A CLIFFORD ALGEBRA REPRESENTATION OF N-DIMENSIONAL MAXWELL EQUATIONS

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"I search, free and free of charge, for something more certain than the fall of dice." [1]

I. Introduction

For several reasons it seems worthwhile developing a theory of multidimensional Maxwell equations. For one, there is the possible use of such equations in the realizability theory of passive n-variable networks where basic, outstanding, and interesting problems are outlined [2, p.830]. To be sure several paths for this development are open, one interesting approach [3, p.485] being through algebraic topology using the differential form representation of Maxwell's equations. Here, though, we follow the techniques of M. Riesz [4], as presented by Hestenes [5, p.29], and represent field quantities in an appropriate Clifford algebra having a suitable metric and associated products and derivative operators introduced. We first review the ideas of Clifford algebras.

"Each thing is pushed and squeezed and held in place by others." [1]

II. Clifford Algebras

A Clifford algebra $\mathcal{C}$ is comprised of a 2^n-dimensional vector space built upon an underlying n-dimensional vector space with products in the algebra defined through the use of an nxn metric matrix. Here the underlying vector space will be real n-space, R^n, with (contravariant) basis vectors $e^1, e^2, \ldots, e^n$ and with an attached scalar product $<,>$. The scalar product is related to the basis vectors of R^n by the real, symmetric (but generally indefinite) metric matrix $M = [m^{ij}]$ through $m^{ij} = <e^i, e^j>$. And by an orthogonalization of the basis vectors we can assume M to be diagonal. The 2^n basis vectors for the Clifford algebra are taken as

$$e^0 = 1,\ e^i,\ e^{ij} = e^i e^j,\ e^{ijk} = e^i e^j e^k,\ \ldots,\ e^{12\ldots n} = e^1 e^2 \ldots e^n \qquad (II.1)$$

Here the superscripts are the 2^n subsets of the set $\{1, 2, \ldots, n\}$ [6, p.186] and e^0 is identified with the real number one. These basis vectors are tensor products, graded by the order of the superscript subset; any vector in the Clifford algebra is represented by $\Sigma a_\alpha e^\alpha$ for $\alpha \subset \{1,2,\ldots,n\}$ and a_α real. Now, an algebra is a vector space with multiplication defined for any two vectors; the multiplication in a Clifford algebra proceeds from $e^i e^j + e^j e^i = 2m^{ij} e^0 = 2m^{ij}$. Multiplication ab for any two vectors a, b is now on hand and in terms of it we also define a dot and a wedge product (giving $ab = a \cdot b + a \wedge b$)

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$$2a\cdot b = ab + ba, \quad 2a\wedge b = ab - ba \tag{II.2}$$

Of particular interest to us will be the special metric

$$M = \text{diagonal}\,[+1, -1, \ldots, -1] \tag{II.3}$$

in which case $(e^1)^2 = 1 = e^1\cdot e^1 = e^0$, $(e^i)^2 = -1 = e^i\cdot e^i$ for $i>1$ and $e^ie^j = -e^je^i = e^{ij} = e^i\wedge e^j$ for $i\neq j$ and $i,j \geq 1$. Through $G = M^{-1} = [g_{ij}]$ the covariant basis e_1, e_2, ..., e_n is defined by $e_i = \sum_{j=1}^{n} g_{ij}e^j$ in which case we can express any vector in the algebra in terms of contravariant, covariant, or mixed components. For example for Clifford 1- and 2-vectors: $x = \sum_{i=1}^{n} x^ie_i = \sum_{j=1}^{n} x_je^j$ with $x_j = \sum_{i=1}^{n} x^ig_{ij}$, $F = \sum_{i=1}^{n}\sum_{j=1}^{n} F^{ij}e_ie_j = \sum_{i=1}^{n}\sum_{j=1}^{n} F^i_je_ie^j = \sum_{i=1}^{n}\sum_{j=1}^{n} F_{ij}e^ie^j$. Note that we just as well assume $F^{ij} = -F^{ji}$ and that we have $e_0 = e^0 = 1$. Returning to (II.1), the basis vectors of $\mathcal{C}$ whose superscripts are the 2^{n-1} even subsets of $\{1,2,\ldots,n\}$ also form a Clifford algebra $\mathcal{C}_s$, $\mathcal{C}_s \subset \mathcal{C}$, the subscript s denoting "space" for its later meaning. We will label the (contravariant) basis of $\mathcal{C}_s$ as (for $i,j,\ldots \geq 2$)

$$e_s^0 = 1,\ e_s^i = e^1e^i,\ e_s^ie_s^j,\ \ldots,\ e_s^{23\ldots n} = e_s^2e_s^3\ldots e_s^n \tag{II.4}$$

