \begin{bmatrix} -1+p & 1 \\ 1 & -4+p+(1/p) \end{bmatrix}$$

Let $p_o = (1/2) + j(\sqrt{3}/2)$, then

$$Y(p_o) = \begin{bmatrix} (-1/2)+j(\sqrt{3}/2) & 1 \\ 1 & -3 \end{bmatrix}$$

We find, using

$$T = \begin{bmatrix} 0 & \sqrt{6} \\ 1/\sqrt{3} & \sqrt{2/3} \end{bmatrix}$$

and Eq. (I. 3), that

$$Y_c(p_o) = \tilde{T}Y(p_o)\,T = -1_2 + j[0 \dotplus 3\sqrt{3}]$$

The Case 2_a synthesis then gives Fig. 5.

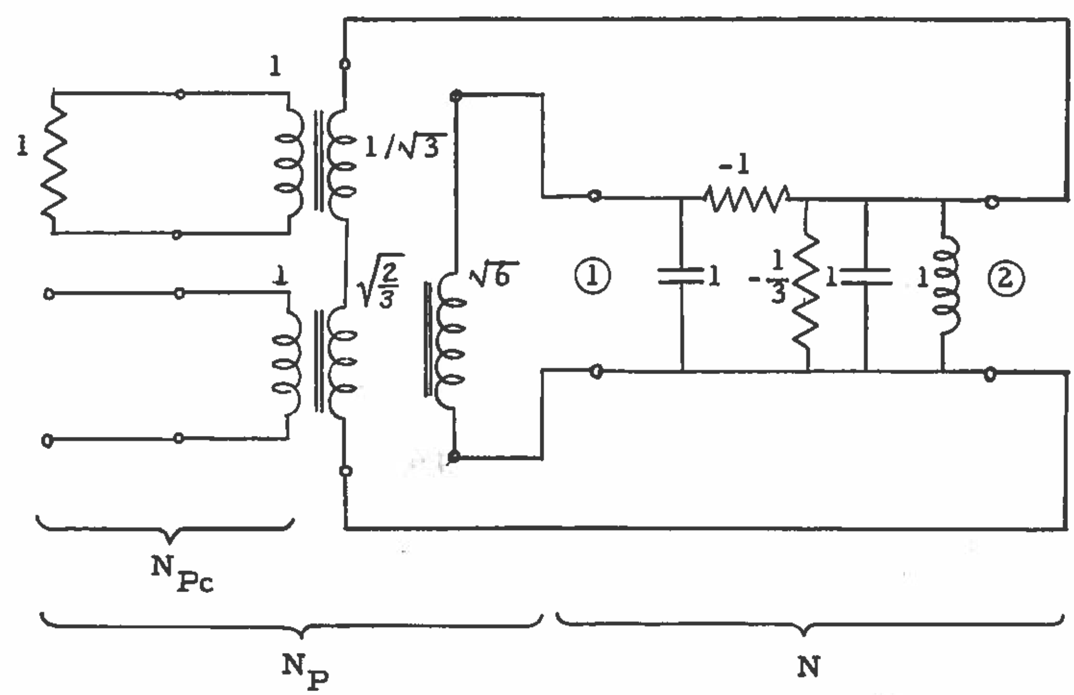

FIG. 5. --Networks for Example E-2.

Case 3: Y_{RS} negative semi-definite (rank 1)

We first diagonalize Y_{RS} to obtain

$$Y'(p_o) = [(-1) \dotplus 0] + g'E + j \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \tag{III. 2}$$

From this we obtain three canonical forms depending upon the vanishing or non-vanishing of b_{22} and b_{12}.

Case 3_a: $b_{22} \neq 0$

In Eq. (III. 2) we can add the second row and column to the first to eliminate the (1, 2) term. If the (1, 1) term of the new imaginary part is non-zero, we can normalize the (2, 2) term to equal the (1, 1), except possibly for sign. We have then

$$Y_c(p_o) = [(-1) \dotplus 0] + gE + j[b_1 \dotplus b_2] \quad \text{where } b_1 = \pm b_2 \text{ or } 0$$

Here $g = g'$ if $b_1 = 0$ or $g = \sqrt{\pm b_1/b_2}\, g'$ otherwise. Because we can find a V with $V_1 \neq 0$ such that $|\tilde{V} Y_c V| = 0$, this case always has $q_+ < 0$. Thus there is no restriction on b_1. However, we have two further cases to consider as far as synthesis is concerned.

Case 3_{a1}: $b_1 b_2 \geq 0$

We here add, for all p,

$$Y_{Pc} = [1 \dotplus 0] + (b_1 - g)E$$

to obtain a zero determinant.

Case 3_{a2}: $b_1 b_2 < 0$

We here add, for all p,

$$Y'_{Pc} = 1 \dotplus 0$$

and then apply feedback by connecting port one to port two to obtain a zero input admittance (compare with Case 2_c).

Case 3_b: $b_{22} = 0$ [in Eq. (III. 2)]

Here we again have two further subcases, this time depending on b_{12}.

Case 3_{b_1}: $b_{12} = 0$

Assuming that $g' = 0$, Eq. (III. 2) then takes the form

$$Y_c(p_o) = [(-1 + jb) \dotplus 0] \text{ where } b = b_{11}$$

Here Y_c already has a zero determinant which corresponds to $Q_+ = 0$ with $V_1 = 0$. Consequently b is not constrained if $q_+ = 0$. However, if $q_+ < 0$ Eq. (II. 2) shows that $b^2 < (\omega_o/\sigma_o)^2$ and synthesis of N_{Pc} results from Eq. (II. 3).

Case 3_{b_2}: $b_{12} \neq 0$

We now add the second row and column of Eq. (III. 2) to the first and then normalize the (1, 2) term to obtain

$$Y_c(p_o) = [(-1) \dotplus 0] + gE + j\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where $g = g'/b_{12}$. Again $q_+ < 0$, since there is a non-zero V such that $|\tilde{V}Y_cV| = 0$. We force the determinant to zero by adding, for all p,

$$Y_{Pc} = [0 \dotplus (1 + g^2)]$$

The following example will serve to illustrate a Case 3 synthesis.

E-3: Let N be as illustrated in Fig. 6. Then

$$Y(p) = \begin{bmatrix} -2+p & 1-p \\ -1-p & -3+2p \end{bmatrix}$$

Let $p_o = 1 + j1$, then

$$Y(p_o) = -\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + j\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} + E$$

A Case 3_{a_1} synthesis is required. Using Eq. (I. 3) with

$$T = (1/5)\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$$

we obtain

$$Y_c(p_o) = [(-1) \dotplus 0] + j(1/5)\, 1_2 + (\overset{1/5}{\cancel{}}/25)\, E$$

The final network is shown in Fig. 6. It should be noted that

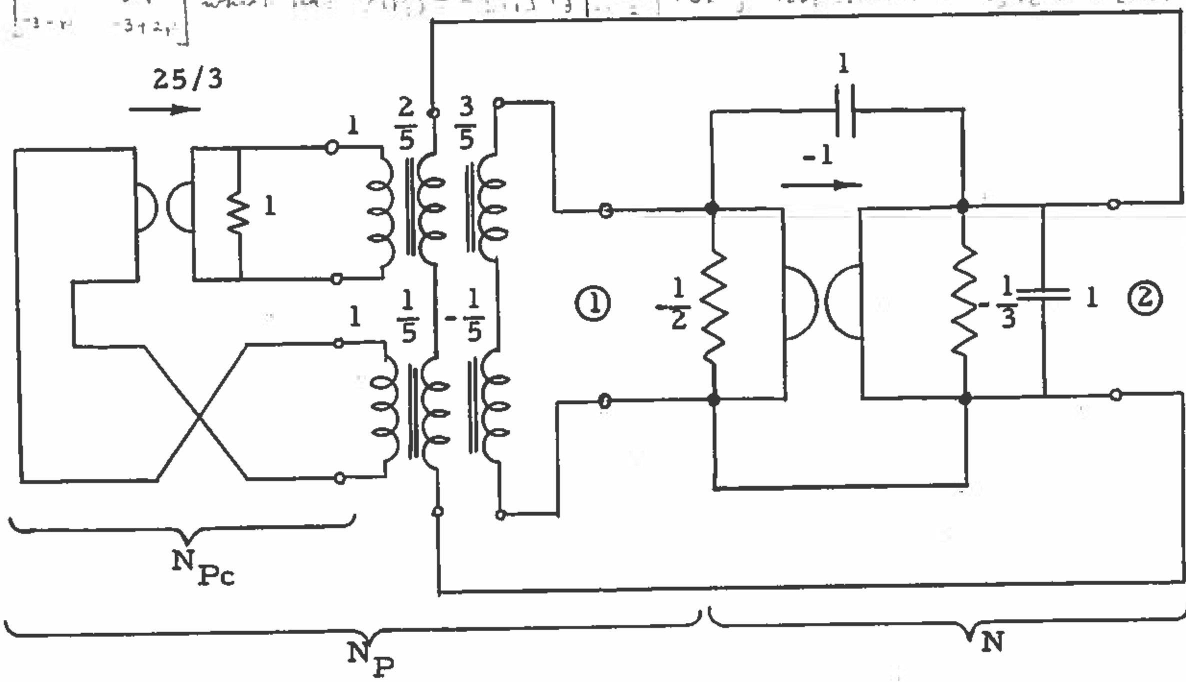

FIG. 6. --Example realization.

we could replace the gyrator-resistor network by a gyrator-L-C network in this case.

Case 4: Y_{RS} indefinite (rank 2)

Depending upon the rank of Y_{IS} we now have several cases.

Case 4_a: Y_{IS} of rank zero

We can diagonalize Y_{RS} to obtain

$$Y_c(p_o) = [1 \dotplus (-1)] + gE$$

We then add rows and columns (connect port one to port two) to get a zero input admittance. It should be noted that here we always have $q_+ < 0$.

Case 4_b: Y_{IS} of rank one

We begin by diagonalizing Y_{IS}

$$Y'(p_o) = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} + j\,[b_{11} \dotplus 0] + g'E \tag{III. 3}$$

$11_2 \dotplus \frac{3}{5} E$
$= [1 \dotplus 0]$
$+ (-\frac{2}{5}) E$
ses the
al circuit
15/3
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1-1 replac
3.

From this will be derived three canonical forms depending upon the value of g_{22}.

Case 4_{b_1}: $g_{22} > 0$

After assuming $g' = 0$ we now use elementary transformations to add the second row and column of Eq. (III. 3) to the first and normalize to obtain

$$Y_c(p_o) = [(-1) \dotplus 1] + j[b \dotplus 0]$$

The requirement $q_+ \leq 0$ yields $b^2 \leq (\omega_o/\sigma_o)^2$, as is seen by choosing $V_2 = 0$. Using Eq. (II. 3) we add a passive network to port one of N_c to obtain a zero determinant.

Case 4_{b_2}: $g_{22} < 0$

Using the same procedure as in the previous case we obtain

$$Y_c\,(p_o) = [1 \dotplus (-1)] + j[b \dotplus 0]$$

This clearly has $q_+ < 0$ (choose $V_1 = 0$), and we add, for all p,

$$Y_{Pc} = 0 \dotplus 1$$

to obtain a zero determinant.

Case 4_{b_3}: $g_{22} = 0$

Since Y_{RS} is indefinite we have $g_{12} \neq 0$. Using the transformation method of the previous two cases we can arrive at

$$Y_c(p_o) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + j[b \dotplus 0] + gE$$

Here we always have $q_+ < 0$ as is shown in Appendix 3. This is substantiated by the fact that we can normalize any non-zero b to ± 1. We here add, for all p,

$$Y_{Pc} = -(g + 1)E$$

to obtain a zero determinant.

Case 4_c: Y_{IS} of rank two

We must divide this case into two further cases depending upon whether Y_{IS} is definite or indefinite. The second of these calls for rather elaborate synthesis methods.

Case 4_{c_1}: Y_{IS} definite

To obtain a canonical form we simultaneously diagonalize Y_{RS} and Y_{IS} and then normalize Y_{RS} to obtain

$$Y_c(p_o) = [1 \dotplus (-1)] + j[b_1 \dotplus b_2] + gE \tag{III. 4}$$

We have two regions for b_2 which are of interest.

Case $4_{c_{1_\alpha}}$: $b_2^2 < (\omega_o/\sigma_o)^2$

Here we always have $q_+ < 0$, as is seen by choosing $V_1 = 0$. After cancelling gE by a gyrator we synthesize a passive network by Eq. (II. 3).

Case $4_{c_{1_\beta}}$: $b_2^2 \geq (\omega_o/\sigma_o)^2$

If $q_+ = 0$, Appendix 3 shows that we require $b_2^2 = (\omega_o/\sigma_o)^2$. As a consequence, a synthesis for $q_+ = 0$ follows that of the preceding case.

$b_1^2 = b_2^2$
also works
([illegible]
if $b_1 = -b_2$
[illegible]
to 2, if $b_1 = b_2$
[illegible]
$Y_{Pc} = gE$
$g = \sqrt{1+b_1^2}$

If $q_+ < 0$, Appendix 3 shows that we require $b_1^2 > b_2^2$ (Appendix 3 also shows that $\sigma_o[(1 + b_1^2)^{1/2} + (1 + b_2^2)^{1/2}] > 2|p_o|$ and that there exists a non-zero V such that $|\tilde{V}Y_cV| = 0$). Because Y_{IS} is definite we have $(b_1/b_2) > 0$ and we can force the determinant to zero by adding, for all p,

$$Y_{Pc} = [(1/b_2)(b_1 - b_2) \dotplus 0] + \{[(b_1/b_2)(b_2^2 + 1)]^{1/2} - g\} E$$

Case 4_{c_2}: Y_{IS} indefinite

We begin by diagonalizing Y_{RS} to get

$$Y'(p_o) = [1 \dotplus (-1)] + j \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} + gE \qquad \text{(III. 5)}$$

Let $B = [b_{ij}]$ be the second matrix on the right. We have three cases depending upon the form into which B can be brought by the congruency transformations of Theorem 1 of Appendix 4 (the subscripts on the following B's refer to the corresponding matrices in Appendix 4).

Case $4_{c_{2_a}}$: $B = B_I = [b_{11} \dotplus b_{22}]$

Letting $b_{11} = b_1$, $b_{22} = b_2$, Eq. (III. 5) is identical to Eq. (III. 4) except that instead of $(b_1/b_2) > 0$ we now have $(b_1/b_2) < 0$. The same subcases occur that were present in Case 4_{c_1}.

Case $4_{c_{2_{a_1}}}$: $b_2^2 < (\omega_o/\sigma_o)^2$

The properties and the synthesis method are identical to that of Case $4_{c_{1_a}}$.

Case $4_{c_{2_{a_2}}}$: $b_2^2 \geq (\omega_o/\sigma_o)^2$

We have the same properties as in Case $4_{c_{1_\beta}}$ except that $(b_1/b_2) < 0$. We now revert to that case, after first assuming $g = 0$ in Eq. (III. 5). To obtain this result we connect a gyrator in cascade with port one (as we did in Fig. 3). The new Y matrix is obtained by using the results of Appendix 2, and then multiplying the first row and column by $1 + b_1^2$; we get

$$Y_c(p_o) = [1 \dotplus (-1)] + j[(-b_1) \dotplus b_2]$$

which now is of the form required for Case $4_{c_{1_\beta}}$. It should be noticed that in general this method uses three gyrators. However, the final two of these are in cascade and can be replaced by a transformer. This replacement shows that N_P has an impedance matrix but no admittance matrix; the natural frequency results from a zero of the determinant of $Z + Z_P$. This procedure is illustrated by example E-4 which follows the remaining cases.

Case $4_{c_{2_\beta}}$:

$$B \Rightarrow B_{II} = \begin{bmatrix} b_{11} - b_{22} & \sqrt{-\Delta_b} \\ \sqrt{-\Delta_b} & 0 \end{bmatrix}; \quad \Delta_b = \text{det. } B$$

Appendix 4 shows that there is a non-singular real T such that

$$Y''(p_o) = \tilde{T} Y'(p_o)\, T = [1 \dotplus (-1)] + jB_{II} + g''E \qquad \text{(III. 6)}$$

From Eq. (III. 6) we will obtain two canonical forms, depending upon $b_{11} - b_{22}$, by elementary transformations.

Case $4_{c_{2_{\beta_1}}}$: $b_{11} - b_{22} \neq 0$

We first add $-\sqrt{-\Delta_b}/(b_{11} - b_{22})$ times the first row and column of Eq. (III. 6) to the second. We then normalize by multiplying the second row and column by $[(b_{11} - b_{22})/\sqrt{-\Delta_b}]$ to get

$$Y_c(p_o) = \begin{bmatrix} 1 & -1 \\ -1 & 1 - \{(b_{11} - b_{22})^2/(-\Delta_b)\} \end{bmatrix} + j(b_{11} - b_{22})\,[1 \dotplus (-1)] + g_c E \qquad \text{(III. 7)}$$

Recalling that $\Delta_b < 0$ by assumption, we now add, for all p,

$$Y'_{Pc} = [\{(b_{11} - b_{22})^2/(-\Delta_b)\} \dotplus 0]$$

and then add rows and columns (connect port one to port two) to obtain a zero input admittance.

Case $4_{c_{2_{\beta_2}}}$: $b_{11} - b_{22} = 0$

We first add 1/2 of the second row and column of Eq. (III. 6) to the first. Following this we subtract the first row and column from the second to get

$$Y_c(p_o) = \begin{bmatrix} 3/4 & -5/4 \\ -5/4 & 3/4 \end{bmatrix} + j\sqrt{-\Delta_b}\,[1 \dotplus (-1)] + g_c E \qquad \text{(III. 8)}$$

We now add, for all p,

$$Y'_{Pc} = [1 \dotplus 0]$$

and then add rows and columns (connect port one to port two) to obtain a zero input admittance.

It should be noticed that the $b_{11} - b_{22} = 0$ and $\neq 0$ cases can be taken care of by a single case. This results from adding $[(1/2\sqrt{-\Delta_b})(b_{22} - b_{11} + \sqrt{-\Delta_b})]$ times the second row and column of Eq. (III. 6) to the first and then subtracting the first row and column from the second. However the canonical form is much messier than those of Eqs. (III. 7) and (III. 8). Further, since we have always found an N_P when $B \Longrightarrow B_{II}$, Result 1 of Section I-B shows that q_+ is always ≤ 0 here. In fact choosing $V_1 = V_2$ shows that $q_+ < 0$. An example of Case 4_{c2_β} will be given as part of E-7, which is worked in Section IV.

Case $4_{c_{2_\gamma}}$:

$$B \Longrightarrow B_{III} = \begin{bmatrix} 0 & -\sqrt{-\Delta_b} \\ -\sqrt{-\Delta_b} & b_{22} - b_{11} \end{bmatrix}; \ \Delta_b = \det. \ B$$

This is the final and worst case of this section. By Appendix 4 we find a real, non-singular T such that Eq. (III. 5) becomes

$$Y''(p_o) = \tilde{T}Y'(p_o)\,T = [1 \dotplus (-1)] + jB_{III} + g''E \qquad \text{(III. 9)}$$

Here we can assume that $b_{22} - b_{11} \neq 0$, as otherwise this is covered by the treatment for B_{II}. We then have two subcases.