For $\mathcal{C}$ we next extend its underlying "field" to differentiable functions in n (contravariant) variables x^1, x^2, ..., x^n, while introducing the partial derivative covariant notation $\partial_i = \partial/\partial x^i$. This allows the introduction into $\mathcal{C}$ of a gradient vector

$$\nabla = \sum_{i=1}^{n} \partial_ie^i \tag{II.5}$$

It is because of this form of the gradient, based upon the covariant ∂_i, that we began with contravariant basis vectors. And since the final basis vectors, called pseudoscalars, in $\mathcal{C}$ and $\mathcal{C}_s$ are somewhat important, we give them labels

$$\underline{i} = e^{12\ldots n}, \quad \underline{i}_s = e_s^{23\ldots n} \tag{II.6}$$

We find

$$\underline{i}_s = (-1)^{[n/2]}(e^1)^n\underline{i}, \quad \underline{i}^2 = \underline{i}_s^2 = (-1)^{\sum_{k=2}^{n} k} \tag{II.7}$$

with $[x]$ denoting the largest integer not exceeding x. We also introduce a conjugation operation, defining for any Clifford algebra vector a, $a^* = e^1ae^1$, from which $\underline{i}^* = (-1)^{n-1}\underline{i}$, $\underline{i}_s^* = (-1)^{n-1}\underline{i}_s$ when considered in $\mathcal{C}$. Finally, we have an "even" and "odd" decomposition (giving $a = a^e + a^o$)

$$2a^e = a + a^*, \quad 2a^o = a - a^* \tag{II.8}$$

in which we can also set $a^e = \underline{i}_sA^s$ with the parity of A^s being the same as that of $\underline{i}_s$.

> "There is no base in the air.
> And I thought this was the order of the world,
> not a speck of dust out of place." [1]

III. Maxwell's Equations

With the Clifford algebra background in hand we can relatively swiftly set up Maxwell's equations. For the R^n coordinate vector $\underline{x} = [x^i]$ to be physical we separate time as the first coordinte, $x^1 = t$. Whithin this privileged coordinate system the differential distance ds at a given point will be taken to be measured according to the fundamental metric law (expressed in terms of covariant components $\underline{x} = [x_i] = \tilde{C}\underline{x}$, $\tilde{}$ = transpose)

$$(ds)^2 = (dx_1)^2 - \sum_{i=2}^{n} (dx_i)^2 = d\tilde{\underline{x}} M d\underline{x} \qquad \text{(III.1)}$$

Equation (III.1) defines M as the metric matrix of (II.3) through which our Clifford algebra is defined.

Next we take as the basic physical law, generalizing [4, p.179] to n dimensions,

$$\nabla F = j \qquad \text{(III.2)}$$

where j is an excitation Clifford algebra 1-vector, F is a field 2-vector, and ∇ is the gradient operator 1-vector, (II.5). Now define using the $\mathcal{C}_s$ basis of (II.4) and the even - odd decomposition of (II.8)

$$e^1\nabla = \partial_t + \underline{\nabla} = \partial_1 + \sum_{i=2}^{n} \partial_i e_s^i \qquad \text{(III.3a)}$$

$$e^1 j = \rho - J = j_1 + \sum_{i=2}^{n} j_i e_s^i \qquad \text{(III.3b)}$$

$$i_s F^s = F^e = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} F_{ij}(e^{ij} + e^1 e^i e^j e^1) = \sum_{i=2}^{n} \sum_{j=2}^{n} F_{ij} e_s^{ji} \qquad \text{(III.3c)}$$

$$F^o = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} F_{ij}(e^i e^j - e^1 e^i e^j e^1) = \sum_{j=2}^{n} F_{1j} e_s^j \qquad \text{(III.3d)}$$