Case $4_{c_{2_{\gamma_1}}}$: $(b_{22} - b_{11})^2 \leq -4\Delta_b$ (recall $\Delta_b < 0$ by assumption)

We follow the procedure used to obtain Eq. (III. 7). Thus we first add $\sqrt{-\Delta_b}/(b_{22} - b_{11})$ times the second row and column to the first and then normalize by multiplying the first row and column by $[(b_{22} - b_{11})/\sqrt{-\Delta_b}]$. This gives

$$Y_c(p_o) = \begin{bmatrix} -1+[(b_{22}-b_{11})^2/(-\Delta_b) & -1 \\ -1 & -1 \end{bmatrix} + j(b_{22} - b_{11})[(-1) \dot{+} 1] + gE$$

We now add, for all p,

$$Y'_{Pc} = [\{4 - [(b_{22} - b_{11})^2/(-\Delta_b)]\} \dot{+} 0]$$

and then add rows and columns to obtain a zero input admittance. Note that again we always have $q_+ \leq 0$, since we have found an N_P.

Case $4_{c_{2_{\gamma_2}}}$: $(b_{22} - b_{11})^2 > -4\,\Delta_b$

We will reduce this to Case $4_{c_{2_a}}$. We apply Theorem 2 of Appendix 4, which shows that there is a real, non-singular T_c such that Eq. (III. 9) takes the form

$$Y_c(p_o) = \tilde{T}_c Y''(p_o) T_c = [1 \dot{+} (-1)] + j[b_1 \dot{+} b_2] + g_c E \qquad \text{(III. 10)}$$

In fact we have

$$b_1 \text{ (or } b_2) = -2t\sqrt{-\Delta_b} + t^2(b_{22} - b_{11})$$

$$b_2 \text{ (or } b_1) = (b_{22} - b_{11}) - 2t\sqrt{-\Delta_b} \qquad \text{(III. 11)}$$

$$t = (1/2)[(b_{22} - b_{11}) \pm \sqrt{(b_{22} - b_{11})^2 + 4\Delta_b}\,]/\sqrt{-\Delta_b}$$

Eq. (III. 10) now falls under the description of Case $4_{c_{2_a}}$. Thus if $q_+ \leq 0$ that case applies and gives a synthesis. However, it has not yet been determined under what constraints on the b_{ij}, satisfying $(b_{22} - b_{11})^2 > -4\Delta_b$, $q_+ \leq 0$. Example E-6 shows that q_+ may be > 0, and E-5 shows that $q_+ \leq 0$ can also occur.

The following instructive examples illustrate that Case 4 synthesis.

E-4: Let N be as shown in Fig. 7. Then

$$Y(p) = [(5 - 4p) \dot{+} (-3 + 2p)]$$

Let $p_o = 1 + j1$ then

$$Y(p_o) = [(1 - j4) \dot{+} (-1 + j2)]$$

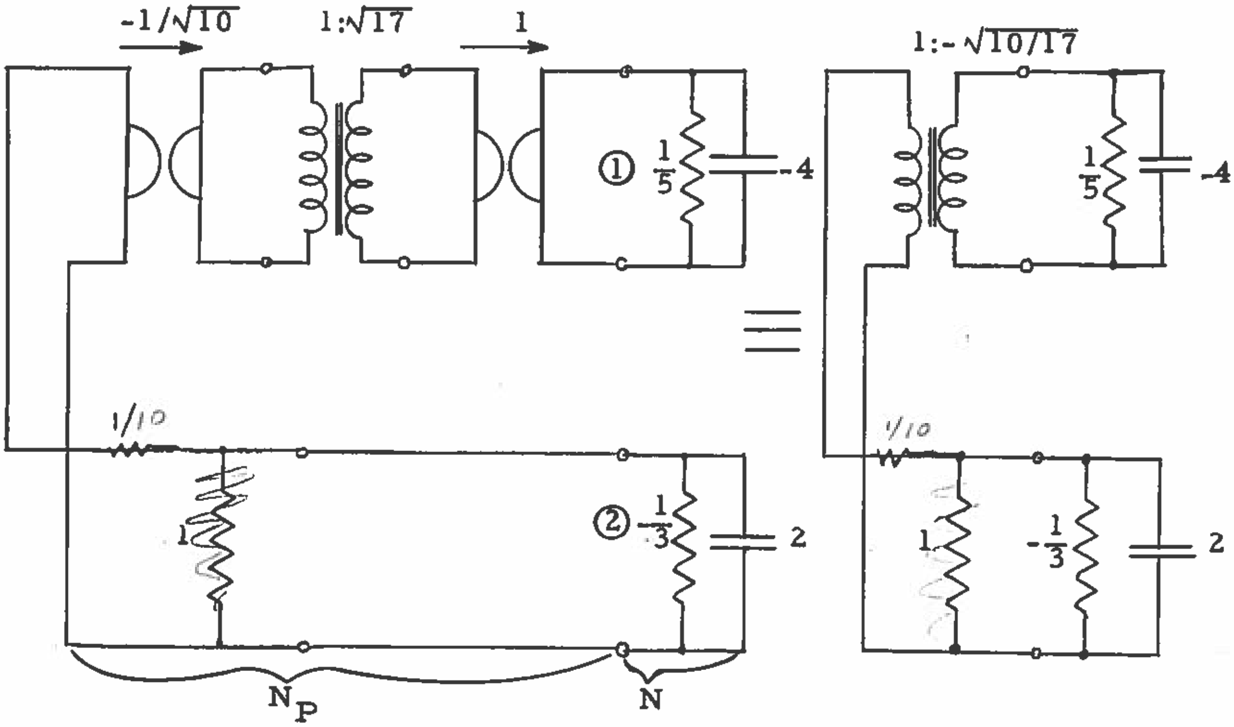

FIG. 7. --Networks for E-4.

which requires a Case $4_{c_{2}a_{2}}$ synthesis. Connecting a gyrator in cascade and multiplying the first row and column of the resulting matrix by $\sqrt{17}$ we get

$$Y_c(p_o) = [(1 + j4) \dotplus (-1 + j2)]$$

Connecting

$$Y_{Pc} = [1 \dotplus 0] + \sqrt{10}\,E$$

in parallel yields a zero determinant. Figure 7 shows the final realization. Here the transformer and the two gyrators in cascade have been replaced by their transformer equivalent. Also N_P has

replace by P. 171 of progress

$$Z_P(p) = \begin{bmatrix} 10/17 & -\sqrt{10/17} \\ -\sqrt{10/17} & 1 \end{bmatrix}$$

which is singular.

E-5: Let N be the network shown in Fig. 8. Then

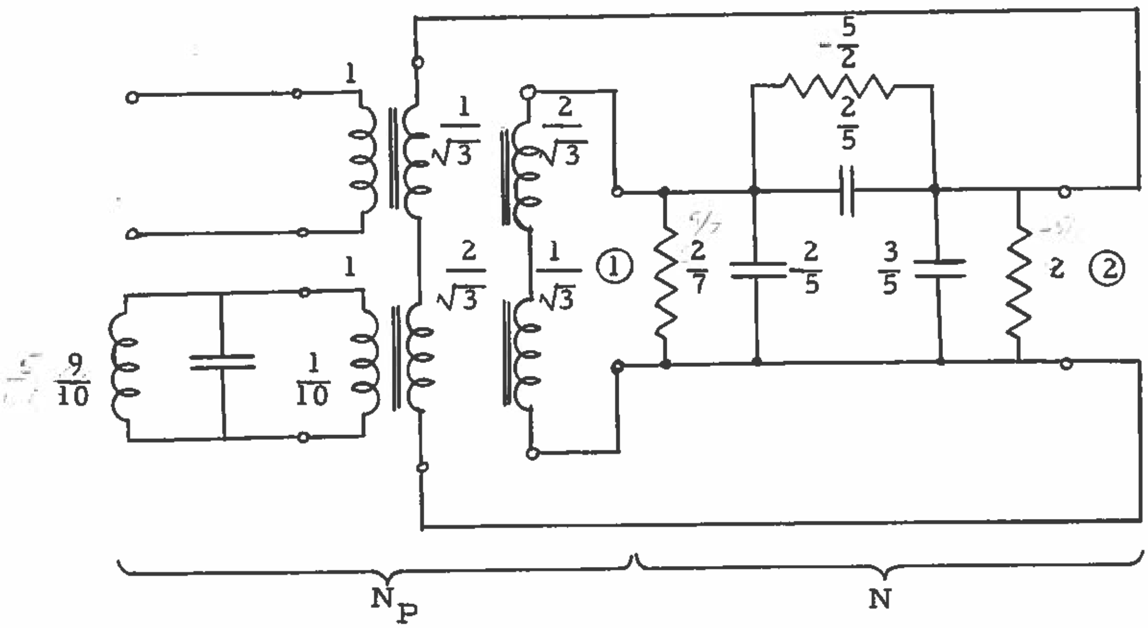

FIG. 8. --Networks for E-5.

$$Y(p) = \begin{bmatrix} 1+p & (2/5)+(2p/5) \\ (2/5)+(2p/5) & -2+p \end{bmatrix}$$

and for $p_o = 1 + j5$ we have

$$Y(p_o) = [1 \dotplus (-1)] + j\begin{bmatrix} 0 & -2 \\ -2 & 5 \end{bmatrix} \qquad \text{(III. 12)}$$

We have Case $4_{c2_{\gamma 2}}$ with $t = 2$ or $1/2$. Let $t = 2$ then with

$$T = (1/\sqrt{3})\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

we get

$$Y' = \tilde{T}YT = [(1 - j1) \dotplus (-1 + j4)] \qquad \text{(III. 13)}$$

which is treated by Case $4_{c2_{\alpha 1}}$. The final network is shown in Fig. 8. Consequently N is active at p_o and Case $4_{c2_{\gamma 2}}$ actually exists.

E-6: Consider the network of Fig. 9. Then

$$Y(p) = \begin{bmatrix} 1 & 2-2p \\ 2-2p & -6+5p \end{bmatrix}$$

Let $p_o = 1 + j1$ then $Y(p_o)$ is the same as given in E-5, Eq. (III. 12). Using the same transformation, Eq. (III. 13) is valid. However, now $4 = b_2 > \omega_o/\sigma_o = 1$ and $b_1^2 = 1 < 16 = b_2^2$. Consequently $q_+ > 0$ at p_o and no passive network exists.

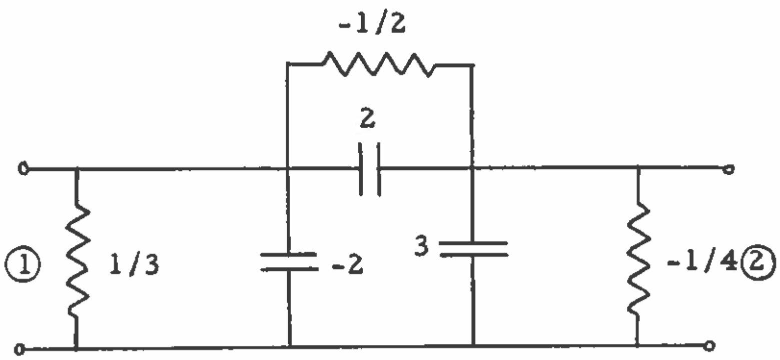

FIG. 9. --Network for E-6.

IV. SYNTHESIS OF N_P; $n = 2$, $Y_{ISS} \neq 0$

In the previous section we have shown how to find N_P if $q_+(p_o) \leq 0$ and $Y_{ISS} = 0$. Although the synthesis procedures were simple enough, we had to consider many different situations. Here we actually have even more situations, owing to the fact that Y_{RS} may be positive definite even when $q_+ < 0$. However, most of these won't have to be investigated completely because the general philosophy will be to convert to the situation of $Y_{ISS} = 0$ when possible. If this isn't possible the synthesis methods of the last section will normally be extended to encompass $Y_{ISS} \neq 0$.

At first glance one might think that the situation $Y_{ISS} \neq 0$ is of only theoretical importance. However a recollection of the fact that the α of a transistor varies with frequency should convince the reader of the practical importance.

By our assumption that $Y(p)$ is rational with real coefficients it is seen that $Y_{ISS} = 0$ for $\omega = 0$, and hence we are only interested in $\omega_o > 0$. Although we could give a separate synthesis for $\sigma_o = 0$, all of the following methods are valid for all $\sigma_o \geq 0$. In those situations where we reduce to $Y_{ISS} = 0$ the separation in terms of σ's is only necessary in applying the methods of Section III.

We again have many cases to consider. The most convenient separation of cases seems to be the following.

Case 5: Y_{RS} or Y_{IS} definite (rank 2) or Y_{RS} and Y_{IS} semi-definite (rank 0 or 1)

Case 6: Y_{RS} indefinite (rank 2) and Y_{IS} semi-definite (rank 0 or 1)

Case 7: Y_{IS} indefinite (rank 2) and Y_{RS} semi-definite (rank 0 or 1)

Case 8: Y_{RS} and Y_{IS} indefinite (rank 2)

Case 5: Y_{RS} or Y_{IS} definite (rank 2) or Y_{RS} and Y_{IS} semi-definite (rank 0 or 1)

We begin by simultaneously diagonalizing Y_{RS} and Y_{IS}. If one of these is definite this is done by known methods, (Ref. 8, p. 10); if both are semi-definite this is done by using a theorem of a companion report.[9] We then have

$$Y'(p_o) = [g_1 \dot{+} g_2] + j[b_1 \dot{+} b_2] + (g + jb)\, E \tag{IV.1}$$

We have two cases to consider

Case 5_a: $g_i + jb_i = 0$ for $i = 1$ & 2

Here $q_+ < 0$ since $(Y_{RS})' = 0$ and $|\tilde{V}Y'V| = 0$ for all V. The passive network connects port one to port two to get a zero input admittance.

Case 5_b: $g_i + jb_i \neq 0$ for $i = 1$ or 2

Assuming, without loss of generality, that $g_1 + jb_1 \neq 0$ we connect a gyrator in cascade with port one, as shown in Fig. 3. From Appendix 2 we find

$$Y_c(p_o) = [1/(g_1 + jb_1)]\begin{bmatrix} 1 & g+jb \\ g+jb & \Delta' \end{bmatrix}$$

where Δ' is the determinant of Y' of Eq. (IV.1). Here $q_+(Y_c) = q_+(Y')$ as is shown in Appendix 1, and since we have $(Y_{ISS})_c = 0$, we can use the procedures of Section III.

The following example illustrates a complete Case 5 synthesis.

E-7: Consider the network N of Fig. 10. This has

$$Y(p) = \begin{bmatrix} -2+p & 1+p \\ 1-p & -1 \end{bmatrix}$$

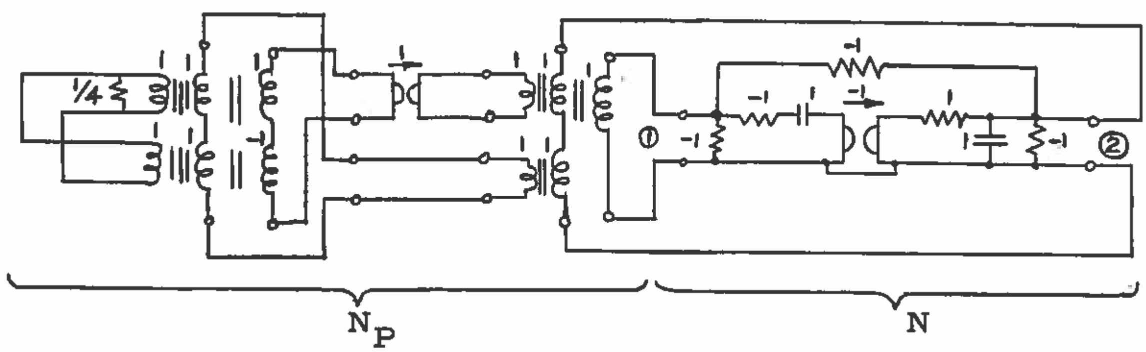

FIG. 10.--Networks for E-7.

Let $p_o = 1 + j1$, then

$$Y(p_o) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} + j[1 \dotplus 0] + (1 + j1)\,E$$

Using

$$T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

we find

$$Y'(p_o) = \tilde{T} Y T = [(j1) \dotplus (-1)] + (1 + j1) E$$

$$\Delta' = -j1$$

Using the connection of Fig. 3 gives

$$Y'_c(p_o) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} - j \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - j \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \right\} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

which fits Case $4_{c_{2_{\beta_1}}}$. Thus we obtain

$$Y_c(p_o) = \begin{bmatrix} 1 & -1 \\ -1 & -3 \end{bmatrix} - j \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} Y'_c \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

We then add $Y_{Pc} = [4 \dotplus 0]$ and connect the two ports together as shown in Fig. 10. In Fig. 10 all but the -1:1 transformer windings can actually be eliminated.

Case 6: Y_{RS} indefinite (rank 2) and Y_{IS} semi-definite (rank 0 or 1)

We must treat Y_{IS} of different ranks separately.

Case 6_a: Y_{IS} of rank zero

We diagonalize Y_{RS} to get

$$Y_c(p_o) = [1 \dotplus (-1)] + (g + jb) E$$

The method of Case 5_b now applies. However, a simple solution consists of connecting port one to port two. This latter shows that such a Y always has $q_+ \leq 0$.

Case 6_b: Y_{IS} of rank one

We first diagonalize Y_{IS} to get

$$Y'(p_o) = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} + j [b_1 \dotplus 0] + (g + jb) E \qquad \text{(IV. 2)}$$

Two further subcases must be considered.

Case 6_{b_1}: $g_{12} = 0$ or if $g_{12} \neq 0$ then $g_{22} \neq 0$

If $g_{12} = 0$ we immediately get, with $g_1 = g_{11}$, $g_2 = g_{22}$

$$Y_c(p_o) = [g_1 \dotplus g_2] + j[b_1 \dotplus 0] + (g + jb)E \tag{IV. 3}$$

If g_{12} and g_{22} are non-zero Eq. (IV. 3) is obtained by adding $-g_{12}/g_{22}$ times the second row and column of Eq. (IV. 2) to the first and then letting $g_{22} = g_2$, $(g_{11} - g_{12}^2/g_{22}) = g_1$. Since $b_1 \neq 0$, the method of Case 5_b applies.