Multiplying (III.2) by e^1 on the left, substituting (III.3) in the result, using $\underline{\nabla} a = \underline{\nabla}\cdot a + \underline{\nabla}\wedge a$ and $\underline{\nabla}\underline{i}_s = (-1)^n \underline{i}_s \underline{\nabla}$ in $\mathcal{C}_s$, and equating respectively terms of dimension 0, 1, 2, 3 in the e_s's, yields

$$\underline{\nabla}\cdot F^o = \rho, \; \partial_t F^o + (-1)^n \underline{i}_s \underline{\nabla}\wedge F^s = -J, \; \underline{i}_s \partial_t F^s + \underline{\nabla}\wedge F^o = 0, \; \underline{i}_s \underline{\nabla}\cdot F^s = 0 \qquad \text{(III.4)}$$

Following Schilt [7], and working in Giorgi units, we define

$$F^o = D = \epsilon_o \sqrt{m}\, E, \quad F^s = \frac{\sqrt{m}}{c} H, \quad B = \mu_o \sqrt{m}\, H \qquad \text{(III.5)}$$

where m is the absolute value of the determinant of the metric matric M, $\epsilon_o \mu_o = 1/c^2$, c = velocity of wave transmission. Defining also the cross product [5, p.22] through

$$a \times b = (-1)^{n-1} \frac{\sqrt{m}}{c} \underline{i}_s a \wedge b \qquad \text{(III.6)}$$

for any a and b in $\mathcal{C}_s$, equations (III.4) yield, on division of the last two by $\sqrt{\epsilon_o/\mu_o}\, i_s$ with $\underline{i}_s^{-2}$ from (II.7),

$$\underline{\nabla}\cdot D = \rho \qquad \text{(III.7a)}$$

$$-\partial_t D + \underline{\nabla}\times H = J \qquad \text{(III.7b)}$$

$$\partial_t B + (-1)^\alpha \underline{\nabla}\times E = 0, \qquad \alpha = \sum_{k=1}^{n} k \qquad \text{(III.7c)}$$

$$\underline{\nabla}\cdot B = 0 \qquad \text{(III.7d)}$$

These are recognized as Maxwell's equations in classical form when n = 4 (as α = 6 then), though now expressed in our Clifford algebra.

"I have seen happiness face to face:
I can still see the hesitant light
fumbling in its fluffy hair." [1]

IV. Discussion

Since Maxwell's equations are fundamental to both electrical engineering and basic physics they have previously been looked at in many ways. Here another formulation is developed, that representing the equations in a Clifford algebra in n dimensions, resulting in (III.7). This development has several advantages. Besides its elegance, it proceeds from a more general physical law, $\nabla F = j$, which yields directly generalization of classical results to n dimensions. Further, it very forcefully focuses attention on the mathematical differences between the four field vectors B, D, E, H with the relationships between the former two and the latter two as brought into perspective by Schilt in an apparently little recognized work; thus $B^i = \mu_o m^{\frac{1}{2}} m^{ij} H_j$, $D^i = \epsilon_o m^{\frac{1}{2}} m^{ij} E_j$. These latter differences are of course very important to engineering constructs but generally ignored by mathematical physicists working in the area whose interests are in quantum fields in the vacuum. In our development we also worked with the equations at a fixed point assuming that the metric was adjusted at the point into the unit diagonal form of (II.3). But by keeping the metric unspecified in certain equations we have allowed for development of details covering variable points and metrics. The Clifford algebra formulation does lead to new and promising research areas, both on a physical basis, such as gravitational fields and n-dimensional holography, and on pure and applied mathematical bases, such as operator theory in Clifford algebras and multivariable network synthesis realizability. Indeed formally the theory is within an operator theory of differential rings, as previously outlined for this series [8] but still to be completely developed.

The search of A. Pfaffenberger, and interest, on the topic is gratefully acknowledged, for which this paper is dedicated.

"Now I see all I can do is love." [1]

"I looked up in the night.
The loom of the past was weaving laws
from glittering threads of change." [1]

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