Case 6_{b_2}: $g_{12} \neq 0$, $g_{22} = 0$

We add $-g_{11}/2g_{12}$ times the second row and column of Eq. (IV. 1) to the first to obtain

$$Y''(p_o) = \begin{bmatrix} 0 & g_{12} \\ g_{12} & 0 \end{bmatrix} + j[b_1 \dotplus 0] + (g + jb)E \tag{IV. 4}$$

Using Eq. (IV. 4) we add 1/2 of the second row and column to the first and follow this by subtracting the first row and column from the second. This gives

$$Y_c(p_o) = g_{12}[1 \dotplus (-1)] + jb_1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + (g + jb)E \tag{IV. 5}$$

Using Eq. (IV. 5) we add rows and columns to obtain a zero input admittance. Since we have found an N_P we know that all Y satisfying the conditions of Case 6_{b2} have $q_+ \leq 0$. Note that this same method can be applied to Case 3_{b2}.

Case 7: Y_{IS} indefinite (rank 2) and Y_{RS} semi-definite (rank 0 or 1)

The methods and results are identical to those for Case 6 except that real symmetric matrices are replaced by imaginary syymetric matrices and vice versa.

Case 8: Y_{RS} and Y_{IS} indefinite (rank 2)

As in Section III, we begin by diagonalizing Y_{RS} to obtain

$$Y'(p_o) = [1 \dotplus (-1)] + j\begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} + (g+jb)E \qquad \text{(IV. 6)}$$

We have the same three cases depending upon which form $B = [b_{ij}]$ can be brought to by Theorem 1 of Appendix 4.

Case 8_a: $B = B_I = [b_{11} \dotplus b_{22}]$

Here the method of Case 5_b can be applied to obtain $Y_{ISS} = 0$.

Case 8_b:

$$B \Rightarrow B_{II} = \begin{bmatrix} b_{11}-b_{22} & \sqrt{-\Delta_b} \\ \sqrt{-\Delta_b} & 0 \end{bmatrix}; \ \Delta_b = \text{det. } B$$

Here the methods of Case $4_{c_{2\beta}}$ in Section III can be taken over word for word to obtain N_P. Since N_P exists we also know that $q_+ \leq 0$.

Case 8_c:

$$B \Rightarrow B_{III} = \begin{bmatrix} 0 & -\sqrt{-\Delta_b} \\ -\sqrt{-\Delta_b} & b_{22}-b_{11} \end{bmatrix}$$

If $(b_{22} - b_{11})^2 \leq -4\Delta_b$ we use the method of Section III to obtain an N_P. If $(b_{22} - b_{11})^2 > -4\Delta_b$ we apply Theorem 2 of Appendix 4 to get

$$Y_c(p_o) = [1 \dotplus (-1)] + j[b_1 \dotplus b_2] + (g_c + jb_c)E$$

The method of Case 5_b then applies to reduce this to the $Y_{ISS} = 0$ situation.

V. SYNTHESIS OF N_P; n = 2, DEGENERATE CASES

In the last two sections we have shown how to find N_P if an N is given which possesses a Y matrix and has $q_+ \leq 0$. Clearly a dual process holds for N which have Z matrices. However, there are devices which have no Z or Y matrices but for which we would still like to find an N_P. An active network of this type which is of practical importance is the negative impedance converter (NIC).

Except for a couple of strange networks, any linear two-port network can be described by, (Ref.5, p.304),

$$AV = BI \tag{V.1}$$

where A and B are 2 x 2 matrices. Further, If Eq. (V.1) is multiplied by a non-singular matrix C on the left, the new equations describe the same two-port. For our purposes we will assume A and B to both be singular, since otherwise a Z or Y matrix exists. We will then premultiply by a non-singular C to put Eq. (V.1) in canonical form. By connecting a gyrator to such a network we will obtain a Z or Y matrix in all except some trivial cases. If $q_+ \leq 0$ for the new network, Sections III and IV apply. Considering the physical meaning of q_+ we know that $\mathcal{Q}_+$ of Eq. (I.1') is invariant under the connections to be given. Consequently we can realize a natural frequency with the given device.

Because of the assumed singularity of A and B, we easily find a C to bring Eq. (V.1) into one of the two following forms.

$$\text{I:} \quad \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$\text{II:} \quad \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{V.2}$$

We will only treat the form I. Form II has three of V_1, V_2, I_1, I_2 arbitrary and isn't of much practical interest. We have three basic connections.

Case a: Cascade Gyrator

Here we connect a gyrator, of gyration resistance γ, in cascade with port one (this is Fig. 3). Letting primed variables refer to the resulting network, we find

$$\begin{bmatrix} 0 & a_{12} \\ b_{21} & 0 \end{bmatrix} \begin{bmatrix} V_1' \\ V_2' \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ 0 & b_{22} \end{bmatrix} \begin{bmatrix} I_1' \\ I_2' \end{bmatrix} \tag{V.3}$$

Thus if one of the pairs (a_{12}, b_{21}) or (a_{11}, b_{22}) has both members non-zero a Z or Y matrix exists. Such is the case for the NIC as shown by the following.

E-8: An NIC is described by

$$\begin{bmatrix} 1 & -n \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1/n \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad , \; n \neq 0 \text{ but real}$$

The connection of Fig. 11 then gives

$$\begin{bmatrix} 0 & -n \\ 1 & 0 \end{bmatrix}\begin{bmatrix} V_1' \\ V_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1/n \end{bmatrix}\begin{bmatrix} I_1' \\ I_2' \end{bmatrix}$$

Thus we have for N' of Fig. 11

$$Y'(p) = \begin{bmatrix} 0 & -n \\ -n & 0 \end{bmatrix}$$

Clearly $q_+ < 0$ for all p.

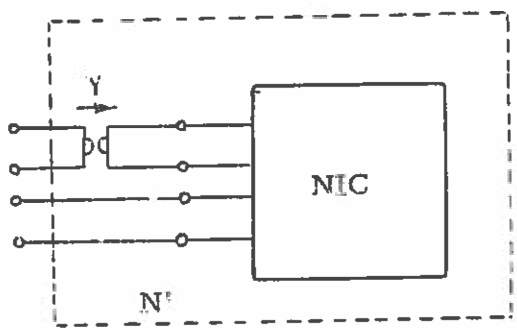

FIG. 11.--Derivation of Y' for an NIC.

Case b: Series Gyrator

Connecting a gyrator, of gyration resistance γ, in series with N gives

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1' \\ V_2' \end{bmatrix} = \begin{bmatrix} -\gamma a_{12} & \gamma a_{11} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} I_1' \\ I_2' \end{bmatrix} \tag{V. 4}$$

Thus if $a_{12}b_{22} + a_{11}b_{21} \neq 0$ a Y matrix exists. As shown by the NIC both Case a and b may lead to a Y matrix. However, the following gives an example of a network covered by Case b but not Case a.

E-8: Let N be the basic active network discovered by Tellegen, (Ref. 10, p. 143), and described by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Connecting a gyrator in series with N yields

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1' \\ V_2' \end{bmatrix} = \begin{bmatrix} 0 & \gamma \\ 1 & 0 \end{bmatrix} \begin{bmatrix} I_1' \\ I_2' \end{bmatrix}$$

This is represented also by

$$Y'(p) = \begin{bmatrix} 0 & 0 \\ \gamma^{-1} & 0 \end{bmatrix}$$

which describes an ideal current amplifier (pentode).

Besides these two important connections the parallel one is sometimes useful.

Case c: Parallel Gyrator

Connecting a gyrator, of gyration resistance γ, in parallel with N gives

$$\begin{bmatrix} a_{11} & a_{12} \\ b_{21}/\gamma & -b_{22}/\gamma \end{bmatrix} \begin{bmatrix} V_1' \\ V_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} I_1' \\ I_2' \end{bmatrix}$$

If $a_{11}b_{22} + a_{12}b_{21} \neq 0$, a Z matrix exists.

Cases a and b are sufficient to cover every network described by Eq. (V. 1) except those for which three or more of V_1, V_2, I_1, I_2 are arbitrary. Besides not being able to cover those N for which three or more of the variables are arbitrary, we can't cover those for which three or more of the variables are constrained to be zero. This is a result of the fact that Eq. (V. 1) can't be written for such networks, as pointed out by Carlin,[11] (we can extend Eq. (V. 1) to cover these cases by letting A and B be rectangular however). When these latter situations occur there is apparently no way of obtaining a Z or Y through (non-degenerate) gyrator-transformer embedding, but these networks appear to be of no practical importance.

VI. SYNTHESIS OF N_P; $n > 2$, $Y_{ISS} = 0$, MOST CASES

In the last four sections we have completely solved the problem of synthesizing N_P for one or two-ports. These methods will be extended to cover most n-ports, essentially by reducing the n-port to a one or two-port.

We assume that a given n x n Y matrix is written in the form of Eq. (I.6) at $p = p_o$ with $Y_{ISS} = 0$ and $q_+(p_o) \leq 0$. Also we will assume $Y_{RSS} = 0$, since it can be cancelled by gyrators. The two previous regions occur.

Region 1: $\omega_o = 0$

Here $Y(p_o) = Y_{RS}$. Y_{RS} is then diagonalized; the condition $q_+ \leq 0$ showing that at least one of the diagonalized elements is ≤ 0. This element is then cancelled by a non-negative element.

Region 2: $\omega_o > 0$

If $\sigma_o = 0$ we diagonalize Y_{IS} and cancel all its terms by inductances and capacitances (note that this doesn't alter q_+ as Y_{IS} doesn't enter into q_+ at $\sigma_o = 0$). The new Y_{RS} is then diagonalized and one of its non-positive elements cancelled by a non-negative one.

If $\sigma_o > 0$ we begin by diagonalizing Y_{RS} to

$$Y'_{RS} = [(-1_r) \dotplus 1_k \dotplus 0_{n-k-r}] \qquad \text{(VI. 1)}$$

We will have to consider three different values for r: $r = 0$, $r = 1$, $r > 1$. The case $r = 0$ will first be disposed of as the others are much harder. If $r = 0$, then only $q_+ = 0$ can occur and then $k < n$ with at least one of the last n-k diagonal terms of the transformed Y_{IS} zero. Assuming this to be the n^{th} element we short out the 2^{nd} thru $n-1^{st}$ terminal pairs (this gives part of N_P) and then just consider the 1^{st} and n^{th} terminal pairs. This reduces the problem to that of a two-port described by

$$Y' = \begin{bmatrix} 1+jb_1 & jb \\ jb & 0 \end{bmatrix}$$

and this is covered in Case 1 of Section III.

If $r > 0$, we diagonalize the upper left r x r submatrix of the transformed Y_{IS} to get

$$Y_c(p_o) = [(-1_r) \dotplus 1_k \dotplus 0_{n-r-k}] + j\begin{bmatrix} \begin{matrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_r \end{matrix} & \\ & \end{bmatrix} \qquad \text{(VI. 2)}$$

The constraints imposed by $q_+ \leq 0$ are in general quite complicated as is seen by reviewing Section III. However, if $r > 1$ we choose $V_3 = V_4 = \cdots = V_n = 0$ (the voltages refer to Y_c of Eq. (VI. 2)) to obtain $q_+ < 0$. For this we find a V_1 and V_2 such that $|\tilde{V} Y_c V| = 0$ (according to Appendix 3a) and in fact get $q_+ = -1$ independently of the values of b_1 and b_2. Further, if $r = 1$ and $b_1^2 \leq (\omega_o/\sigma_o)^2$ then we know, by choosing $V_2 = \cdots = V_n = 0$, that $q_+ \leq 0$.

If $r = 1$ and $b_1^2 > (\omega_o/\sigma_o)^2$ the constraints imposed by $q_+ \leq 0$ are still unknown and we are also unable to give a general synthesis.

This is the only unsolved case for $n > 2$ and $Y_{ISS} = 0$.

Having established these facts the synthesis of N_{Pc} is quite simple. If $r = 1$ and $b_1^2 \leq (\omega_o/\sigma_o)^2$ we short out ports 2 through n and then, ignoring these ports, connect a passive network to port one through the use of Eq.(II. 3). If $r > 1$ we short out ports 3 through n and, ignoring these ports, apply Case 2 of Section III to get a passive network. The following example illustrates this procedure.

E-9: Let N be described by

$$Y(p) = \begin{bmatrix} -3/2 & -2 & -2 \\ -2 & 0 & -1 \\ -2 & -1 & 0 \end{bmatrix} + p \begin{bmatrix} 1/2 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

with $p_o = 1 + j1$ we have

$$Y(p_o) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + j \begin{bmatrix} 1/2 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Shorting ports 2 and 3 gives (as $V_2 = V_3 = 0$)

$$I_1 = (-1 + j(1/2))\, V_1$$

$$I_2 = (j2)\, V_1$$

$$I_3 = (j2)\, V_1$$

Ignoring I_2 and I_3, we connect $1 - j(1/2)$ onto port one to obtain a natural frequency. The process is shown in Fig. 12.

From these methods it should be noted that, if $r = 1$ and $b_1^2 > (\omega_o/\sigma_o)^2$ and there is some principal 2 x 2 submatrix of Y_c which contains the (1, 1) element and also has $q_+ \leq 0$, then we can obtain an N_P using the methods of Section III. Likewise, if $Y_{ISS} \neq 0$ we can obtain Eq. (VI. 2) with a Y_{ISS} term. If this has some principal 1 x 1 or 2 x 2 submatrix for which $q_+ \leq 0$, then, after shorting all but these one or two ports, Section IV can be applied.

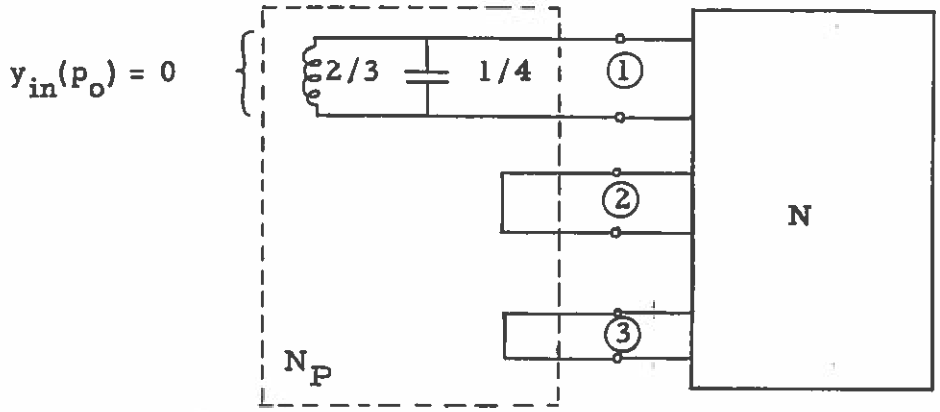

FIG. 12. --n-port synthesis.

CONCLUSIONS

In Sections II-V the problem posed in the introduction was completely solved for the one and two-port. That is, given a one or two-port for which $q_+(p_o) \leq 0$ we have shown how to find a passive embedding network N_P to give a natural frequency at p_o, even when the active network is highly degenerate. By reducing the n-port to the one or two-port case we have also shown how to find N_P for many active n-ports. Although the problem is still unsolved for the general n-port, it may turn out that, after connection of transformers, there is a subnetwork of one or two ports which has $q_+ \leq 0$. In this case Section VI would give a general solution even when $Y_{ISS} \neq 0$.

OPEN PROBLEMS

The primary unsolved problems are:

1. Under what conditions do matrices for Case $4c2_{Y2}$ have $q_+ \leq 0$?

2. For the n-port, if $b_1^2 > (\omega_c/\sigma_o)^2$ in Eq. (VI. 2), when is $q_+ \leq 0$?

3. How do we handle $Y_{ISS} \neq 0$ for the n-port?

4. Is there a better synthesis that avoids all the cases we had to consider?

5. Can the degenerate 2-ports with three arbitrary variables be forced to give a natural frequency?

6. How do we handle degenerate n-ports?

7. If a network has $q_+(p_o) \leq 0$ and $q_+(p_1) \leq 0$, does <u>one</u> N_P exist which will give a natural frequency at <u>both</u> p_o and p_1?

8. Under what conditions can an active network be put in parallel with a passive network such that the result is passive for all p? The networks with $Y(p) = p^2$ or $\begin{bmatrix} p & 0 \\ -p & p \end{bmatrix}$ show that this may not be possible.

APPENDIX 1: INVARIANCE OF q_+ FOR THE CASCADE GYRATOR CONNECTION

Let a gyrator be connected in cascade with port one of an n-port N as shown in Fig. A. 1.

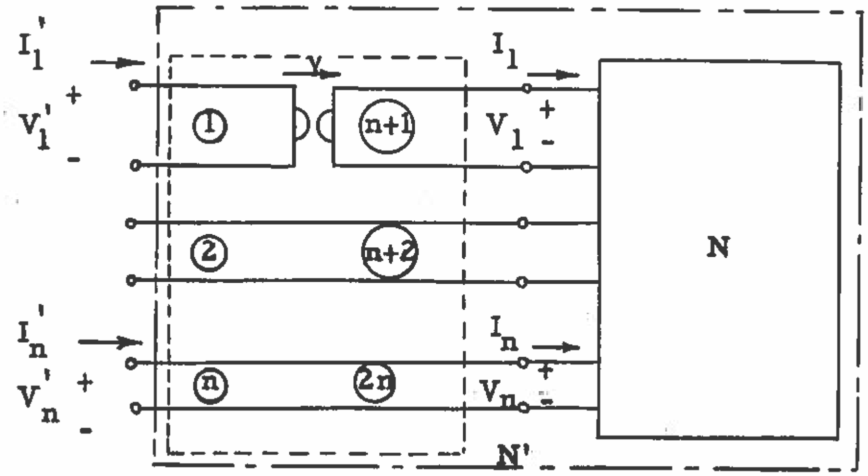

FIG. A. 1. --Cascade connection.

For Eq. (I. 1') we find

$$\mathcal{Q}_+(V', I', p) = \mathcal{Q}_+(V, I, p) \tag{A_1. 1}$$

since $V_1'I_1' = (-\gamma I_1)(-V_1/\gamma) = V_1 I_1$. Assuming N to have a Y matrix, Appendix 2 shows that N' also has a Y' matrix (if $y_{11} \neq 0$). As a consequence Q_+ is the same for both N and N' and hence the q_+'s are identical.

It should also be noted that Eq. (I. 1') shows that if a network has Z, Y or S (scattering matrix) then the q_+'s, defined analogously to Eq. (I. 2), are all identical (for Z we minimize over currents, for S we minimize over incident waves).

APPENDIX 2: DETERMINATION OF Y FOR THE CASCADE GYRATOR CONNECTION

We wish to find Y' for N' of Fig. A.1. For this first consider the more general situation of Fig. A.2.

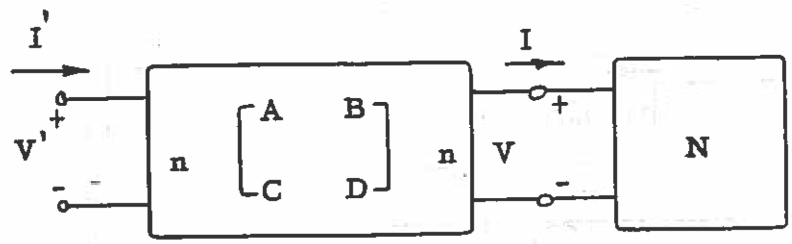

FIG. A.2.--General cascade connection.

Here we can write

$$\begin{bmatrix} V' \\ I' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V \\ I \end{bmatrix}, \quad I = YV \qquad (A_2.1)$$

where A, B, C, D are n x n matrices. Solving this we can obtain

$$Y' = [C + DY]\,[A + BY]^{-1} \qquad (A_2.2)$$

Applying this to Fig. A.1 we find

$$\begin{aligned} A &= D = 0 \dot{+} 1_{n-1} \\ B &= -\gamma \dot{+} 0_{n-1} \\ C &= (-1/\gamma) \dot{+} 0_{n-1} \end{aligned} \qquad (A_2.3)$$

If $y_{11} \neq 0$, Eq. $(A_2.2)$ is easily evaluated to give

$$Y' = \left[\begin{array}{c|c} 1/(\gamma^2 y_{11}) & y_{12}/(\gamma y_{11}) \;\cdots\; y_{1n}/(\gamma y_{11}) \\ \hline -y_{21}/(\gamma y_{11}) & \\ \vdots & y'_{ij} = (y_{ij}y_{11} - y_{i1}y_{1j})/y_{11} \\ & \text{for } i, j \geq 2 \\ -y_{n1}/(\gamma y_{11}) & \end{array}\right] \qquad (A_2.4)$$

Note that if $y_{ij} = y_{ji}$ for all i, j then $y'_{ij} = y'_{ji}$ for i, $j \geq 2$.

APPENDIX 3: Q_+ FOR VARIOUS CASES

Here we will prove some of the statements made about Q_+ and q_+ in the body of the report.

a) Case 2

We have $Y_c = -1_2 + j\,[b_1 \dotplus b_2] + gE$ or

$$Q_+ = -|V_1|^2 - |V_2|^2 + (\sigma_o/|p_o|)\,|V_1^2(-1 + jb_1) + V_2^2(-1 + jb_2)|$$

Choosing

$$(V_1/V_2)^2 = -(-1 + jb_2)/(-1 + jb_1)$$

gives $|\tilde{V} Y_c V| = 0$. As a consequence $q_+ = -1$, no matter what values b_1 and b_2 assume.

b) Case 4_{b_3}

We have $Y_c = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + j[b \dotplus 0] + gE$ or

$$Q_+ = 2\,\mathrm{Re}(V_1^* V_2) + (\sigma_o/|p_o|)\,|jbV_1^2 + 2V_1V_2|$$

$$= |V_1|^2 \{2\,\mathrm{Re}\,(V_2/V_1)^* + (\sigma_o/|p_o|)\,|jb + 2(V_2/V_1)|\} \quad \text{if } V_1 \neq 0$$

We now choose

$$(V_2/V_1) = u - j(b/2) \text{ with u arbitrary but} < 0$$

Then

$$Q_+ = (-2u)\,[-1 + (\sigma_o/|p_o|)] < 0$$

As a consequence we know $q_+ < 0$, independently of b.

c) Case 4_{c_1}

We have $Y_c = [1 \dotplus (-1)] + j[b_1 \dotplus b_2] + gE$. We wish to find the constraints put on b_1 and b_2 by $q_+ \leq 0$.

$$Q_+ = |V_1|^2 - |V_2|^2 + (\sigma_o/|p_o|)\,|V_1^2(1 + jb_1) + V_2^2(-1 + jb_2)|$$

Let

$$|V_1|^2 = \epsilon |V_2|^2$$

$$\phi_i = \text{phase of } V_i^2 (1 + jb_i), \quad i = 1 \ \& \ 2$$

$\varphi_1 \to V_1^2(1+jb_1)$; $\varphi_2 \Rightarrow V_2^2(-1+jb_2)$

Since we wish to minimize Q_+ we require

$$0 \leq \epsilon < 1$$

if $\epsilon = 1$
$q_+ = 0$
$b_1^2 = b_2^2$

$$\phi_1 = \phi_2 + \pi$$

Then, restricting Q_+ to $|V_1|^2 + |V_2|^2 = 1$, we have

$$Q_+ = \frac{1}{1+\epsilon} \{-1 + \epsilon + (\sigma_o / |p_o|) \, | (1 + b_2^2)^{1/2} - \epsilon (1 + b_1^2)^{1/2} | \}$$

We have two cases to consider.

Case I: $b_1^2 \leq b_2^2$

Then

$$Q_+ = \frac{1}{1+\epsilon} \{[-1 + (\sigma_o / |p_o|)(1 + b_2^2)^{1/2}] - \epsilon [-1 + (\sigma_o / |p_o|)(1 + b_1^2)^{1/2}]$$

$$\geq [(1 - \epsilon)/(1 + \epsilon)][-1 + (\sigma_o / |p_o|)(1 + b_2^2)^{1/2}]$$

Consequently for $q_+ \leq 0$ we clearly require

$$b_2^2 \leq (\omega_o / \sigma_o)^2$$

Further, this inequality on b_2 is sufficient to insure $q_+ \leq 0$ as is seen by taking $\epsilon = 0$.

Case II: $b_1^2 > b_2^2$

Letting

$$\epsilon_o = [(1 + b_2^2)/(1 + b_1^2)]^{1/2}$$

then

$$(1+\epsilon) Q_+ = \begin{cases} \{[-1 + (\sigma_o / |p_o|)(1 + b_2^2)^{1/2}] + \epsilon [1 - (\sigma_o / |p_o|)(1 + b_1^2)^{1/2}]\} & \text{for } \epsilon < \epsilon_o \\ \{[-1 - (\sigma_o / |p_o|)(1 + b_2^2)^{1/2}] + \epsilon [1 + (\sigma_o / |p_o|)(1 + b_1^2)^{1/2}]\} & \text{for } \epsilon \geq \epsilon_o \end{cases}$$

On differentiating we find

$$(1+\epsilon)^2[dQ_+/d\epsilon] = \begin{cases} 2-(\sigma_o/|p_o|)[(1+b_1^2)^{1/2}+(1+b_2^2)^{1/2}] & \text{for } \epsilon < \epsilon_o \\ 2+(\sigma_o/|p_o|)[(1+b_1^2)^{1/2}+(1+b_2^2)^{1/2}] & \text{for } \epsilon \geq \epsilon_o \end{cases}$$

Two situations can occur.

Case II_α: $\sigma_o[(1+b_1^2)^{1/2} + (1+b_2^2)^{1/2}] \leq 2|p_o|$

In this case $(dQ_+/d\epsilon) \geq 0$ for all ϵ and hence the minimum occurs at $\epsilon = 0$. This requires

$$b_2^2 \leq (\omega_o/\sigma_o)^2 \quad \text{for } q_+ \leq 0$$

Case II_β: $\sigma_o[(1+b_1^2)^{1/2} + (1+b_2^2)^{1/2}] > 2|p_o|$

Here $dQ_+/d\epsilon$ changes from negative to positive (as ϵ increases) at $\epsilon = \epsilon_o$. The minimum is then at ϵ_o. This always has $q_+ < 0$, since at $\epsilon = \epsilon_o$, $(1+\epsilon_o)\,Q_+ = -1 + \epsilon_o < 0$.

From these results we conclude that if $q_+ < 0$ and $b_2^2 \geq (\omega_o/\sigma_o)^2$ then $b_1^2 > b_2^2$.

APPENDIX 4: CANONICAL FORMS FOR TWO INDEFINITE MATRICES

Theorem 1: Let $G = [1 \dotplus (-1)]$ and let $B = [b_{ij}]$ be a real, symmetric, indefinite matrix. Then there exists a real, non-singular matrix T such that $\tilde{T}GT = G$ and $\tilde{T}BT$ is one of the following matrices

$$B_I = [b_{11} \dotplus b_{22}], \quad B_{II} = \begin{bmatrix} b_{11}-b_{22} & \sqrt{-\Delta_b} \\ \sqrt{-\Delta_b} & 0 \end{bmatrix}, \quad B_{III} = \begin{bmatrix} 0 & -\sqrt{-\Delta_b} \\ -\sqrt{-\Delta_b} & b_{22}-b_{11} \end{bmatrix}$$

where $\Delta_b = \det. B$

Proof: Consider the two matrices

$$\begin{aligned} T_1 &= (1/\sqrt{1-t^2})\begin{bmatrix} 1 & t \\ t & 1 \end{bmatrix} \text{ if } t^2 < 1 \\ T_2 &= (1/\sqrt{t^2-1})\begin{bmatrix} t & 1 \\ 1 & t \end{bmatrix} \text{ if } t^2 > 1 \end{aligned} \qquad (A_4.1)$$

where t will later be defined. Then we have $\tilde{T}_1GT_1 = \tilde{T}_2GT_2 = G$ and

$$\begin{aligned} (1-t^2)\,\tilde{T}_1BT_1 &= \begin{bmatrix} b_{11}+2tb_{12}+t^2b_{22} & b_{12}(1+t^2)+(b_{11}+b_{22})t \\ b_{12}(1+t^2)+(b_{11}+b_{22})t & b_{22}+2tb_{12}+t^2b_{11} \end{bmatrix} \\ (t^2-1)\,\tilde{T}_2BT_2 &= \begin{bmatrix} b_{22}+2tb_{12}+t^2b_{11} & b_{12}(1+t^2)+(b_{11}+b_{22})t \\ b_{12}(1+t^2)+(b_{11}+b_{22})t & b_{11}+2tb_{12}+t^2b_{22} \end{bmatrix} \end{aligned} \qquad (A_4.2)$$

Now if $b_{12} = 0$, $B = B_I$ already and if $b_{22} = 0$ then $B = B_{II}$ already. Thus assume that $b_{12} \neq 0$ and $b_{22} \neq 0$, then we choose t such that one of the diagonal members in Eq.(A_4.2) is zero, i.e., choose

$$t = -(1/b_{22})\,[b_{12} \pm \sqrt{-\Delta_b}\,]$$

Since $b_{12} \neq 0$ and $\Delta_b < 0$, we can choose the $\pm$ sign in t such that $t^2 \neq 1$. We then choose the T of the theorem to be one of T_1, T_2

depending upon the value of t^2. B_{II} or B_{III} then results as a short calculation shows, after perhaps normalizing the (1, 2) elements by -1. Q. E. D.

Theorem 2: Let G and $B = B_{III}$ be given, as in Theorem 1, and let $(b_{22} - b_{11})^2 > -4\Delta_b$. Then there exists a real, non-singular matrix T such that $\tilde{T}GT = G$ and $\tilde{T}B_{III}T$ is diagonal.

Proof: Consider Eqs. (A_4. 1) and (A_4. 2) where in this latter we make the replacement

$$b_{11} \Rightarrow 0$$

$$b_{12} \Rightarrow -\sqrt{-\Delta_b}$$

$$b_{22} \Rightarrow b_{22} - b_{11}$$

We then choose t in Eq. (A_4. 2) to make the (1, 2) term zero. Such a t is

$$t = (1/2)[(b_{22} - b_{11}) \pm \sqrt{(b_{22} - b_{11})^2 + 4\Delta_b}\,]/\sqrt{-\Delta_b}$$

t is real and $\neq 1$ by the assumption made on $b_{22} - b_{11}$. Q. E. D.

at one

APPENDIX 5: ALTERNATIVE SYNTHESIS METHODS

As is illustrated by E-1, it is advantageous to have more than one synthesis method available for N_P for each case. For the one-port some alternate methods are contained in Ref. 9 . In many cases some alternate methods are obvious, however, we will give three which are not very easy to come by.

Case 4_{c_2} : Ignoring the skew-symmetric term, we have

$$Y''(p_o) = [1 \dotplus (-1)] + j\begin{bmatrix} b_{11}-b_{22} & \sqrt{-\Delta_b} \\ \sqrt{-\Delta_b} & 0 \end{bmatrix}$$

We add

$$Y_1(p) = (p/\omega_o)\begin{bmatrix} b_1 & -\sqrt{-\Delta_b} \\ -\sqrt{-\Delta_b} & b_2 \end{bmatrix}$$

Here b_1 and b_2 are chosen to make Y_1 realizable and $Y_1 + Y''$ have $q_+ \leq 0$. For this let

$$b_2 = \omega_o/2\sigma_o$$

$$b_1 = -2\sigma_o\Delta_b/\omega_o \qquad \text{i. e., det. } Y_1 = 0$$

At p_o we then have

$$Y_1 + Y'' = \begin{bmatrix} 1-(2\sigma_o^2\Delta_b/\omega_o^2) & -\sigma_o\sqrt{-\Delta_b}/\omega_o \\ -\sigma_o\sqrt{-\Delta_b}/\omega_o & -1/2 \end{bmatrix} + j\begin{bmatrix} -2\sigma_o\Delta_b/\omega_o + b_{11} - b_{22} & 0 \\ 0 & \omega_o/2\sigma_o \end{bmatrix}$$

We now cancel the (2, 2) term by a passive network, Eq. (II. 3), and cancel the (2, 1) term by a gyrator to get a zero determinant.

Case 7_b: The situation Y_{IS} indefinite and Y_{RS} of rank one will be treated. We begin by diagonalizing Y_{RS} to get

$$Y = [g_1 \dotplus 0] + j\begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} + (g + jb)E \qquad (A_5.1)$$

This leads to several cases.

Case 7_{b_1}: $b_{12} = 0$ or if $b_{12} \neq 0$ then $b_{22} \neq 0$

We either have directly, or obtain by adding $-b_{12}/b_{22}$ times the second row and column of Eq. $(A_5.1)$ to the first,

$$Y_c = [g_1 \dotplus 0] + j[b_1 \dotplus b_2] + (g + jb)E$$

The method of Case 5_b can now be applied.

Case 7_{b_2}: $b_{12} \neq 0$ and $b_{22} = 0$

After perhaps adding the second row and column to the first and normalizing we get for Eq. $(A_5.1)$ (after dropping the gE term)

$$Y_c = [g_c \dotplus 0] + j\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + jb_c E\,; \qquad g_c = \pm 1 \qquad (A_5.2)$$

There are then three synthesis methods depending upon the value of b_c.

Case $7_{b_{2_\alpha}}$: $b_c^2 = 1$

In this case either the last row or column of Eq. $(A_5.2)$ is already zero.

Case $7_{b_{2_\beta}}$: $b_c^2 > 1$

We add, for all p,

$$Y_{Pc} = [2 \dotplus 0] + [0 \dotplus (b_c^2 - 1)]$$

where the matrix $[2 \dotplus 0]$ is omitted if $g_c = +1$, to obtain a zero determinant.

Case $7_{b_{2_\gamma}}$: $b_c^2 < 1$

If $g_c = -1$ we add, for all p,

$$Y_{Pc} = [0 \dotplus (1 - b_c^2)]$$

to obtain a zero determinant.

If $g_c = +1$ we force the Y_{ISS} term to zero in the following way. We first add a gyrator of gyration resistance

$$\gamma_1 = -1/\sqrt{1 - b_c^2}$$

in parallel and then follow this by adding a gyrator of gyration resistance

$$\gamma_2 = (1/2)/\sqrt{1 - b_c^2}$$

in series. The resultant network, N_o, has an admittance matrix Y_o with $(Y_{ISS})_o = (Y_{RSS})_o = 0$.

Case 8_b: We have

$$Y_c = [1 \dotplus (-1)] + j\begin{bmatrix} b_{11}-b_{22} & \sqrt{-\Delta_b} \\ \sqrt{-\Delta_b} & 0 \end{bmatrix} + (g + jb)E$$

We then consider two situations.

Case 8_{b_1}: $-(1 + b^2 + \Delta_b) + ([b_{11} - b_{22}]/2b)^2 < 0$

We add, for all p,

$$Y_{Pc} = [g_1 \dotplus g_2]$$

and choose g_1, g_2 and g such that the new determinant is zero. The new determinant is

$$\Delta = [-1 - b^2 - \Delta_b - g_1 + g_1 g_2 + g^2] + j[(b_{11} - b_{22})(-1 + g_2) + 2gb] \quad (A_5.3)$$

Setting the imaginary term equal to zero gives

$$g = (1 - g_2)(b_{11} - b_{22})/2b$$

Plugging this into the real part set equal to zero yields

$$g_1 - g_2 - g_1 g_2 + (2g_2 - g_2^2)([b_{11} - b_{22}]/2b)^2 = -(1 + b^2 + \Delta_b) + ([b_{11} - b_{22}]/2b)^2 \quad (A_5.4)$$

Arbitrarily setting this equal to $-g_1$ we get

$$g_1 = (1 + b^2 + \Delta_b) - ([b_{11} - b_{22}]/2b)^2$$

This results from the right of Eq. (A_5. 4). From the left side we find

$$g_2 = \begin{cases} 2g_1/(1+g_1) & \text{if } b_{11} = b_{22} \\ 1-(1+g_1)2b^2/(b_{11}-b_{22})^2+(1/2)\sqrt{[-2+(1+g_1)4b^2/(b_{11}-b_{22})^2]^2+32g_1b^2/(b_{11}-b_{22})^2} & \text{if } b_{11} \neq b_{22} \end{cases}$$

The values for the passive network are then determined.

Case 8_{b_2}: $-(1 + b^2 + \Delta_b) + ([b_{11} - b_{22}]/2b)^2 \geq 0$

Here we add, for all p,

$$Y_{Pc} = [g_1 \dotplus 0]$$

and choose g_1 and g such that $\Delta = 0$. Here Eq. (A_5. 3) remains valid with $g_2 = 0$ and we find

$$g = (b_{11} - b_{22})/2b$$

$$g_1 = -(1 + b^2 + \Delta_b) + ([b_{11} - b_{22}]/2b)^2$$

Case 8_c: We have

$$Y_c = [1 \dotplus (-1)] + j\begin{bmatrix} 0 & -\sqrt{-\Delta_b} \\ -\sqrt{-\Delta_b} & b_{22}-b_{11} \end{bmatrix} + (g + jb)\,E$$

Almost the same two situations that were treated in Case 8_b occur here.

Case 8_{c_1}: $-(1 + b^2 + \Delta_b) + ([b_{22} - b_{11}]/2b)^2 \leq 0$

We add, for all p,

$$Y_{Pc} = [g_1 \dotplus 0]$$

2 and choose g_1 and g to force the new determinant to zero. This gives

$$g = -(b_{22} - b_{11})/2b$$

$$g_1 = (1 + b^2 + \Delta_b) - ([b_{22} - b_{11}]/2b)^2$$

Case 8_{c_2}: $-(1 + b^2 + \Delta_b) + ([b_{22} - b_{11}]/2b)^2 > 0$

Here we put a gyrator, of gyration resistance γ, in series and then adjust γ and g such that the new Y_{ISS} is zero. The values required are

$$g = - [(b_{22} - b_{11})/2b] + 1/2\gamma$$

$$\gamma^2 = \frac{1}{4} [-(1 + b^2 + \Delta_b) + ([b_{22} - b_{11}]/2b)^2]^{-1}$$

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